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On the Numerical Solutions of the Abel's Differential Equation by Padé Series

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Abstract

In this paper, we propose a matrix method to solve the Abel's differential equation in terms of Taylor polynomials and a formula for calculation of mth order derivative of y^n. Moreover, numerical solutions of Abel's differential equation has been presented by Padé approximation. The solution is calculated in the form of a series with easily computable components. The numerical results show the effectiveness of the method for this type of equation. Comparing the method with some known techniques indicates that the present approach is relatively easy and highly accurate.

Keywords: Abel's differential equation; Taylor matrix method; Padé approximation; non linear differential equations

Mathematics Subject Classification: 65L80, 34B15 and 65H10

1. Introduction

Consider the Abel equation of the form [10]

$$y'(x) = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3 + \ldots, \qquad a \le x \le b \tag{1}$$

or

$$A(x)y'(x) = E(x) + B(x)y + C(x)y^2 + D(x)y^3 + \ldots$$

with the mixed condition

$$\alpha_1 y(a) + \beta_1 y(b) + \gamma_1 y(c) + \alpha_2 y^2(a) + \beta_2 y^2(b) + \gamma_2 y^2(c) + \alpha_3 y^3(a) + \beta_3 y^3(b) + \gamma_3 y^3(c) + \cdots = \lambda \tag{2}$$

where $A(x) = 1, B(x) = f_1(x), C(x) = f_2(x), D(x) = f_3(x), E(x) = f_0(x)$.

We assume that the solution is expressed in the form

$$y(x) = \sum_{n=0}^{N} \frac{y^{(n)}(c)}{n!}(x-c)^n, \qquad a \le c \le b, \tag{3}$$

which is Taylor polynomial of degree N at $x = c$, where $y^{(n)}(c)$, $n = 0, 1, \ldots, N$ are the coefficients to be determined.

The nonlinear differential equations are indispensable tools for modelling many physical phenomenon such as chemical reactions, spring-mass system and bending of beams. These equations are also useful in ecology and economics. Therefore the solution methods for these equations have gained importance for engineers and scientists. Taylor matrix method is one of the methods used to solve a wide class of algebraic, difference and partial differential equations without linearization or smallness assumptions [4-9].

In this study, we consider nonlinear Abel differential equation of first type and the solution is obtained by Taylor matrix method and is also expanded into padé series. The numerical results obtained are given in tables.

2. Taylor matrix method

Let us consider the Abel's differential equation (1) and find the truncated Taylor series expansions of each term in expression (1) at $x = c$ and their matrix representations. We first consider the desired solution $y(x)$ of Eq. (1) defined by a truncated Taylor series (3). Then we can put series (3) in the matrix form

$$[y(x)] = \mathbf{XMY} \tag{4}$$

where

$$\mathbf{X} = \left[1 \ (x-c) \ (x-c)^2 \cdots (x-c)^N\right]$$

$$\mathbf{M} = \begin{bmatrix} \frac{1}{0!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1!} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{N!} \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} y^{(0)}(c) \\ y^{(1)}(c) \\ y^{(2)}(c) \\ \vdots \\ y^{(N)}(c) \end{bmatrix}.$$

Now we consider the term $B(x)y(x)$ of Eq. (1) and can write it as the truncated Taylor series expansions of degree N at $x = c$ in the form

$$B(x)y(x) = \sum_{n=0}^{N} \frac{1}{n!} [B(x)y(x)]_{x=c}^{(n)} (x-c)^n. \tag{5}$$

By the Leibnitz's rule we evaluate

$$[B(x)y(x)]_{x=c}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} B^{(n-m)}(c) y^{(m)}(c)$$

and substitute in (5). Thus expression (5) becomes

$$B(x)y(x)=\sum_{n=0}^{N}\sum_{m=0}^{n}\frac{1}{n!}\binom{n}{m}B^{(n-m)}(c)y^{(m)}(c)(x-c)^{(n)}$$

and its matrix form

$$[B(x)y(x)]=\mathbf{XBY} \tag{6}$$

where

$$\mathbf{B}=\begin{bmatrix} \frac{B^{(0)}(c)}{0!0!} & 0 & 0 & \cdots & 0 \\ \frac{B^{(1)}(c)}{1!0!} & \frac{B^{(0)}(c)}{0!1!} & 0 & \cdots & 0 \\ \frac{B^{(2)}(c)}{2!0!} & \frac{B^{(1)}(c)}{1!1!} & \frac{B^{(0)}(c)}{0!2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{B^{(N)}(c)}{N!0!} & \frac{B^{(N-1)}(c)}{(N-1)!1!} & \frac{B^{(N-2)}(c)}{(N-2)!2!} & \cdots & \frac{B^{(0)}(c)}{0!N!} \end{bmatrix}.$$

By analogy we obtain

$$A(x)y'(x)=\sum_{n=0}^{N}\sum_{m=0}^{n}\frac{1}{n!}\binom{n}{m}A^{(n-m)}(c)y^{(m+1)}(c)(x-c)^{(n)}, \tag{7}$$

$$E(x)=\sum_{n=0}^{N}\frac{1}{n!}E^{(n)}(c)(x-c)^{n}, \tag{8}$$

$$C(x)Y_1(x)=\sum_{n=0}^{N}\sum_{m=0}^{n}\frac{1}{n!}\binom{n}{m}C^{(n-m)}(c)Y_1^{(m)}(c)(x-c)^{(n)} \tag{9a}$$

$$D(x)Y_2(x)=\sum_{n=0}^{N}\sum_{m=0}^{n}\frac{1}{n!}\binom{n}{m}D^{(n-m)}(c)Y_2^{(m)}(c)(x-c)^{(n)} \tag{9b}$$

so that

$$Y_1(x)=(y(x))^2, \qquad Y_1^{(m)}(c)=\sum_{i=0}^{m}\binom{m}{i}y^{(m-i)}(c)y^{(i)}(c) \tag{9c}$$

$$Y_2(x)=(y(x))^3, \qquad Y_2^{(1)}(c)=3(y(c))^2y^{(1)}(c)$$

for $m\geq 2$ $\qquad Y_2^{(m)}(c)=3\sum_{i=0}^{m-1}\binom{m-1}{i}y^{(m-i)}(c)Y_1^{(i)}(c)$

In general we write

$$Y_{n-1}(x)=(y(x))^n, \qquad Y_{n-1}^{(1)}(c)=n(y(c))^{n-1}y^{(1)}(c).$$

for $m\geq 2$ $\qquad Y_{n-1}^{(m)}(c)=n\sum_{i=0}^{m-1}\binom{m-1}{i}y^{(m-i)}(c)Y_{n-2}^{(i)}(c)$ (9d)

and their matrix representations, respectively.

$$[A(x)y'(x)] = \mathbf{XAY}, \tag{10}$$

$$[C(x)y^2(x)] = [C(x)Y_1(x)] = \mathbf{XC}\overline{Y_1}, \tag{11a}$$

$$[D(x)y^3(x)] = [D(x)Y_2(x)] = \mathbf{XD}\overline{Y_2} \tag{11b}$$

$$[E(x)] = \mathbf{XME}, \tag{12}$$

where

$$\mathbf{A}=\begin{bmatrix} 0 & \frac{A^{(0)}(c)}{0!0!} & 0 & \cdots & 0 \\ 0 & \frac{A^{(1)}(c)}{1!0!} & \frac{A^{(0)}(c)}{0!1!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{A^{(N-1)}(c)}{(N-1)!0!} & \frac{A^{(N-2)}(c)}{(N-2)!1!} & \cdots & \frac{A^{(0)}(c)}{0!(N-1)!} \\ 0 & \frac{A^{(N)}(c)}{N!0!} & \frac{A^{(N-1)}(c)}{(N-1)!1!} & \cdots & \frac{A^{(1)}(c)}{1!(N-1)!} \end{bmatrix}, \quad \mathbf{E}=\begin{bmatrix} E^{(0)}(c) \\ E^{(1)}(c) \\ \vdots \\ E^{(N)}(c) \end{bmatrix},$$

$$\mathbf{C}=\begin{bmatrix} \frac{C^{(0)}(c)}{0!0!} & 0 & 0 & \cdots & 0 \\ \frac{C^{(1)}(c)}{1!0!} & \frac{C^{(0)}(c)}{0!1!} & 0 & \cdots & 0 \\ \frac{C^{(2)}(c)}{2!0!} & \frac{C^{(1)}(c)}{1!1!} & \frac{C^{(0)}(c)}{0!2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{C^{(N)}(c)}{N!0!} & \frac{C^{(N-1)}(c)}{(N-1)!1!} & \frac{C^{(N-2)}(c)}{(N-2)!2!} & \cdots & \frac{C^{(0)}(c)}{0!N!} \end{bmatrix}, \quad \overline{Y_1} = \begin{bmatrix} Y_1^{(0)} \\ Y_1^{(1)} \\ \vdots \\ Y_1^{(N)} \end{bmatrix},$$

$$\mathbf{D}=\begin{bmatrix} \frac{D^{(0)}(c)}{0!0!} & 0 & 0 & \cdots & 0 \\ \frac{D^{(1)}(c)}{1!0!} & \frac{D^{(0)}(c)}{0!1!} & 0 & \cdots & 0 \\ \frac{D^{(2)}(c)}{2!0!} & \frac{D^{(1)}(c)}{1!1!} & \frac{D^{(0)}(c)}{0!2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{D^{(N)}(c)}{N!0!} & \frac{D^{(N-1)}(c)}{(N-1)!1!} & \frac{D^{(N-2)}(c)}{(N-2)!2!} & \cdots & \frac{D^{(0)}(c)}{0!N!} \end{bmatrix}, \quad \overline{Y_2} = \begin{bmatrix} Y_2^{(0)} \\ Y_2^{(1)} \\ \vdots \\ Y_2^{(N)} \end{bmatrix},$$

Substituting the matrix forms (6), (10), (11) and (12) corresponding to the functions $B(x)y(x)$, $A(x)y'(x)$, $C(x)y^2(x)$, $D(x)y^3(x),\cdots$ and $E(x)$, into Eq. (1), and then we have the matrix representation.

$$\mathbf{(A+B)Y+C}\overline{Y_1}+\mathbf{D}\overline{Y_2}+\cdots=\mathbf{ME}. \tag{13}$$

The matrix representation (13) is fundamental relation for Abel's differential equation (1). Using the relation (4), we find the matrix representation of the functions at the points a, b and c in the forms

$$\begin{aligned} [y(a)] &= \mathbf{A}_1\mathbf{MY}, \\ [y(b)] &= \mathbf{B}_1\mathbf{MY}, \\ [y(c)] &= \mathbf{C}_1\mathbf{MY}, \end{aligned} \tag{14a}$$

where

$$\mathbf{A}_1 = [1 \quad (a-c) \quad (a-c)^2 \quad (a-c)^3 \quad \cdots \quad (a-c)^N],$$
$$\mathbf{B}_1 = [1 \quad (b-c) \quad (b-c)^2 \quad (b-c)^3 \quad \cdots \quad (b-c)^N],$$
$$\mathbf{C}_1 = [1 \quad 0 \quad 0 \quad \cdots \quad 0].$$

By taking $C(x) = 1$ in (11a), we have

$$[y^2(x)] = [Y_1(x)] = \mathbf{XM}\overline{Y_1}.$$

Therefore, for the points a, b and c

$$\begin{aligned} [y^2(a)] &= [Y_1(a)] = \mathbf{A}_1\mathbf{M}\overline{Y_1}, \\ [y^2(b)] &= [Y_1(b)] = \mathbf{B}_1\mathbf{M}\overline{Y_1}, \\ [y^2(c)] &= [Y_1(c)] = \mathbf{C}_1\mathbf{M}\overline{Y_1}. \end{aligned} \tag{14b}$$

By taking $D(x) = 1$ in (11b), we have

$$[y^3(x)] = [Y_2(x)] = \mathbf{XM}\overline{Y_2}.$$

Therefore, for the points a, b and c

$$\begin{aligned} [y^3(a)] &= [Y_2(a)] = \mathbf{A}_1\mathbf{M}\overline{Y_2}, \\ [y^3(b)] &= [Y_2(b)] = \mathbf{B}_1\mathbf{M}\overline{Y_2}, \\ [y^3(c)] &= [Y_2(c)] = \mathbf{C}_1\mathbf{M}\overline{Y_2}. \end{aligned} \tag{14c}$$

and so on.

Substituting the matrix representations (14a), (14b) and (14c) into Equation (2), we obtain

$$\mathbf{RY} + \mathbf{V}\overline{Y_1} + \mathbf{W}\overline{Y_2} + \cdots = [\lambda], \tag{15}$$

where

$$\mathbf{R} = (\alpha_1\mathbf{A}_1 + \beta_1\mathbf{B}_1 + \gamma_1\mathbf{C}_1)\mathbf{M} \equiv [r_0 \quad r_1 \quad \cdots \quad r_N],$$
$$\mathbf{V} = (\alpha_2\mathbf{A}_1 + \beta_2\mathbf{B}_1 + \gamma_2\mathbf{C}_1)\mathbf{M} \equiv [v_0 \quad v_1 \quad \cdots \quad v_N],$$

$$\mathbf{W}=(\alpha_3\mathbf{A}_1+\beta_3\mathbf{B}_1+\gamma_3\mathbf{C}_1)\mathbf{M}=[w_0 \quad w_1 \quad \cdots \quad w_N],$$

and so on.

Thus, the augmented matrix of (15) becomes

$$[\mathbf{R};\mathbf{V};\mathbf{W};\cdots;\lambda]=[r_0 \quad r_1 \quad \cdots \quad r_N \;;\; v_0 \quad v_1 \quad \cdots \quad v_N \;;\; w_1 \quad w_2 \quad \cdots \quad w_N;\cdots;\; \lambda]. \tag{16}$$

3. Method of solution

Let us consider the fundamental matrix Eq. (13) corresponding to the Abel's differential equation (1). We can write Eq. (13) in the form

$$\mathbf{UY}+\mathbf{C}\overline{Y_1}+\mathbf{D}\overline{Y_2}+\cdots=\mathbf{ME}, \tag{17}$$

Where

$$\mathbf{A}+\mathbf{B}=\mathbf{U}=[u_{ij}], \quad \mathbf{C}=[c_{ij}], \quad \mathbf{D}=[d_{ij}], \cdots \quad i,j=0,1,...,N. \tag{18}$$

The augmented matrix of Equation (17) becomes

$$\mathbf{Q}=[\mathbf{U};\mathbf{C};\mathbf{D};\cdots;\mathbf{ME}] \tag{19}$$

$$\mathbf{U}=\begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} \\ u_{10} & u_{11} & \cdots & u_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ u_{N0} & u_{N1} & \cdots & u_{NN} \end{bmatrix}, \quad \mathbf{C}=\begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0N} \\ c_{10} & c_{11} & \cdots & c_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ c_{N0} & c_{N1} & \cdots & c_{NN} \end{bmatrix},$$

$$\mathbf{D}=\begin{bmatrix} d_{00} & d_{01} & \cdots & d_{0N} \\ d_{10} & d_{11} & \cdots & d_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ d_{N0} & d_{N1} & \cdots & d_{NN} \end{bmatrix}, \quad \mathbf{ME}=\begin{bmatrix} \dfrac{E^{(0)}(c)}{0!} \\ \dfrac{E^{(1)}(c)}{1!} \\ \vdots \\ \dfrac{E^{(N)}(c)}{N!} \end{bmatrix}$$

We now consider the matrix Eq. (15) corresponding to the condition (2). Then the augmented matrix of (15) becomes

$$[\mathbf{R};\mathbf{V};\mathbf{W};\cdots;\lambda]=[r_0 \quad r_1 \quad \cdots \quad r_N \;;\; v_0 \quad v_1 \quad \cdots \quad v_N \;;\; w_1 \quad w_2 \quad \cdots \quad w_N \;;\cdots;\; \lambda], \tag{20}$$

Where the elements $r_0, r_1, \cdots, r_N; v_0, v_1, \cdots, v_N; w_1, w_2, \cdots, w_N, \cdots$ are defined in Equation (16).

Consequently, to find the unknown Taylor coefficients $y^{(n)}(c)$, related with the approximate solution of the Equation (1) with condition (2), by replacing the last row of the matrix (19) by the augmented matrix (20), we have new augmented matrix

$$\mathbf{Q}^*=[\mathbf{U}^*;\mathbf{C}^*;\mathbf{D}^*;\ldots;\mathbf{E}^*]$$

or the corresponding matrix equation

$$\mathbf{U}^{*}\mathbf{Y}+\mathbf{C}^{*}\overline{Y_1}+\mathbf{D}^{*}\overline{Y_2}+\ldots=\mathbf{E}^{*} \tag{21}$$

so that

$$\mathbf{U}^{*}=\begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} \\ u_{10} & u_{11} & \cdots & u_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ u_{N-1,0} & u_{N-1,1} & \cdots & u_{N-1,N} \\ r_0 & r_1 & \cdots & r_N \end{bmatrix},\quad \mathbf{C}^{*}=\begin{bmatrix} c_{00} & c_{01} & \cdots & c_{0N} \\ c_{10} & c_{11} & \cdots & c_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ c_{N-1,0} & c_{N-1,1} & \cdots & c_{N-1,N} \\ v_0 & v_1 & \cdots & v_N \end{bmatrix},$$

$$\mathbf{D}^{*}=\begin{bmatrix} d_{00} & d_{01} & \cdots & d_{0N} \\ d_{10} & d_{11} & \cdots & d_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ d_{N-1,0} & d_{N-1,1} & \cdots & d_{N-1,N} \\ w_0 & w_1 & \cdots & w_N \end{bmatrix},\quad \mathbf{E}^{*}=\begin{bmatrix} \dfrac{E^{(0)}(c)}{0!} \\ \dfrac{E^{(1)}(c)}{1!} \\ \vdots \\ \dfrac{E^{(N-1)}(c)}{(N-1)!} \\ \lambda \end{bmatrix}.$$

Since the Taylor polynomial (3) is an approximate solution of Eq. (1),after necessary substitution we have the solution $y(x)$ and its derivatives are substituted in Equation (1), the resulting equation must be satisfied approximately; that is, for x_r belong to [a,b].

$$F(x_r)=\left|A(x_r)y'(x_r)-B(x_r)y(x_r)-C(x_r)y^2(x_r)-D(x_r)y^3(x_r)-\cdots-E(x_r)\right|\cong 0$$

or

$$F(x_r)\le 10^{-l_r}\quad (l_r \text{ is any positive integer}).$$

4. Padé Series

The Taylor series can be transformed into Padé series is defined in the following:

$$a_0+a_1x+a_2x^2+\cdots=\frac{p_0+p_1x+\cdots+p_Mx^M}{1+q_1x+\cdots+q_Lx^L}. \tag{22}$$

Multiply both sides of (22) by the denominator of right-hand side in (22) and compare the coefficients of the sides in (22). We have

$$a_l+\sum_{k=1}^{M}a_{l-k}q_k=p_l\qquad (l=0,\cdots,M), \tag{23}$$

$$a_l+\sum_{k=1}^{L}a_{l-k}q_k=p_l\qquad (l=M+1,\cdots,M+L). \tag{24}$$

Solving the linear equation in (24), we have q_k $(k=1,\cdots,L)$. And substituting q_k into (22), we have p_k $(l=1,\cdots,M)$ [8,9,10].

5. Numerical examples

In this section, two initial Abel's differential equations are considered. The algorithm is performed by MATLAB.

Example 1.

Consider $y' = -2e^{3x} + y + e^{x}y^{2} + y^{3}$, over [0,1] with y(0)=1.

The exact solution of this problem is e^{x}. If we approximate the solution $y(x)$ by the Taylor polynomial,where $N = 5$,$c = 0, A(x) = 1$, $B(x) = -1, C(x) = -e^{x}, D(x) = -1,\ E(x) = -2e^{3x}$.

Then, for $N = 5$, the matrix Eq. (13) becomes

$$\mathbf{(A+B)Y+C}\overline{Y_1}\mathbf{+D}\overline{Y_2}\mathbf{=ME}.$$

For the condition $y(0) = 1$, the augmented matrix becomes [**R**; **V**; **W**; λ] from (16). $\overline{Y_1}$ and $\overline{Y_2}$ are obtained from (9c) and (9d).

From the system defined in Eq. (21), the coefficients $y^{(n)}(0)$ $(n = 0,1,2,3,4,5)$ are found as $y^{(0)}(0) = 1,\ y^{(1)}(0) = 1,\ y^{(2)}(0) = 1,\ y^{(3)}(0) = 1,\ y^{(4)}(0) = 1,\ y^{(5)}(0) = 1$. Substituting these coefficients in (25) we have the solution

$$y_T(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5. \tag{26a}$$

For N=7 and N=9 obtained

$$y_T(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 \tag{26b}$$

$$y_T(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \frac{1}{9!}x^9. \tag{26c}$$

The Taylor series $y_T(x)$ can be transformed into following Padé series:

$$y_p = [3/2] = (1 + 0.6x + 0.15x^2 + 0.01666667x^3)/(1 - 0.4x + 0.05x^2) \tag{27a}$$

$$y_p = [4/3] = (1 + 0.5714285714x + 0.1428571428x^2 + 0.0190476190x^3 + 0.0011904762x^4)$$
$$/(1 - 0.4285714285x + 0.0714285714x^2 - 0.0047619047x^3) \tag{27b}$$

$$y_p = [5/4] = (1 + 0.5555555555x + 0.1388888889x^2 + 0.0198412698x^3 + 0.0016534391x^4$$
$$+ 0.0000661375x^5)/(1 - 0.4444444445x + 0.0833333334x^2 - 0.0079365079x^3$$
$$+ 0.0003306878x^4). \tag{27c}$$

Table 1. Numerical solutions obtained for $y' = -2e^{3x} + y + e^x y^2 + y^3$ for N=5 and $y_P = [3/2]$

	Taylor Matrix Method	Padé Approximation	Exact Solution	Absolute Error	
x_i	y_T	y_P	y_G	$\lvert y_G - y_T \rvert$	$\lvert y_G - y_P \rvert$
0.00	1.0000000000	1.0000000000	1.0000000000	0.0000000000	0.0000000000
0.10	1.1051709167	1.1051709179	1.1051709181	0.0000000014	0.0000000002
0.20	1.2214026667	1.2214027477	1.2214027582	0.0000000915	0.0000000105
0.30	1.3498577500	1.3498586772	1.3498588076	0.0000010576	0.0000001304
0.40	1.4918186667	1.4918238994	1.4918246976	0.0000060310	0.0000007983
0.50	1.6486979167	1.6487179487	1.6487212707	0.0000233540	0.0000033220
0.60	1.8220480000	1.8221079692	1.8221188004	0.0000708004	0.0000108312
0.70	2.0135714167	2.0137228565	2.0137527075	0.0001812908	0.0000298510
0.80	2.2251306667	2.2254681648	2.2255409285	0.0004102618	0.0000727637
0.90	2.4587582500	2.4594415871	2.4596031112	0.0008448612	0.0001615241
1.00	2.7166666667	2.7179487180	2.7182818285	0.0016151618	0.0003331105

Table 2. Numerical solutions obtained for $y' = -2e^{3x} + y + e^x y^2 + y^3$ for N=7 and $y_P = [4/3]$

	Taylor Matrix Method	Padé Approximation	Exact Solution	Absolute Error	
x_i	y_T	y_P	y_G	$\lvert y_G - y_T \rvert$	$\lvert y_G - y_P \rvert$
0.00	1.0000000000	1.0000000000	1.0000000000	0.0000000000	0.0000000000
0.10	1.1051709181	1.1051709181	1.1051709181	0.0000000000	0.0000000000
0.20	1.2214027581	1.2214027581	1.2214027582	0.0000000000	0.0000000000
0.30	1.3498588062	1.3498588076	1.3498588076	0.0000000014	0.0000000000
0.40	1.4918246830	1.4918246982	1.4918246976	0.0000000147	0.0000000006
0.50	1.6487211793	1.6487212749	1.6487212707	0.0000000914	0.0000000042
0.60	1.8221183942	1.8221188206	1.8221188004	0.0000004061	0.0000000202
0.70	2.0137512757	2.0137527837	2.0137527075	0.0000014317	0.0000000763
0.80	2.2255366651	2.2255411723	2.2255409285	0.0000042634	0.0000002438
0.90	2.4595919454	2.4596037981	2.4596031112	0.0000111657	0.0000006870
1.00	2.7182553957	2.7182835814	2.7182818285	0.0000264328	0.0000017530

Table 3. Numerical solutions obtained for $y' = -2e^{3x} + y + e^x y^2 + y^3$ for N=9 and $y_p = [5/4]$

	Taylor Matrix Method	Padé Approximation	Exact Solution	Absolute Error	
x_i	y_T	y_P	y_G	$\lvert y_G - y_T \rvert$	$\lvert y_G - y_P \rvert$
0.00	1.0000000000	1.0000000000	1.0000000000	0.0000000000	0.0000000000
0.10	1.1051709181	1.1051709181	1.1051709181	0.0000000000	0.0000000000
0.20	1.2214027582	1.2214027582	1.2214027582	0.0000000000	0.0000000000
0.30	1.3498588079	1.3498588076	1.3498588076	0.0000000003	0.0000000000
0.40	1.4918246999	1.4918246976	1.4918246976	0.0000000023	0.0000000000
0.50	1.6487212816	1.6487212707	1.6487212707	0.0000000109	0.0000000000
0.60	1.8221188386	1.8221188003	1.8221188004	0.0000000382	0.0000000001
0.70	2.0137528167	2.0137527073	2.0137527075	0.0000001092	0.0000000002
0.80	2.2255411960	2.2255409278	2.2255409285	0.0000002675	0.0000000007
0.90	2.4596036893	2.4596031091	2.4596031112	0.0000005782	0.0000000020
1.00	2.7182829530	2.7182818225	2.7182818285	0.0000011245	0.0000000059

Example 2.

Consider $y' = y + y^2 + y^3$ on [0,0.1] with y(0)=1.

The analytical solution is

$x = \ln y - 0.5\ln(1 + y + y^2) - 3^{-1/2} a\tan(3^{-1/2}(2y+1)) + \ln 3/2 + 3^{-1/2}\pi/3$.

If we approximate the solution $y(x)$ by the Taylor polynomial (3)

where $N = 5, c = 0, A(x) = 1, B(x) = -1, C(x) = -1, D(x) = -1, E(x) = 0$. Then, for $N = 5$, the matrix Equation (13) becomes

$$\mathbf{(A+B)Y+C}\overline{Y_1}\mathbf{+D}\overline{Y_2}\mathbf{=ME},$$

For the condition $y(0) = 1$, the augmented matrix becomes

[R; V; W; λ **]** from (16). $\overline{Y_1}$ and $\overline{Y_2}$ are obtained from (9c) and (9d).

From the obtained system, the coefficients $y^{(n)}(0)$ $(n = 0,1,2,3,4,5)$ are found as $y^{(0)}(0) = 1$, $y^{(1)}(0) = 3$, $y^{(2)}(0) = 18$, $y^{(3)}(0) = 180$, $y^{(4)}(0) = 2538$, $y^{(5)}(0) = 46116$.

Subtituting these coefficients in (28a) we have the solution

$$y_T(x) = 1 + 3x + \frac{18}{2}x^2 + \frac{180}{6}x^3 + \frac{2538}{24}x^4 + \frac{46116}{120}x^5. \tag{28a}$$

The Taylor series $y_T(x)$ can be transformed into following Padé series:

$$y_p = [3/2] = (1 - 2.5304347826x - 0.9065217391x^2 + 0.284347826x^3) \\ /(1 - 5.5304347826x + 6.6847826086x^2) \tag{28b}$$

Table 4. Numerical solutions obtained for $y' = y + y^2 + y^3$ for N=5 and $y_p = [3/2]$

	Taylor Matrix Method	Padé Approximation	Exact Solution	Absolute Error	
x_i	y_T	y_P	y_G	$\lvert y_G - y_T \rvert$	$\lvert y_G - y_P \rvert$
0.00	1.00000000	1.00000000	1.00000000	0.00000000	0.00000000
0.01	1.03093109	1.03093109	1.03093109	0.00000000	0.00000000
0.02	1.06385814	1.06385824	1.06385824	0.00000010	0.00000000
0.03	1.09900499	1.09900616	1.09900616	0.00000117	0.00000000
0.04	1.13663007	1.13663689	1.13663693	0.00000686	0.00000004
0.05	1.17703103	1.17705826	1.17705843	0.00002740	0.00000017
0.06	1.22054935	1.22063456	1.22063516	0.00008581	0.00000060
0.07	1.26757495	1.26780065	1.26780245	0.00022750	0.00000180
0.08	1.31855079	1.31908041	1.31908519	0.00053440	0.00000478
0.09	1.37397751	1.37511146	1.37512311	0.00114560	0.00001165
0.10	1.43441800	1.43667865	1.43670524	0.00228724	0.00002659

Numerical solutions of Eq. (1) is shown in Table 1-4. The numerical values in Table 1-4 is coinciding with the exact solution of (1).

6. Conclusion

In this paper we have presented Taylor matrix method to solve the Abel's differential equation and introduced an expression given by (9d) from which one can compute mth order derivative of y^n where $n \geq 2$ is an integer. The matrix method avoids the difficulties and massive computational work by determining the analytic solution. The Taylor matrix method provides a reliable technique that requires less work and highly accurate results if compared with the traditional techniques and existing numerical methods. The Padé method that often show superior performance over series approximants. Results show the avantages of the method. A considerable avantage of the method is that Taylor coefficients of the solution are found very easily by using the computer programs.

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