FUNCTIONS

In this chapter we will examine some special functions, their properties and graphs. Firstly we shall give some fundamental definitions.

**Definition:** A function \( f \) from a set \( D \) to a set \( Y \) is a rule that assigns a unique element \( f(x) \in Y \) to each element \( x \in D \).

In other words;

Let \( D \) and \( Y \) be non-empty sets.

The rule of \( f \) from \( D \) to \( Y \), is said to function if it satisfies the following conditions.

![Diagram](image)

1) for every \( x \in D \), there exist the element \( y \) in the set \( Y \).

2) \( f \) assign a unique element \( y \in Y \) to each elements in the set \( Y \).

\[ D(f) = D \text{ is the set of all the possible input values of } f \]

\[ R(f) \subseteq Y \text{ is the set of all possible output values of } Y \text{ which is subset of } Y \]
\[
f: D \rightarrow Y \\
x \rightarrow y = f(x)
\]
the function \( f \) is denoted by in this way.

Here the letter \( x \) is independent variable, \( y \) is dependent variable or output value of \( x = f(x) \).

**Example:** find the domain of the following each function.

\[
y = x^2 \\
D(t) = \mathbb{R} , (-\infty, \infty)
\]
\[
y = \frac{1}{x} \\
D(t) = \mathbb{R} \setminus \{0\} , (-\infty,0) \cup (0,\infty)
\]
\[
y = \sqrt{x} \\
D(t) = [0,\infty)
\]
\[
f(x) = \sqrt[3]{x} \\
D(f) = \mathbb{R} \quad (R(f) = \mathbb{R})
\]
\[
f(x) = \sqrt{x^2 - 4} \\
D(f) = \{ x \in \mathbb{R} ; x^2 - 4 \geq 0 \}
\]
\[
f(x) = \log x \\
D(f) = [0,\infty) \quad \text{positive real numbers}
\]
Examples: Let's verify the natural domains and associated ranges of some simple functions.
The domains in each case are the values of $x$ for which the formula makes sense:

1) $y = f(x) = x^2$
   The formula $y = x^2$ gives a real $y$-value for any real number $x$ so domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is non-negative.

2) $y = \frac{1}{x}$
   The formula $y = \frac{1}{x}$ gives a real $y$-value for every real number except $x = 0$. So
   Domain = $\{x | x \neq 0\}$
   Range = $(-\infty, 0) \cup (0, \infty)$
   The range of $y = \frac{1}{x}$ is the set of all non-zero real numbers.

3) $y = \sqrt{x}$
   The formula $y = \sqrt{x}$ gives a real $y$-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$. Because every non-negative number $y$ is the square of its own square root, $(y = (\sqrt{y}))^2$
   Domain = $[0, \infty)$
   Range = $[0, \infty)$
4) \( y = \sqrt{4-x} \)

The quantity \( 4-x \) can not be negative.

Namely \( 4-x \geq 0 \) or \( x \leq 4 \).

The formula gives real \( y \)-values for all \( x \leq 4 \).

The range of \( \sqrt{4-x} \) is \([0, \infty)\) the set of all non-negative numbers.

5) \( y = \sqrt{1-x^2} \)

This formula gives real \( y \)-values for all \( 1-x^2 \geq 0 \) \( \Rightarrow x^2 \leq 1 \).

Every \( -1 \leq x \leq 1 \) (every \( x \) in the closed interval from \(-1\) to \(1\).)

The values of \( 1-x^2 \) vary from \( 0 \) to \( 1 \) on given domain and square roots of these values do the same: \([0,1]\).
A function can be defined as a set

\[ f = \{(x, f(x)) \mid x \in A\} \]

In this case the function of \( f \) is the subset of the cartesian product \( A \times B \).

**Example:** Let consider these sets;

\[ A = \{-1, 0, 1, 2\} \quad B = \{-2, -1, 0, 2, 3, 4, 6\} \]

Let the function \( f \) be defined by in this way

\[ f : A \rightarrow B \]

\[ f(x) = x^2 + 2 \]

Let us determine the set of \( f \)

\[ f = \{(-1, 3), (0, 2), (1, 3), (2, 6)\} \quad (f \text{ is open bracket}) \]

This set of ordered pairs

\[ R(f) = \{2, 3, 6\} \subseteq B \]

\( A \) is domain of \( f \)

Let us show these ordered pairs in the following figures.
**Vertical Line Test**

A curve refers to a function, if each line $x = a$ (where $a$ is an element of its domain set) cuts the curve in only one point $a \in D(f)$.

$\text{f}$ and $g$ are functions but $h$ is not a function because this line cuts the curve in two points $A$ and $B$.

A curve doesn’t always specify a function. We can determine whether or not a curve refers to a function with vertical line test. If each vertical line intersects the curve only one point then the curve refers to a function.

**Example:** The circle doesn’t graph of unique function. It fails the vertical line test.

- $x^2 + y^2 = 1$  
- The upper semi-circle is a graph of a function $y = \sqrt{1-x^2}$
- The lower semi-circle is a graph of a function $y = -\sqrt{1-x^2}$
Domain of Functions:

We know the domain of a function $f(x)$ is the set of all possible values of $x \in \mathbb{R}$.
To find the domain of a function, we may apply the following steps:

1-) A polynomial function of

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is defined for all real numbers. So the domain is $D = \mathbb{R}$

2-) Let $P(x)$ and $Q(x)$ be polynomial functions. The domain of $f(x) = \frac{P(x)}{Q(x)}$ is all real numbers except some $x$-values which make $Q(x)$ zero.

That is $D = \mathbb{R} - \{x \mid Q(x) = 0\}$

3-) Let $f(x) = \sqrt[n]{P(x)}$ be a function (n root of $P(x)$)

a-) If $n$ is an even number, then domain

$$D = \{x \mid P(x) \geq 0, x \in \mathbb{R}\}$$

This inequality must be satisfied

(b) Domain of $f$ consist $x \in \mathbb{R}$ such that $P(x) \geq 0$ (greater than or equal to zero)

b-) If $n$ is an odd number, then domain

$D = \mathbb{R}$. 


4) Let \( f(x) = \log[P(x)] \).
Then its domain \( D = \left\{ x \mid P(x) > 0, x \in \mathbb{R} \right\} \).

5) Let \( y = f(x) + g(x) \).
Then the domain of this function is the lowest common multiple of the domains of \( f(x) \) and \( g(x) \).

Examples:

1) \( f(x) = x^3 + 2x^2 - x + 3 \) Polynomial function is defined for all real numbers \( D(f) = \mathbb{R} \).

2) \( f(x) = \frac{x^2 + 2x + 5}{x^3 - 5x} \) rational functions is defined except numbers that make the denominator zero so
\[
x^3 - 5x = x(x^2 - 5) = 0 \quad x = 0 \quad x = \pm \sqrt{5}
\]
\( D = \mathbb{R} \setminus \{ 0, \pm \sqrt{5} \} \)

3) \( f(x) = \frac{\sqrt{x^2 - x - 12}}{(2-x)^2} \) square root function is defined for zero and positive numbers so the function in square root must be positive.
\[
\frac{x^2 - x - 12}{(2-x)^2} \geq 0 \quad \text{(This inequality must be solved.)}
\]
Quadratic Equation:

Solve by factoring: It has two roots. Denominator is positive.

\[ x^2 - x - 12 = 0 \quad (x-4)(x+3)=0 \quad x=4 \]
\[ (2-x)^2 = 0 \quad x=2 \quad x=2 \]

The solution set is \([-3, 4]\).

Let us study the table:

<table>
<thead>
<tr>
<th></th>
<th>-3</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>+</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>-</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>D = (-\infty, -3] \cup [4, \infty)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Quadratic equation can solve by quadratic formula:

\[ ax^2 + bx + c = 0 \]

\[ x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

4-) \[ f(x) = \frac{\sqrt{2+x-x^2}}{x-1} \] is defined both

\[ 2+x-x^2 \geq 0 \] and \[ x-1 \neq 0 \]

\[ x^2 - x - 2 \leq 0 \]
\[ (x-2)(x+1) \leq 0 \]

\[ D(f) = [-1, 2] \setminus \{1\} \]
5) \( f(x) = \log(x^2 - 3x - 4) \) is a logarithmic function, this is defined for only positive numbers. So

\[
x^2 - 3x - 4 > 0 \\
(x+1)(x-4) > 0
\]

\[
\begin{array}{c|cc}
& -1 & 4 \\
- & + & - & + \\
\end{array}
\]

\( D(f) = (-\infty, -1) \cup (4, \infty) \)

Note: fraction \( \frac{a}{b} \rightarrow \text{numerator} \)
\( \frac{a}{b} \rightarrow \text{denominator} \)

\( a \over b = a \text{ divided by } 3 \)

by multiplying the \( a \) by 6 \( \rightarrow 6a \)

\( \underline{\text{to multiply}} \)
Graphs of Functions

If \( f \) is a function with domain \( D \), its graphics consist of the points in the cartesian plane whose coordinates are input-output pairs for \( f \). In set notation:

\[
\text{graph } f = \{ (x, f(x)) \mid x \in D \}
\]

The graph of \( f \) is the set of pairs \( (x, f(x)) \in \mathbb{R}^2 \).

Ex: The graph of the function \( f(x) = x + 2 \) is the set of points with coordinates \( (x, y) \) for which \( y = x + 2 \). The graph is the straight line sketched in this figure:

\[
\text{graph } f = \{ (x, x+2) \mid x \in \mathbb{R} \}
\]

This function is linear function so we know that the graph of \( f \) is a straight line, exactly. But, if we doesn't know the function how we can we streatch of its graph, we will see subject of "curve sketching" in chapter of Applications of Derivatives.
let us remember the some special functions.

Types of Functions:

One to One Functions: (Injective functions)

Let $f$ be a function from $A$ to $B$. If $f$ assigns the different member of $A$ to different member of $B$, in other words, if the function $f$ satisfies this condition:

- $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ or
- $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$

is said to be a $(1-1)$ or injective function.

For example $f(x) = 3x-1$ and $g(x) = 3x$ are one to one functions, but $h(x) = x^2$ isn’t one to one function.

A horizontal line through a particular $f(x)$-value should intersect the graph of a one to one function in just one point.
Constant Function:
If the range of a function \( f \) consist of only one number, then \( f \) is called constant function. In other words:

\[ f : A \rightarrow B \text{ is a constant function if the same element in } B \text{ is assigned to every element in } A. \]

Symbolically

\[ f(x) = a, \ a \in R \]

represents a constant function.

For example; \( f(x) = 4 \) is a constant func.

```
\begin{array}{c}
| y |
\end{array}
\begin{array}{c}
| f(x) = 4 |
\end{array}
\begin{array}{c}
| x |
\end{array}
```

Onto Function:
Let \( f : A \rightarrow B \) be a function. If \( f(A) = B \), \( f \) is an onto or surjective function. That is, for each \( y \in B \) there exist at least one element \( x \in A \) such that \( f(x) = y \) then \( f(x) \) is onto.

Range of \( f = R(f) = B \)
Identity Function:

Let A be a set. The function \( I: A \to A \) defined by \( I: x \mapsto x \) for all \( x \in A \) is called the identity function.

In other words, the particular linear function defined by \( f(x) = x \) is called the identity function.

For example, each of following functions is an identity function:

\[
\begin{array}{c|c|c}
A & \mapsto & B \\
-1 & \mapsto & -1 \\
0 & \mapsto & 0 \\
1 & \mapsto & 1 \\
2 & \mapsto & 2 \\
\end{array}
\]

\[
\begin{array}{c|c}
y & 2 \\
\hline
x & 0 \\
\hline
x & 1 \\
\hline
x & 2 \\
\hline
1 & -2 \\
\hline
2 & -1 \\
\hline
1 & 1 \\
\hline
2 & 2 \\
\hline
\end{array}
\]

Let's consider and examine the following functions:

Ex: \( f(x) = x^2 - 2x \) \[ f: [1, \infty) \to [-1, \infty) \]

This function is 1-1 and onto.

Proof of 1-1: \( \forall x_1, x_2 \in [1, \infty) \)

\[
\begin{align*}
 f(x_1) &= x_1^2 - 2x_1 \quad \text{suppose that} \quad f(x_1) = f(x_2) \\
 f(x_2) &= x_2^2 - 2x_2 \\
 x_1^2 - 2x_1 &= x_2^2 - 2x_2 \\
 (x_1 - x_2)(x_1 + x_2 - 2) &= 0 \\
 \end{align*}
\]

since \( x_1 \geq 1 \) \( x_2 \geq 1 \) not equal than \( x_1 + x_2 > 2 \)
So \( x_1 - x_2 = 0 \) and then \( x_1 = x_2 \).

**Proof of onto:**

\[
y = x^2 - 2x \\
x = 1 + \sqrt{1 + y} \\
\text{This inequality must be satisfied.}
\]

for \( y \geq -1 \) \( \Rightarrow x \in [1, \infty) \) \( \Rightarrow f \) onto.

**Ex:** \( f(x) = |x| \)

\( f: [-1, 1] \rightarrow [0, 1] \)

for \( x_1 \neq x_2 \) : \( f(x_1) \neq f(x_2) \) must be different.

\[
|x_1| \neq |x_2| \\
|\text{but equal.}
\]

\( f \) \( \rightarrow \) not one to one.

**Ex:** \( f: R \rightarrow R \)

\( f(x) = x^3 \)

\(
\text{for } x_1 \neq x_2 \\
x_1^3 \neq x_2^3 \quad (\text{one to one})
\)

\( f \) is one to one and onto.

**Ex:** \( f(x) = 2x + 1 \)

\( f: R \rightarrow R \) is one to one and onto function.
Inverse Function: bijective

If \( f \) is one to one and onto function from \( A \) to \( B \), then the relation \( f^{-1} \) is also a function from \( B \) to \( A \) and \( f^{-1} \) is called the inverse of \( f \). Symbolically, if \( f \) is bijective and

\[ f: \{(x,y) \mid x \in A, \ y \in B\} \quad \text{then} \quad f^{-1}: \{(y,x) \mid x \in A, \ y \in B\} \]

\[ y = f(x) \iff x = f^{-1}(y) \]

\( (a,b) \in f \Rightarrow (b,a) \in f^{-1} \)

The graphs of \( f \) and \( f^{-1} \) are symmetric to line \( x = y \).

Ex: \( f(x) = x - 2 \Rightarrow f^{-1}(x) = x + 2 \quad f: \mathbb{R} \to \mathbb{R} \)
**Ex:** 
\[ f(x) = x^2 \implies f^{-1}(x) = \sqrt{x} \]

\[ f: [0, \infty) \to [0, \infty) \quad f^{-1}: [0, \infty) \to [0, \infty) \]

---

**Increasing and Decreasing Functions:**

A function \( f \) defined on an interval is said to be increasing on that interval if and only if 
\[ f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ where } x_1 \text{ and } x_2 \]
are any numbers in the interval.

A function \( f \) defined on an interval is said to be decreasing on that interval if and only if 
\[ f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ where } x_1 \text{ and } x_2 \]
are any numbers in the interval.

If a function \( f \) is either increasing or decreasing on an interval, then \( f \) is said to be monotone on the interval.

**Ex:** Write the intervals in which the function \( f \)
is increasing or decreasing.

for \( x \in [-\infty, -2] \) \( f \) is inc.

in \( [-2, 6] \) \( f \) is dec.

in \( [6, \infty) \) \( f \) is inc.
Ex: \( f(x) = \frac{1}{x} \) \( f: \mathbb{R}^+ \rightarrow \mathbb{R} \) is decreasing
\[ x_1 < x_2 \implies \frac{1}{x_1} > \frac{1}{x_2} \]

Ex: \( f(x) = x^2 \) \( f: [1, 2] \rightarrow \mathbb{R} \) is increasing
\[ 1 < x_1 < x_2 \quad x_1^2 < x_2^2 \]

Ex: \( f(x) = \frac{1}{1 + x^2} \) \( f: (-\infty, 0] \rightarrow \mathbb{R} \) is increasing.

For if \( x_1, x_2 < 0 \) let \( x_1 < x_2 \)
\[ x_1 - x_2 < 0 \quad x_1 + x_2 < 0 \]
\[ (x_1 - x_2)(x_1 + x_2) > 0 \]
\[ x_1^2 - x_2^2 > 0 \implies x_1^2 > x_2^2 \]

Thus:
\[ x_1 < x_2 \implies x_1^2 > x_2^2 \implies 1 + x_1^2 > 1 + x_2^2 \]
\[ \frac{1}{1 + x_1^2} < \frac{1}{1 + x_2^2} \]
\[ f(x_1) < f(x_2) \]
\( f \) is increasing.

**Homework:** \( f(x) = \frac{x^2 - 1}{x} \quad x \neq 0 \) increasing.
Determine whether this function is increasing or decreasing.
**Even and Odd Functions:**

A function $f$ is an **even function** of $x$, if $f(-x) = f(x)$ for every value of $x$, and **odd** if $f(-x) = -f(x)$.

The graph of an even function is symmetric with respect to the y-axis while the graph of an odd function is symmetric with respect to the origin.

For example; the function $f(x) = x^2$ is even since

$$f(-x) = (-x)^2 = x^2 = f(x)$$

and the function $f(x) = x^3$ is odd since

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

**Ex:** Discuss whether the following each function is odd or even.

1.) $f(x) = 3x^3 + 4x$

   $f(-x) = 3(-x)^3 + 4(-x)$
   
   $$= -3x^3 - 4x = -f(x)$$
   
   odd.

2.) $g(x) = x^2 + \cos x + 7$

   $\cos(-x) = \cos x$ **even**
   
   $\sin(-x) = -\sin x$ **odd**

   $g(-x) = (-x)^2 + \cos(-x) + 7$
   
   $$= x^2 + \cos x + 7 = g(x)$$ **even**

3.) $h(x) = |x|$ **even**

4.) $H(x) = \frac{x^3 - x}{\sin x - 1}$

   $H(x) = \frac{-x^3 + x}{-\sin x - 1}$ **odd**.
Types of Functions

Piecewise Defined Function: (Definition is 20. sheet.)

Sometimes a function is described in pieces by using different formulas on different parts of its domain.

Ex: Absolute Value Function:

\[ |x| = \begin{cases} 
  x & x \geq 0 \\
  -x & x < 0 
\end{cases} \]

Ex: Greatest Integer Function: (Tam deger fonk.)

\[ [x] = \begin{cases} 
  1 & 1 \leq x < 2 \\
  0 & 0 \leq x < 1 \\
  -1 & -1 \leq x < 0 
\end{cases} \]

The function whose value at any number \( x \) the greatest integer less than or equal to \( x \) is called the greatest integer function.

Definition:

If the value of function at any real number \( x \) is the greatest integer number less than or equal to \( x \), this function defined by in this way is called the greatest integer function.

Ex: \( [2.5] = 2 \) \( [3] = 3 \) \( [-0.1] = -1 \) \( [-1.5] = -2 \)

Note: We may define the least integer function similarly.
Example:

\[
\text{Sign } x = \text{Sgn } x = \begin{cases} 
1 & x > 0 \\
0 & x = 0 \\
-1 & x < 0
\end{cases}
\]

\[
\text{Sign } : \mathbb{R} \to \{-1, 0, 1\}
\]

\*

Example:

\[
u(x) = \begin{cases} 
1 & x \geq 0 \\
0 & x < 0
\end{cases}
\]

\*

Unit step function

Heaviside function

Example:

\[
f(x) = \begin{cases} 
-x & x < 0 \\
x^2 & 0 \leq x \leq 1 \\
1 & x > 1
\end{cases}
\]

If \( x \) is less than zero, graph of \( y = -x \) line
in this way straight.

If \( x \) is between zero and 1, graph of this quadratic function like this.

If \( x \) is greater than 1, graph of this constant function can draw easily.
Piecewise Function:

The functions which are defined as different functions on the sub intervals of the domain are called piecewise functions.

For example:

\[ f(x) = \begin{cases} 
  h(x) & \text{if } x \geq a \\
  g(x) & \text{if } b < x < a 
\end{cases} \]

is a piecewise function where \( x = a \) and \( x = b \) are critical points. These points must be determined when the points of function are drawn. Here \( g(x) \) and \( h(x) \) are the parts of the function \( f(x) \) in different intervals.
Bounded Function:

let the function \( f \) be defined in an interval \( I \).

1) If there is a number \( M \) such that \( f(x) \leq M \) for every \( x \in I \) then \( f \) is called upper bounded function in the interval \( I \) and \( M \) is said to be upper-bound of \( f \).

2) If there is a number \( m \) such that \( f(x) \geq m \) for every \( x \in I \) then \( f \) is called lower bounded function in the interval \( I \) and \( m \) is said to be lower-bound of \( f \).

3) If \( f \) is both upper bounded and lower bounded function than \( f \) is bounded function.

Similarly,

Let a function be defined as \( f(x): A \rightarrow B \).

If we can find two real numbers \( m \) and \( M \) such that

\[ m \leq f(x) \leq M \quad \text{for} \quad x \in I \]

then \( f(x) \) is called the bounded function. \( m \) and \( M \) are called the lower-bound and upper-bound of \( f(x) \) respectively.

The range of \( f(x) \) is \([m, M]\) in this interval.

However, if \( m \) and \( M \) or either of them is not defined (i.e. infinite) then \( f(x) \) is said to be unbounded function.
Ex: If the function is defined as
\[ f(x) = x^2 \quad f: [-2, \infty) \rightarrow \mathbb{R} \]
then \( f \) is unbounded,

\[ f(x) = x^2 \quad f: [0, 1] \rightarrow \mathbb{R} \]
then \( f \) is bounded.

Ex: \( f(x) = x^2 + 2x - 1 \quad f: \mathbb{R} \rightarrow \mathbb{R} \)
\[ f(x) = (x+1)^2 - 2 \geq -2 \]
\[ \{ \begin{align*} & f \text{ is only lower bounded func.} \\ & \text{not bounded func.} \end{align*} \]

Ex: \( f(x) = \frac{1}{1+x^2} \quad R \rightarrow R \)
\[ x \in R \quad f(x) > 0 \quad \Rightarrow \quad \text{so } f \text{ is lower bounded} \]
\[ x^2 > 0 \quad \Rightarrow \quad 1+x^2 > 1 \quad \Rightarrow \quad \frac{1}{1+x^2} < 1 \quad \Rightarrow \quad f(x) < 1 \quad \Rightarrow \]
Therefore \( f \) is bounded.
EX: Let us consider the polynomial function defined by in this way.

\[ f(x) = x^3 - 3x^2 + 6x - 2 \]
\[ f: [-1, 1] \to \mathbb{R} \]

Determine whether the function is bounded or not.

Since \( x \in [-1, 1] \) \( |x| < 1 \), hence

This inequality can be written

\[ |f(x)| \leq |x^3| + 3|x|^2 + 6|x| + |2| = 12 \]

\[ -12 \leq f(x) \leq 12 \]

\( f \) is both upper bounded and lower bounded function.

This is a rough solution exactly

\[ \begin{align*}
\quad & f(-1) = -12 \quad \quad -12 \leq f(x) \leq 12 \\
\quad & f(1) = 2
\end{align*} \]

EX:

\[ f(x) = x^4 - 2x^2 + 5 \]
\[ f: [-2, 2] \to \mathbb{R} \]

\[ (x^2 - 1)^2 \geq 0 \]

\[ f(x) = (x^2 - 1)^2 + 4 \geq 4 \]

\[ f(\pm 2) = 13 \]

\( \Rightarrow \) the value of \( f \) at 2 is thirteen.
Explicit and Implicit Function:

An explicit function is one which is given in terms of independent variables. For example:

\[ y = 7x + 5 \] is an explicit function, as \( y \) can be explicitly expressed in terms of \( x \).

\[ x \cdot y = 1 \] is an explicit function, as \( y \) can be expressed explicitly in terms of \( x \). (i.e. \( y = \frac{1}{x} \))

\[ y = x^2 + 3x - 8 \] \( y \) is the dependent variable and is given in terms of the independent variable \( x \).

An implicit function is a function in which one variable can not be explicitly expressed in terms of the other. For example:

The equation \( x^2 + xy - y^2 = 1 \) represents an implicit relation. Because the dependent variable is not isolated on one side of the equation.

If dependent variable \( y \) is defined by the equation \( F(x,y) = 0 \), then \( y \) is called an implicit function.

For example; unit circle is defined by the equation \( x^2 + y^2 = 1 \) \( x \) square plus \( y \) square equal to 1.

In order to get explicit function \( y \) can be expressed in terms of \( x \).
\[ y = \pm \sqrt{1-x^2} \quad -1 \leq x \leq 1 \]

Finally the function \( y = f(x) \) can be given by \( x^2 + y^2 = 1 \) as implicit, the equations \[ y = \pm \sqrt{1-x^2} \] are explicit of \( y = f(x) \).

**Ex:** \[ x^2 y^3 + y + \sqrt{y^2+1} = 0 \]

- \( y \) is not isolated on one side of the eq.
- \( y \) can not be expressed in terms of \( x \).
- \( y \) is implicit function.

Functions are examined in two parts. They are "basic functions".

1) Algebraic functions.
2) Transcendental functions.

1) Algebraic functions:

Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division and taking roots) lies within the class of algebraic functions.

- Polynomials: It's only created by total and multiplication.
  - Linear function: First order polynomial, the graph of linear function is a straight line.
  - Quadratic function: Second order polynomial is a graphical parabola.
  - Cubic function: Third degree polynomial.
Rational Functions: A ratio of two polynomials.

N-root

Square root functions:

Cube root functions:

2) Transcendental Functions:

Transcendental functions are non-algebraic functions.

- Exponential functions.
- Hyperbolic functions.
- Force function (Power Function)
- Logarithmic function: Inverse of exponential functions.
- Periodic functions.
  - Trigonometric functions: Sinus, Cosine, tangent, ...
  - Inverse trigonometric functions.
Combining Functions

There are several ways of combining two functions to create a new function.

Like numbers, functions can be added, subtracted, multiplied and divided. (except where the denominator is zero)

If \( f \) and \( g \) are functions, then every \( x \) that belongs to the domains of both \( f \) and \( g \) (that is for \( x \in D(f) \cap D(g) \)), we define function the sum \( f+g \), the difference \( f-g \), the product \( f \cdot g \) and quotient \( \frac{f}{g} \) by the formulas:

\[
(f+g)(x) = f(x) + g(x) \\
(f-g)(x) = f(x) - g(x) \\
(f \cdot g)(x) = f(x) \cdot g(x) \\
(\frac{f}{g})(x) = \frac{f(x)}{g(x)} \quad \text{(where } g(x) \neq 0) 
\]

If \( c \) is a real number then the function \( c \cdot f \) is defined for all \( x \) in the domain of \( f \) by

\[
(c \cdot f)(x) = c \cdot f(x) \quad \text{scalar product multiply by constant.}
\]
Example: The functions defined by the formulas $f(x) = \sqrt{x}$ and $g(x) = \sqrt{1-x}$ have domains

$$D(f) = [0, \infty) \text{ and } D(g) = (-\infty, 1]$$

The points common to these domains are the points:

$$[0, \infty) \cap (-\infty, 1] = [0, 1]$$

(intersection of these intervals is the closed interval zero-one.)

Let's determine the formulas and domains for the algebraic combinations of the two functions:

$$(f+g)(x) = \sqrt{x} + \sqrt{1-x} \quad D(f+g) = D(f) \cap D(g) = [0, 1]$$

$$(f-g)(x) = \sqrt{x} - \sqrt{1-x} \quad D(f-g) = D(f) \cap D(g) = [0, 1]$$

$$(g-f)(x) = \sqrt{1-x} - \sqrt{x} \quad D(g-f) = D(g) \cap D(f) = [0, 1]$$

$$(f \cdot g)(x) = \sqrt{x(1-x)}$$

square root of the product of $x$ and $(1-x)$

$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}} \quad D(\frac{f}{g}) = [0, 1)$$

$$(\frac{g}{f})(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}} \quad D(\frac{g}{f}) = (0, 1]$$
Composite Functions:
Composition is another method for combining functions.

Definition: If $f$ and $g$ are functions, the composite function $f \circ g$ (f composed with g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all $x$ in the domain of $g$, such that $g(x)$ is in the domain of $f$.

$$D(f \circ g) = \{ x \in D(g) : g(x) \in D(f) \}$$

To evaluate the composite function $f \circ g$, first we find $g(x)$ and then $f(g(x))$.

For functions $g: X \to Y$ and $f: Y \to Z$, the composite of $f$ with $g$ $f \circ g: X \to Z$

$$D(f \circ g) \subseteq D(g)$$
Example: If \( f(x) = x^2 \) and \( g(x) = \sqrt{x+1} \), find the composite of \( f \) with \( g \):

- \( (fog)(x) \)
- \( (gof)(x) \)
- \( (fop)(x) \)
- \( (gog)(x) \)

\[ D(f) = \mathbb{R} \quad D(g) = [-1, \infty) \]

\[ (fog)(x) = f(g(x)) = (\sqrt{x+1})^2 = x+1 \]

\[ D(fog) = \{ x \in D(g) : g(x) \in D(f) \} = D(g) = [-1, \infty) \]

\[ (gof)(x) = g(f(x)) = \sqrt{x^2+1} \]

\[ D(gof) = \{ x \in D(f) : f(x) \in D(g) \} = D(f) = \mathbb{R} \]

\[ (fop)(x) = f(f(x)) = (x^2)^2 = x^4 \]

\[ D(fop) = \mathbb{R} \]

\[ (gog)(x) = g(g(x)) = \sqrt{\sqrt{x+1}+1} \]

\[ D(gog) = [-1, \infty) \]
Example: find the following composite functions and domain of them.

* \( f(x) = \frac{x^2 + 2}{x^2 - 1} \)  
  \((f \circ f)(x) = ? \)  \( D(f \circ f) = ? \)

\[
(f \circ f)(x) = \frac{\left(\frac{x^2 + 2}{x^2 - 1}\right)^2 + 2}{\left(\frac{x^2 + 2}{x^2 - 1}\right)^2 - 1} = \frac{3x^4 + 6}{6x^2 + 3}
\]

\( D = \mathbb{R} \setminus \{-1, 1\} = D(f) \)

* \( f(x) = \frac{1-x}{1+x} \)  
  \((f \circ f)(x) = ? \)  \( D(f \circ f) = ? \)

\[
(f \circ f)(x) = \frac{1 - \frac{1-x}{1+x}}{1 + \frac{1-x}{1+x}} = x \quad D(f \circ f) = \mathbb{R} \setminus \{-1\}
\]

* \( f(x) = \sqrt{x} \) and \( g(x) = x + 1 \)

\[
(f \circ g)(x) = \sqrt{x + 1} \quad D(f \circ g) = [-1, \infty)
\]
\[
(g \circ f)(x) = \sqrt{x + 1} \quad D(g \circ f) = [0, \infty)
\]
\[
(f \circ f)(x) = \sqrt{x} \quad D(f \circ f) = [0, \infty)
\]
\[
(g \circ g)(x) = x + 2 \quad D(g \circ g) = \mathbb{R}
\]
**Trigonometric Functions**

**Angles:** Angles are measured in degrees or radians. Let us consider the unit circle;

\[
\widehat{AD} = s \\
s(\widehat{AOP}) = \theta
\]

The radian measure of the central angle \( AOP \) is the length of arc \( PA \). \( (PA = s) \)

The degree measure of the central angle \( AOP \) is \( \theta \).

Since one complete revolution of the unit circle is \( 360^\circ \), then

\[
360^\circ = 2\pi \quad \text{or} \quad 180^\circ = \pi \text{ radians.}
\]

An angle in the \( xy \)-plane is said to be standard position if its vertex lies at the origin and its initial ray lies along the positive \( x \)-axis.
Angles measured counter clockwise from the positive x-axis are assigned positive measures.

**Basic Trigonometric Functions:**

Let's consider ΔABC right triangle and (dik üçgen) measure of acute angle of BCA be θ.

1-) The defining of the six basic trigonometric functions of an acute angle is given as follows:

\[
\begin{align*}
\sin \theta &= \frac{\text{opposite}}{\text{hypotenuse}} \\
\cos \theta &= \frac{\text{adjacent}}{\text{hypotenuse}} \\
\tan \theta &= \frac{\text{opposite}}{\text{adjacent}} \\
\cot \theta &= \frac{\text{adjacent}}{\text{opposite}} \\
\sec \theta &= \frac{\text{hypotenuse}}{\text{adjacent}} \\
\csc \theta &= \frac{\text{hypotenuse}}{\text{opposite}}
\end{align*}
\]

We extended this definition to obuse angle and negative angles in a circle of radius r.

2-) Let's define the trigonometric functions in terms of the coordinates of the point P(x,y) in a circle of radius r as follows:

\[
\begin{align*}
\text{Sine} &\rightarrow \sin \theta = \frac{y}{r} \\
\text{Cosine} &\rightarrow \cos \theta = \frac{x}{r} \\
\text{Tangent} &\rightarrow \tan \theta = \frac{y}{x} \\
\text{Cotangent} &\rightarrow \cot \theta = \frac{x}{y}
\end{align*}
\]
Note: If the angle is acute, then this definition agree with right triangle definition.

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \quad \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}
\]

\[
\sec \theta = \frac{1}{\cos \theta} \quad \quad \csc \theta = \frac{1}{\sin \theta}
\]

As you can see, \( \tan \theta \) and \( \sec \theta \) are not defined if \( \cos \theta = 0 \). This means they are not defined if \( \theta = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots \).

Similarly, \( \cot \theta \) and \( \csc \theta \) are not defined if \( \theta = 0, \pm \pi, \pm 2\pi, \ldots \).

In first quadrant all positive:
- \( x = \cos \theta > 0 \)
- \( y = \sin \theta > 0 \)

In second "\sin" 

In third "\tan" 

In forth "\cos"
Radian angles and side lengths of two common right triangles:

![Isosceles right triangle](image1)

![30-60-90 triangle](image2)

The exact values of trigonometric ratios for some angles can be read from these triangles. For instance:

\[
\sin 45 = \cos 45 = \frac{\sqrt{2}}{2}
\]
\[
\sin 60 = \frac{\sqrt{3}}{2}
\]
\[
\cos 30 = \frac{\sqrt{3}}{2}
\]
\[
\tan 30 = \frac{\sqrt{3}}{3}
\]

\[
\sin 30 = \frac{1}{2}
\]
\[
\cos 60 = \frac{1}{2}
\]
\[
\tan 45 = 1
\]
\[
\tan 60 = \sqrt{3}
\]

Calculate the cosine/sine of \( \frac{2\pi}{3} \) radians:

From definition we can write:

\[
\cos \frac{2\pi}{3} = \frac{x}{r} = -\frac{1}{2}
\]
\[
\sin \frac{2\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}}{2}
\]
Coordinates of P are \((-\frac{1}{2}, \frac{\sqrt{3}}{2})\)

\[x = \cos 30^\circ\]
\[-x = \cos 150^\circ\]
\[y = \sin 30^\circ = \sin 150^\circ\]

**Examples:** Determine the values of \(\sin \theta\), \(\cos \theta\) for given below values of \(\theta\).

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>(\sin \theta)</th>
<th>(\cos \theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>30°</td>
<td>(\frac{1}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
</tr>
<tr>
<td>45°</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(\frac{\sqrt{2}}{2})</td>
</tr>
<tr>
<td>60°</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>90°</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>120°</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(-\frac{1}{2})</td>
</tr>
<tr>
<td>135°</td>
<td>(\frac{\sqrt{2}}{2})</td>
<td>(-\frac{\sqrt{2}}{2})</td>
</tr>
<tr>
<td>150°</td>
<td>(\frac{1}{2})</td>
<td>(-\frac{\sqrt{3}}{2})</td>
</tr>
<tr>
<td>180°</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>210°</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>270°</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>315°</td>
<td>(-\frac{\sqrt{2}}{2})</td>
<td>(-\frac{\sqrt{2}}{2})</td>
</tr>
<tr>
<td>330°</td>
<td>(-\frac{\sqrt{3}}{2})</td>
<td>(-\frac{1}{2})</td>
</tr>
<tr>
<td>450°</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>-90°</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>-180°</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>-270°</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>-315°</td>
<td>(-\frac{\sqrt{2}}{2})</td>
<td>(-\frac{\sqrt{2}}{2})</td>
</tr>
</tbody>
</table>

To do this we should use

1) Isosceles right triangle and 30-60-90 right triangle.
2) Unit circle
3) The definition of trig. funct.

If the angle is acute angle, than we use two common triangle

We can see the angles and side lengths of two common triangles:

\[\frac{1}{2} \text{ opp} \quad \frac{\sqrt{3}}{2} \text{ adj} \quad \frac{\sqrt{2}}{2} \]

If the angle is obuse angle than we should check the sign of the function by using the unit circle

\[P(x,y) = P(\cos \theta, \sin \theta)\]

\[\sin 60^\circ = y\]

We define the trig. function in terms of the coordinates of the point \(P(x,y)\) where the angles terminal ray intersects the circle.

The terminal rays corresponding to the angles 180° and -180° coincide. In that case values of the trig. functions 180° and -180° are same.
Trigonometric Identities

Trigonometric identities can be proved by using mainly the geometry of the right triangle. Let's consider the unit circle:

\[ P(x, y) = P(\cos \theta, \sin \theta) \]

\[ |AB| = |\cos \theta| \]

\[ |AP| = |\sin \theta| \]

The length of AB

Unit circle \( x^2 + y^2 = 1 \) \( \Rightarrow \) we obtain the equation;

\[ \cos^2 \theta + \sin^2 \theta = 1 \]

We can proof the addition formulas by using right triangle definition of sine and cosine functions

Addition Formulas:

\[ \cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B \]

\[ \sin(A+B) = \sin A \cdot \cos B + \sin B \cdot \cos A \]

Double-Angle Formulas:

\[ \cos 2\theta = \cos^2 \theta - \sin^2 \theta \]

\[ \sin 2\theta = 2 \sin \theta \cdot \cos \theta \]

\[ \cos 2\theta = 2 \cos^2 \theta - 1 \]

\[ = 1 - 2 \sin^2 \theta \]
Law of Cosines:

\[ a^2 = c^2 + b^2 - 2bc \cdot \cos \theta \]

Proof: using \( \triangle ABC \) triangle

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \Rightarrow \text{opposite} = \text{hypotenuse} \cdot \sin \theta
\]

\[
\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \Rightarrow \text{adjacent} = \text{hypotenuse} \cdot \cos \theta
\]

using \( \triangle ABD \) right triangle

\[
a^2 = b^2 \sin^2 \theta + (c - bc \cos \theta)^2
\]

\[
a^2 = b^2 \sin^2 \theta + c^2 - 2bc \cos \theta + b^2 \cos^2 \theta
\]

\[
a^2 = c^2 + b^2 - 2bc \cos \theta
\]

The proof of each of trigonometric identity follows from the definitions of all trigonometric functions.
Periodicity and Graphs of the Trigonometric Functions.

When an angle of measure $\theta$ and an angle of measure $\theta + 2\pi$ are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function's values:

\[
\sin (\theta + 2\pi) = \sin \theta \quad \tan (\theta + \pi) = \tan \theta \\
\cos (\theta + 2\pi) = \cos \theta \quad \cot (\theta + \pi) = \cot \theta
\]

We describe this repeating behavior by saying that the six basic trigonometric functions are periodic.

**Definition:** A function $f(x)$ is periodic if there is a positive number $p$ such that $f(x+p)=f(x)$ for every value of $x$.

The smallest such value of $p$ is the period of $f$.

Sine and cosine functions have period $p = 2\pi$.

Tangent and cotangent \(\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
Examples: find the periods of following functions.

(1) \( f(x) = \cos 2x \)

Def: \( f(x+p) = f(x) \Rightarrow f \) periodic.
Can we find, for which \( p \) satisfies

\[
 f(x+p) = \cos 2(x+p) = \cos 2x = f(x) .
\]

We know that the period of \( \cos x \) is \( 2\pi \).
Thus:

\[
 2x + 2p = 2x + 2\pi \\
 2p = 2\pi \Rightarrow p = \pi
\]

So the period of \( \cos 2x \) function is \( \pi \).

(2) \( y = \frac{\cos 2x + \sin 3x}{p_1 + p_2} \Rightarrow p = ? \)

\[
 p_1 = \pi \quad \text{and} \quad p_2 = \frac{2\pi}{3} \Rightarrow \text{period of } y \Rightarrow \text{LCM}(\pi, \frac{2\pi}{3})
\]

LCM \( \Rightarrow \) the Least Common Multiple

GCD \( \Rightarrow \) the Greatest Common Divisor

(3) \( y = \sin^2 x = \frac{1 - \cos 2x}{2} \Rightarrow p = \pi \)

(4) \( y = \frac{\sin 3x}{\cotg 2x} \Rightarrow p_1 = \frac{2\pi}{3}, \quad p_2 = \pi/2 \)

\( \Rightarrow p = 2\pi \)

Note That: Since period of cosine func is \( 2\pi \),
if \( \cos A = \cos B \) then \( A - B = 2k\pi \), \( k \in \mathbb{Z} \)
Graphs of the Trigonometric Functions

We can see the graphs of the six trigonometric functions in the coordinate plane in this section. These functions are not one to one, for this reason they have not inverse on their domains. However, we can restrict their domains to intervals on which they are 1-1.

\[
y = \cos x
\]

- **Domain:** \(-\infty < x < \infty \rightarrow [0, \pi]
- **Range:** \(-1 \leq y \leq 1
- **Period:** \(2\pi

We make the cosine function 1-1 by restricting its domain to \([0, \pi]\) so that it has an inverse.

\[
y = \sin x
\]

- **Domain:** \(-\infty < x < \infty \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]
- **Range:** \(-1 \leq y \leq 1
- **Period:** \(2\pi

Domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$

Range: $-\infty < x < \infty$

Period: $\pi$

---

Domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$

Range: $y \leq -1$ or $y \geq 1$

Period: $2\pi$

---

Domain: $x \neq 0, x \neq \pm \pi, x \neq \pm 2\pi, \ldots$

Range: $y \leq -1$ or $y \geq 1$

Period: $2\pi$
Graphs of the six basic trigonometric functions using radian measure are like these. The shading for each trigonometric function indicates its periodicity.

The six basic trigonometric functions are not one to one, their values repeat periodically. However, we can restrict their domains to intervals on which they are one to one.

Ex: The sine function increases from -1 at $x = -\frac{\pi}{2}$ to +1 at $x = \frac{\pi}{2}$. By restricting its domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ we make it one to one. So that it has an inverse $\sin^{-1}x$. 
Domain restrictions make trigonometric functions one to one. Since these restricted functions are now one to one, they have inverse which denote by as below.

**Inverse Trigonometric Functions**

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles.

Inverse trigonometric functions denote by:

\[
\begin{align*}
y &= \sin^{-1}x \\
y &= \cos^{-1}x \\
y &= \tan^{-1}x \\
y &= \cot^{-1}x
\end{align*}
\]

\[
\begin{align*}
y &= \arcsin x \\
y &= \arccos x \\
y &= \arctan x \\
y &= \arccot x
\end{align*}
\]

The graph of \( \sin^{-1}x \) obtained by reflection across the line \( y = x \), is a portion of the curve \( x = \sin y \). The graphs of these inverse trigonometric functions are obtained by reflecting the graphs of the restricted trig functions through the line \( y = x \).
Ex: Evaluate the \( \sin^{-1} \frac{\sqrt{3}}{2} \) and \( \cos^{-1}(\frac{-1}{2}) \)

If \( \sin^{-1} \frac{\sqrt{3}}{2} = x \) then \( \sin x = \frac{\sqrt{3}}{2} \) so \( x = \frac{\pi}{3} \) radian.

from the definition of the inverse function.

\[
\begin{align*}
\sin x &= \frac{\sqrt{3}}{2} \\
x &= \frac{\pi}{3}
\end{align*}
\]

Similarly:

If \( \cos^{-1}(\frac{-1}{2}) = x \) then \( \cos x = -\frac{1}{2} \) \( x \in [0, \pi] \)

Range of \( \arccos x = \cos^{-1} x = [0, \pi] \). Since \( \cos x \) is negative then \( x \in [-\frac{\pi}{2}, \pi] \).

\( x \) in the second region (quadrant)

\[
\begin{align*}
\cos x &= -\frac{1}{2} \\
x &= \frac{2}{3} \pi
\end{align*}
\]

\[
\begin{align*}
\sin 30 &= \cos 60 = \frac{1}{2} \\
\cos 120 &= -\frac{1}{2}
\end{align*}
\]

* We can create the following table of common values for the arcsine and arccosine functions

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \arcsin x )</th>
<th>( \arccos x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{3}/2 )</td>
<td>( \pi/3 )</td>
<td>( \pi/6 )</td>
</tr>
<tr>
<td>( \sqrt{2}/2 )</td>
<td>( \pi/4 )</td>
<td>( \pi/4 )</td>
</tr>
<tr>
<td>( 1/2 )</td>
<td>( \pi/6 )</td>
<td>( \pi/3 )</td>
</tr>
<tr>
<td>( -1/2 )</td>
<td>( -\pi/6 )</td>
<td>in 2. quadrant</td>
</tr>
<tr>
<td>( -\sqrt{3}/2 )</td>
<td>( -\pi/4 )</td>
<td>( 3\pi/4 ), ( 45^\circ )</td>
</tr>
<tr>
<td>( -\sqrt{3}/2 )</td>
<td>( -\pi/3 )</td>
<td>( 5\pi/6 ), ( 300^\circ )</td>
</tr>
</tbody>
</table>
General Exponential Functions

The function \( f(x) = a^x \) is called an exponential function with base \( a \). Where \( a \) is any constant (number) such that \( a > 0, a \neq 1 \), exponent \( x \) is any real number. \( D(f) = \mathbb{R} \) (if \( a = 1 \) then \( f(x) = 1^x = 1 \) this is constant function.)

Rules of Exponents: We can raise:

\[
\begin{align*}
    a^x \cdot a^y &= a^{x+y} \\
    \frac{a^x}{a^y} &= a^{x-y} \\
    (a^x)^y &= a^{x \cdot y} \\
    (a \cdot b)^x &= a^x \cdot b^x \\
    (\frac{a}{b})^x &= \frac{a^x}{b^x} \\
    a^{-x} &= \frac{1}{a^x} \\
    a^0 &= 1 \\
    a = a
\end{align*}
\]

Graphs of Exponential Functions:

\[
\begin{array}{c|c}
    x & 2^x \\
    \hline
    -2 & \frac{1}{4} \\
    -1 & \frac{1}{2} \\
    0 & 1 \\
    1 & 2 \\
    2 & 4 \\
    3 & 8
\end{array}
\]

\[
\begin{array}{c|c}
    x & \left(\frac{1}{2}\right)^x \\
    \hline
    -3 & 8 \\
    -2 & 4 \\
    -1 & 2 \\
    0 & 1 \\
    1 & \frac{1}{2} \\
    2 & \frac{1}{4}
\end{array}
\]
Properties: \( f(x) = a^x \)

- Domain of \( f \): \( D(f) = (-\infty, \infty) \)
- Range of \( f \): \( R(f) = (0, \infty) \)
- \( f(0) = 1 \)
- If \( a > 1 \) then graph rises,
  If \( 0 < a < 1 \) then graph falls
- Get \( a > 1 \), if \( x \) becomes more and more negative, than graph approaches the \( x \)-axis.
- Get \( 0 < a < 1 \), if \( x \) becomes more and more positive, than graph approaches the \( x \)-axis.

Natural Exponential Function

**Number \( e \):** Euler: \( 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots = e = 2.71828 \ldots \)

The number \( e \) is an important constant irrational number.

**Definition:** If \( n \) is positive and very big number than the function \( (1 + \frac{1}{n})^n \) approaches the irrational constant number 2.71828... denoted by the letter \( e \). (It has been found by Euler)

The exponential function with base \( e \) is called natural exponential function denoted by

\[ f(x) = e^x = \exp x \]
Logarithmic Function

Get the function \( f(x) = a^x \) (exponential func.) 
\( (a > 0, a \neq 1) \).

Inverse of this function is called logarithmic function base \( a \) and denoted by

\[ f^{-1}(x) = \log_a x \]

\[ y = \log_a x \quad \text{if and only if} \quad a^y = x \]

We have the following fundamental equations.

* \( \log_a a^x = x \) \quad (\text{for all } x \text{ in } (-\infty, \infty) = \mathbb{R})
* \( a^{\log_a x} = x \) \quad (\text{for } x > 0)

\( f(x) = a^x : (-\infty, \infty) \rightarrow (0, \infty) \), it follows that 
\( (0, \infty) \) is domain of logarithm function base \( a \) 
\( (-\infty, \infty) \) is range of logarithm function base \( a \).

\[ f(x) = \log_a x : (0, \infty) \rightarrow (-\infty, \infty) \]

The graph of \( \log_a x \) is drawn with symmetry rule.

Note that the graph of \( f \) and inverse of \( f \) are symmetric to the line \( x = y \).
Properties of Logarithms

These can be proof by using the definition of log function

* \( \log_a (m \cdot n) = \log_a m + \log_a n \quad m, n > 0 \)
* \( \log_a \left( \frac{m}{n} \right) = \log_a m - \log_a n \quad m, n > 0 \)
* \( \log_a m^r = r \cdot \log_a m \quad m > 0 \)
* \( \log_a 1 = 0 \quad \log_a a = 1 \quad a > 0 \)
* If \( a > 1 \) then log function is increasing function.
  If \( 0 < a < 1 \) " " " " decreasing " "

* From (2) \( \log_a \frac{1}{n} = -\log_a n \)
* \( \log_a m = \frac{\log_b m}{\log_b a} \)

**Common Logarithms:** Logarithms to the base 10 are common logarithms, denoted by

\[ \log_{10} x = \log x \]

**Natural Logarithms:** Logarithms to the base e are natural logarithms, denoted by

\[ \log_e x = \ln x \]

- \( \ln 1 = 0 \)
- \( \ln e = 1 \)
- \( \ln e^{-1} = -1 \)
Some Examples:

* Solve the equation $e^{2x-6} = 4$ for $x$.

In order to solve this equation, let’s apply the function $\ln$ (natural logarithmic function) to both sides of the equation.

$$\ln e^{2x-6} = \ln 4$$

We find

$$2x - 6 = \ln 4 \quad \Rightarrow \quad 2x = 6 + 2\ln 2$$

$$x = 3 + \ln 2$$

* Proof the law $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$

Let $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$. Then we obtain

$x_1 = \ln y_1$ and $x_2 = \ln y_2$ inverse.

$$x_1 + x_2 = \ln y_1 + \ln y_2$$

$$= \ln (y_1 \cdot y_2) \quad \text{(Product rule for log.)}$$

$$e^{x_1+x_2} = e^{\ln (y_1 \cdot y_2)}$$

$$\Rightarrow y_1 \cdot y_2 = e^{x_1+x_2}$$

* $(e^{x_1})^r = e^{x_1 \cdot r}$

$$y = (e^{x_1})^r \quad \Rightarrow \quad \ln y = r \cdot \ln e^{x_1} = r \cdot x_1$$

$$\ln y = r \cdot x_1 \quad \Rightarrow \quad y = e^{r \cdot x_1}$$