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YILDIZ TECHNICAL UNIVERSITY
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CONSTRUCTING HIGHLY CONSISTENT MATRICES IN THE ANALYTIC HIERARCHY PROCESS

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There was a question that lead to a tsunami of questions. Sometimes I asked them, the others I have been asked. Sometimes I easily floated through the waves, sometimes I was pulled deep by the currents.

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LIST OF SYMBOLS

$\mathbf{a_{ij}}$	Estimated weight ratio of specimens i and j
$\mathbf{E_G}$	Set of edges for a graph G
$\mathbf{V_G}$	Set of vertices for a graph G
$\mathbf{w_i}$	Weight of specimen i
r	Number of matrices to be generated randomly
n	Order of matrices to be generated randomly
m	Number of direct pairwise comparisons in a random PCM
s(m)	Range of direct pairwise comparisons for a random PCM
i	Row indice of a PCM
j	Column indice of a PCM
k	Row and column indice of a PCM for deriving indirect pairwise comparisons

LIST OF ABBREVIATIONS

AHP	Analytic Hierarchy Process
CI	Consistency Index
CR	Consistency Ratio
CPCM	Complete Pairwise Comparison Matrix
IPCM	Incomplete Pairwise Comparison Matrix
PC	Pairwise Comparison
PCM	Pairwise Comparison Matrix
PRM	Positive Reciprocal Matrix
RPCM	Random Pairwise Comparison Matrix
RI	Random Index
RIS	Random Index Study

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ABSTRACT

CONSTRUCTING HIGHLY CONSISTENT MATRICES IN THE ANALYTIC HIERARCHY PROCESS

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MSc. Thesis

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AHP provides a decision making framework by quantifying the decision elements in order to evaluate alternative solutions with respect to a specified objective in multiple criteria decision making problems. AHP uses pairwise comparison (PC) data for generating weight vectors of decision elements for final results, which limits the validity of the result with the consistency of PC matrices. Final results (resulting weight vector) can only be considered as a reliable reflection of the evaluators opinion, if and only if relevant data is sufficiently consistent. By determining the causes of inconsistency, we develop a new method for constructing highly consistent PC matrices. This study investigates underlying reasons of inconsistency, and explores new a faciliting elicitation process to derive consistent matrices.

Keywords: Inconsistency, pairwise comparison, consistency improvement

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ÖZET

ANALITIK HIYERARŞI PROSESI'NDE YUKSEK TUTARLILIKLI MATRISLERIN OLUŞTURULMASI

Şahika KOYUN

Endüstri Mühendisliği Bölümü
Yüksek Lisans Tezi

Danışman: Yrd. Doç. Dr. Vildan Çetinsaya Özkır

Analitik Hiyerarşi Prosesi, Çok Kriterli Karar Problemlerinde belirli bir amaca yönelik olarak alternatif çözümleri değerlendirebilmek için karar elemanlarını sayısal olarak değerlendirerek bir karar çatısı oluşturur. Analitik Hiyerarşi Prosesi karar elemanlarına ait ağırlık vektörünü oluşturmak için ikili karşılaştırma metodunu kullanmaktadır. Bu durum karar sonucunun geçerliliğini ikili karşılaştırma matrislerinin tutarlılık seviyesi ile sınırlamaktadır. Analitik Hiyerarşi Prosesi sonucu olarak elde edilen ağırlık vektörünün değerlendiricinin fikirlerinin güvenilir bir yansıması olarak değerlendirilebilmesi ilgili verinin yeterli derecede tutarlı olmasına bağlıdır. Tutarsızlığa yol açan nedenleri belirleyerek, yüksek tutarlılığa sahip ikili karşlaştırma matrisleri oluşturmak için yeni bir metot geliştirilmiştir. Bu çalışmada tutarsızlığa sebep olan nedenler ve tutarlı matrisler elde edebilmek için yeni veri toplama süreçleri araştırılmaktadır.

Anahtar Kelimeler: Tutarsızlık, ikili karşılaştırma, tutarlılık iyileştirme

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CHAPTER 1

Introduction

In our daily lives, we constantly face choices and decisions. Let it be what to eat for dinner or which university to apply, we have to make decisions in order to move forward in life. Inspite of the wide range of subjects, a choice is always a *goal directed activity* and *decision theory* is concerned with *goal directed behaviour in the presence of options* [1].

There are two different approaches on decision theory: *normative* and *descriptive*. *Normative Decision Theory* is about *how decisions should be made in order to be rational*, and *Descriptive Decision Theory* is about *how decisions are actually made*.

Regardless of the type of decision theory, decision-making is always a process that evaluates the options as regard to a goal in order to choose an option which returns the most utility value. The *Analytic Hierarchy Process (AHP)* is also such a process that evaluates some sets of alternatives and criteria, puts them in hierarchical order as regard to a goal in order to single out an alternative with the highest utility value. AHP utilizes experts' opinions as raw data, and returns two types of results:

- How much each criterion affects the goal
- How much each alternative satisfies the goal

AHP is a very useful decision aid tool to evaluate both *qualitative* and *quantitative* criteria but the human factor utilized in the process makes it vulnarable to inconsistency.

Behavioural decision-making studies [2, 3, 4] specifies five different decision-making styles: *rational, intuitive, dependent, avoidant* and *spontaneous*. These decision-making styles are characterized as follows [4]

- *Rational decision-making style* is characterized by a thorough search for and logical evaluation of alternatives.
- *Intuitive decision-making style* is characterized by the use of hunches and feelings in decision-making.
- *Dependent decision-making style* is characterized by a reliance on the advice of others.

- *Avoidant decision-making style* is characterized by attempts to avoid decision making.
- *Spontaneous decision-making style* is characterized by a sense of immediacy and desire to complete the decision-making as soon as possible.

By definition *rational decision-making style* is the most suitable decision-making style to be utilized for decision aid tools. Studies [2, 5] also show that personality traits and individual differences have effect on decision-making. For example individuals who are open to experience and conscientous are more likely to be rational decision makers [2]. Since we cannot be in control of which decision-making style the decision-maker will show tendency to, we have to assume they are all rational decision-makers. Inspite of this assumption inconsistencies will occur due to various reasons. Regardless of the reasons, inconsistencies must be eliminated or decreased to a tolerable level.

AHP is a decision aid tool that can deal with inconsistency, and locate the reasons of inconsistency. Inspite of this quality, *AHP* is still vulnarable to inconsistency up to a level. Mentioned level is defined as *inconsistency threshold*, and there are many studies [6, 7, 8, 9, 10, 11, 12, 13, 14] as to what it should be. Saaty [15] defines inconsistency threshold as 0.10, however many studies state that 0.10 threshold is rigid and non-applicable to every decision-making situation.

When inconsistency level of a reported judgment matrix is higher than the stated threshold, consistency improvement methods are utilized. Saaty [15] and several other researchers [16, 17, 18, 19, 20, 21, 22] proposes consistency improvement methods. However such methods do not necessarily help increasing the validity of the result.

As the most basic definition, inconsistency is the distance of reported judgment matrix from the real (unknown) judgment matrix. The *pairwise comparison method* lets us to estimate the real weights of specimens. The reported judgment matrix is constituted of *pairwise comparison* values which are the estimated ratio of weight for two specimens. The real judgment matrix represents the judgment matrix if the exact value of each specimen's weight were known. Since we do not know the exact weight of each specimen, and it is our aim to find out the most proximate weight for each specimen, there is no way to be certain if the reported judgment matrix reflects the reality, indeed. Therefor when consistency of a judgment matrix is questioned, the result can only be interpreted as if the matrix in question is consistent or not in itself. In case of a judgment matrix is found to be inconsistent, and consistency improvement methods are utilized, an *adjusted judgment matrix* is acquired from the reported judgment matrix. It was earlier stated that there is no way to be certain if a given judgment matrix truly reflects the reality, this is also in effect for adjusted judgment matrices. Regardless of consistency level of the adjusted judgment matrix, it is still unknown if it reflects the reality. Gaul and Gastes's [23] study shows that any adjusted matrix acquired from a reported judgment matrix, converges to *a* consistent matrix but not necessarily to the real judgment matrix. This can be interpreted as consistency improvement methods only improve the consistency of given judgment matrix but its effect on the validity of the decision cannot be predicted.

The main objects of decision aid tool is to guide the decision-maker to a valid and

rational choice. This objective is disputed when the validity of the judgment matrices are questionable. Since they are the only means to understand and evaluate the raw data, their validity is directly connected to the validity of the result. Therefor, instead of adjusting inconsistent judgment matrices to achieve consistency, consistent matrices should be builded in the first place.

1.1 Literature Review

Saaty defines a hierarchy as *an abstraction of the structure of a system to study the functional interactions of its components and their impacts on the entire system*[15]. Since criteria, alternatives and the goal are interactive components of decision problems, they can be represented as a hierarchy. The *Analytic Hierarchy Process (AHP)* developed by Saaty [15] utilizes hierarchical structure in order to see interactions between each level and their impact on the goal. *Pairwise Comparison Method* is utilized for gathering data, and *Pairwise Comparison Matrice*s (PCM) are constituted accordingly. Generally *Eigenvalue Method* is utilized to derive the weight vector but other methods (*additive nozrmalization method, weighted least-squares method, logarithmic least-squares method, logarithmic goal programming method, fuzzy preference programming method* [24]) are available. A weight vector shows how much each criteria affects the goal or how much each alternative satisfies the goal. Different levels of criteria are evaluated amongst themselves, and alternatives are evaluated as regard to criteria. As a result each alternative obtains a score showing how much it satisfies the goal. The alternative with the highest score is usually suggested as the answer for decision problem, considering it is a utility problem.

It is important to know and identify the causes of inconsistency in order to eliminate them. Consistency of a *Pairwise Comparison Matrix* is affected by intransitivity and error in judgment. Graph representation of a *PCM* is a directed and complete graph, incase of all pairwise comparisons are present. Intransitivity occurs if the graph consists of cycles, leading to inconsistency. While intransitivity distrubs logical structure of the process, it also adds to the mathematical error. Other cause of inconsistency is measurement error, the mathematical error from the real weight values.

The consistency level of a *PCM* is defined as *Consistency Index (CI)* and it is the deviation from the maximum eigenvalue of the given *PCM*. Inconsistency causes an increase in the the maximum eigenvalue which is equal to the order of the matrix incase of total consistency. *Consistency index* increases with the level of inconsistency; the more inconsistent the matrix higher the *CI*.

$$CI = \frac{\lambda_{\max} - n}{n - 1} \tag{1.1}$$

In order to measure inconsistency Saaty developed a deterministic benchmark. *Consistency Rate* is defined as the ratio of *CI* over respective *Random Index (RI)*[1], and if *CR*

[1]RI: Average of *CI* values for randomly generated *PCM*s

is higher than 10 percent given *PCM* is accepted as inconsistent [15]. However Saaty's 10 percent threshold is widely criticized as being too rigid and not suitable for every decision problem. Studies [7, 10, 9, 8, 6, 12, 13, 14] proposes both deterministic and probabilistic consistency thresholds.

$$CR = \frac{CI}{RI} \tag{1.2}$$

There are two methods of investigating a *PCM*'s consistency level: *deterministic* and *probabilistic*. A study by Dong et al. [25] show that different prioritization methods give similar results when same numerical scale is used. Therefor method of evaluating consistency level is independent from eigenvector elicitation method. With deterministic approach *consistency index* of a *PCM* is compared against a benchmark which is the expected value derived by random matrices. Divergently from deterministic approach, probabilistic method tests if *consistency index* of a *PCM* belongs to the relevant random consistency index distribution.

Vargas [14] shows that while the size of set of elements increases natural inconsistency level of a *PCM* converges to the average random consistency index. This study also shows that two methods are not comparable without the use of same benchmark. Dodd et al. [9] propose a probabilistic approach and significance levels for different confidence levels.

There are various methods to decrease the level of inconsistency. Revision of the *pairwise comparisons* is the most used method amongst them. In order to increase the consistency level of a *PCM* by revision of the *pairwise comparisons*, first the origin point of inconsistency must be located. The largest absolute difference between a_{ij} and w_i/w_j gives an indication for the origin point of the inconsistency. By replacing a_{ij} by w_i/w_j consistecy level of a *PCM* is repaired. However excessive use of this process may distort the answer[15].

Dadkhah et al. [18] assume inconsisteny is a measurement error, proposes a revision method smilar to Saaty's. Proposed method is also based on locating the elements that will contrubute the most to increasing consistency as Saaty's. Differently from Saaty's revision method, the elements which will decrease the maximum eigenvalue are investigated; in order to do so the first derivation of λ_A of a_{ij} is used. The element with the largest absolute value for $\frac{\delta\lambda_A}{\delta a_{\mathrm{ij}}}$ is revised; if negative decision maker is instructed to increase a_{ij}, if positive decision maker is instructed to decrease a_{ij}.

Benitez et al. [21] use a linearization technique that provides the closest consistent matrix to a given inconsistent matrix using orthogonal projection in a linear space. In order to measure the closeness of two given matrices, Frobenius norm is utilized. The proposed method improves consistency in a direct and straightforward way differently from iterative methods.

Gomez et al. [20] propose a multi-layer perceptron-based model to improve consistency of a given *PCM*. Given model is capable of computing missing values while also improving consistency.

In order to improve consistency of a *PCM*, Siraj et al. [19] propose a three part objective function that minimizes *total deviation* (sum of all quadratic errors between the estimated priority vector and the set of given judgments), *mean absolute deviation* (average of all absolute error between the estimated priority vector and the ideal preference vector), and *the number of priority violations*. Their method is usefull when the revision of judgments is not possible.

As earlier stated by Saaty [15], Bozoki et al. [17] also state that modification of elements more than three times may destroy the correct judgments, and give weights to alternatives that does not reflect the reality. Some tools are presented to elicit if consistency of a *PCM* can be improved by modification of a few $(1, 2$ or $3)$ elements. In order to make consistent matrices with modification of a few elements two approaches are proposed. First is a mixed $0-1$ programming problem, and second is graph theoric analysis of the graph G.

Rahmami et al. [22] propose a method that will improve consistency with less cost and better results than other methods in practice. A generalization of Purcell method is used to solve a system of homogenous linear equations. Their method iteratively reduces $\lambda_{\max}$ of a given inconsistent *PCM*, reaching a limit value.

Although there are many studies focusing on mathematical treatment methods in order to improve consistency, study of Gaul and Gastes [23] shows that such methods are not often fit for purpose. Their study explains the disadvantages of consistency adjustments for computing acceptable weights for the determination of the underlying overall objective function. While mathematical treatment methods can improve consistency of a given *PCM*, their affect on the validity of the result is unknown. Gaul and Gastes's study states that consistency of a *PCM* can be improved with mathematical treatment methods but it cannot be understood that if in needed direction.

Harker [26] proves that non-negative, quasi-reciprocal matrices can be used as exactly the same manner as positive, reciprocal matrices. Allowing decision maker to answer questions as "I do not know" or "I am not sure' makes the questioning process shorter. Therefore, better representation of the responses to certain stimuli may be derived. Harker [26] states that *the ability to skip certain direct questions may make the decision maker more willing to participate in a structured decision analysis experience*. Some of the inconsistency of a given *PCM* may be caused by decision maker giving dishonest answers. Ability to skip certain questions may encourage decision makers to be more honest, eliminating another reason of inconsistency. Also reducing the number of questions to be answered will help to build more consistent matrices because the time needed to answer these questions will also shorten, helping decision maker to stay focused.

Harker's [26] proval of the use of non-negative quasi-reciprocal matrices is supported with the graph theory. The graph theory shows that any spanning tree of a complete *PCM* is sufficient to acquire a *Complete Pairwise Comparison Matrix (CPCM)*, indirectly. With n being the order of *PCM*, $(n-1)$ pairwise comparisons is enough to derive remaining missing judgments. There are some studies [27, 28] on the subject of how and which $(n-1)$ pairwise comparisons should be chosen.

Wedley et al. [27] investigate the effect of different reference items for the first (n-1) PCs. Different reference items are selected as regard to their place in relative ranking order. The empirical results show that significantly greater initial accuracy is achieved if the lowest ranked item is used as the common reference. However Ishizaka and Lusti [28] argue that the use of a common reference point compromizes psychological independency of pairwise comparisons. They propose an expert module which uses the second diagonal of the *PCM* as the first $(n-1)$ *PC*s to eliminate the dependency of pairwise comparisons.

1.2 Objective of the Thesis

This study focuses on gathering data rather than adjusting inconsistent matrices in order to achieve highly consistent *PCM*s. A new elicitation method for gathering pairwise evaluations from decision makers is proposed. The below given aspects of the proposed method summarizes how it will contribute to *AHP*.

- Relative ordering of specimens,
- Randomization of asking order of questions,
- Derivation of intervals for missing judgments,
- Reduced number of questions.

To further explain these aspects, *relative ordering of specimens* will be used to eliminate intransitivity issues in check during gathering data. The decision maker will be required to revise a judgment instantly if given judgment creates an intransitivity amongst specimens. *Randomization of asking order of questions* will ensure to preserve psychological independency of judgments. With the *derivation of intervals for missing judgments* will be used to guide the decision maker for evaluation of these judgments. If a given judgment is not in the derived interval the decision maker will be required to revise the evaluation as regard to interval. *Reduced number of questions* will help decision maker to stay focused, and complete the elicitation process in shorter time periods.

In order to reduce the number of questions, consistency index distributions of random *PCM*s with different number of direct pairwise comparisons will be analyzed. The first objective of this analysis is to determine an interval for the number of direct *PC*s, and random index to evaluate such judgment matrice's consistency. The second object is to determine a specific random index for each level of *direct pairwise comparisons* for *PCMs* with the orders of 4 through 11.

1.3 Original Contribution

The main objective of this study is to design a new data elicitation process that will construct highly consistent *Pairwise Comparison Matrices* with reliable information. In order to achieve this objective causes of inconsistency are investigated. Solutions that will target each cause of inconsistecy are explored, and combined in one process.

However each solution has a different impact level on the targeted objective. While some of these solutions offer high impact, therefor more effect on constructing highly consistent *Pairwise Comparison Matrices* other have low impact but helps to construct reliable and consistent *Pairwise Comparison Matrices*.

Suggested solutions are *limited number direct pairwise comparisons*, *initial relative ordering of specimens*, *randomized questioning order* and *limited measurement error allowance*. Limiting the number direct pairwise comparisons brings along another issue which affects the decision if a given *PCM* is accepted or rejected as consistent. The literature offers various *Random Index Studies* with one important common property: *Random Index* is derived through *Random Pairwise Comparison Matrices (RPCM)*[2] which do not have missing judgments. Whether dealing with *PCM*s with missing judgments or not, in order to accept a given *PCM* as a consistent one, the *consistency index* is compared against the *random index* which is derived from *RPCMs* with complete set of judgments. However this study shows that the number of missing judgments affects the mean *consistency index* of a *RPCM* population. Therefor comparing *consistecy index* of a given *IPCM* against a *random index* derived by such a population will lead to misguided results.

The proposed method utilizes *PCM*s with missing judgments. In order to evaulate a resulting *PCM* correctly, a *Random Index Study (RIS)* is conducted. As a result of *RIS*, for *PCM*s with the orders of 4 through 11 a *RI* for each level of number of direct pairwise comparisons is derived.

[2] *Random Pairwise Comparison Matrix*: A randomly generated *PCM*

CHAPTER 2

Analytic Hierarchy Process

A decision aid tool is designed to help the decision maker to see the interactions between criteria and alternatives. There are several different methods supplies such help in different ways. The *Analytic Hierarchy Process* is such an aid tool which utilizes hierarchical structure to visualize the interactions mentioned before.

There are different levels of a hierarchy: *Focus, forces, actors, objectives and scenarios* [15]. *Pairwise comparison* method is utilized to show the interactions between the components of the hierarchy. These interactions can be within criteria clusters, between alternatives and criteria clusters, and between sequential hierarchical levels. *PCM*s which are positive reciprocal matrices by definition are utilized to visualize these interactions mathematically.

In the following chapters *pairwise comparison method* and *consistency analysis* will be described. Steps of *AHP* (gathering data and their evaluation) will be described with the use of numerical examples.

2.1 Pairwise Comparisons

Every decision making process basicly states a set of alternatives in relative ordering as regard to a number of criteria. This value pattern [1] can be expressed both verbally and numerically. The latter is more preferable and reliable from engineering point of view because of its mathematical properties.

Verbal relational representations are commonly used in everyday language. In order to compare alternatives phrases such as *'better than', 'worse than', 'equally good', 'equally worse'* etc. are used. Such phrases compare alternatives in pairs. However verbal representations are not trustworthy because of their non-symmetrical attributes, for example phrases *'better than'* and *'worse than'* does not necessarily conclude to same decision. Although decision theory is abstracted from non-symmetrical attributes of verbal representation, it does not show the distance between alternatives. Numerical

Table 2.1: Crowdedness levels of local bus routes

Name of the bus route	Crowdedness level
30A	10
30M	9
DT1	5
DT2	4

representations of value pattern is more valid for their capability of showing the specific distance between each alternative.

There are different numerical representations for value patterns. For example for each alternative from the same domain[1] a numerical value from a pre-specified interval can be assigned. With such usage, the minimum value of the interval usually represents the least satisfactory level while the maximum value represents the most satisfactory level for the domain in question.

To clarify with a numerical example Table 2.1 shows the crowdedness levels for the local bus routes from Besiktas to Mecidiyekoy. The crowdedness level of a route is assigned to a value between 0 and 10. Table 2.1 shows that the most crowded bus route is 30A, and the least crowded bus route is DT2. The relative ordering as regard to crowdedness is $30A > 30M > DT1 > DT2$. Therefore to avoid crowd, DT lines are more preferable than 30 lines.

In such process every alternative is evaluated individually as regard to remaining alternatives. Result is usually an improved relative ordering than verbal representations. However such evaluation results are highly unclear to interpret as to what numbers represent. Although a scale of measurement is used, the interval between each step of the scale is amibiguous. When this coincides with differantial of human cognition, any assignment of numbers became arbitrary.

There are more adequate numerical representation methods. The most commonly used method amongst modern decision aid tools is *pairwise comparison method*. This method is generalized by Thurstone [29], and implemented in the AHP by Saaty [15].

The main principle of pairwise comparison method is to take two alternatives at a time, and estimate their weight ratio to derive the real weights of each specimen[2].

Decision making requires choosing an alternative within a group of alternatives as regard to a common attribute. Evaluation process is commonly conducted as a comparative process. An attribute which should be able to considered as 'more' or 'less' for each specimen should be specified for comparative process. Thurstone [29] states that *'The law of comparative judgment is applicable not only to the comparison of physical stimulus intensities but also to qualitative comparative judgments such as those of excellence of specimens in an educational scale and it has been applied in the measurement of such psychological values as a series of opinions on disputed public issues.'*.

[1]Domain: common attribute in question for alternatives.

[2]Specimen: An alternative or a criterion

Let B the set of balls with different amounts of weight. In order to create the relative ordering of the balls as regard to their weight, they must be compared in pairs.

$$B = \{i| \quad i \in \{b_1, b_2, b_3, b_4\}\}$$

If the weight of the balls were known, they could be put in increasing order easily. In this example let the weight of each ball can be measured. Exact measurement of the attributes is possible for quantitative attributes but this is not possible for qualitative attributes considering the lack of a common measurement scale to do so.

In order to show the working principles of pairwise comparison, the weight of each ball is measured in kilograms as shown below:

$$\begin{array}{ll} w_1 = 1kg & w_3 = 4kg \\ w_2 = 2kg & w_4 = 8kg \end{array}$$

Let w_i the weight of ball i and w_j the weight of ball j.

Let N be the set of specimens, and denoted as:

$$N = \{1, 2, \ldots, n\}$$

Let a_{ij} be the weight ratio of ball i over ball j, and denoted as:

$$a_{\mathrm{ij}} = \frac{w_i}{w_j} \quad a_{\mathrm{ij}} > 0, \qquad i, j \in N \tag{2.1}$$

A *Pairwise Comparison Matrix (PCM)* is a square matrix with the order of n that is constituded of pairwise comparison values a_{ij}.

$$PCM = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1\mathrm{n}} \\ a_{21} & a_{22} & \cdots & a_{2\mathrm{n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\mathrm{n}1} & a_{\mathrm{n}2} & \cdots & a_{\mathrm{nn}} \end{pmatrix}$$

The first diagonal of a *PCM* is constituded of ones by convention. Since a_{ij} is ratio for the weight of balls i and j when i is equal to j

$$a_{\mathrm{ij}} = \frac{w_{\mathrm{i}}}{w_{\mathrm{j}}} = 1 \qquad \forall i = j, \quad i, j \in N$$

$$PCM = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1\mathrm{n}} \\ a_{21} & 1 & \cdots & a_{2\mathrm{n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\mathrm{n}1} & a_{\mathrm{n}2} & \cdots & 1 \end{pmatrix}$$

A *PCM* is a reciprocal matrix, that is symmetrical a_{ij} values are reciprocal of each other. Therefor product of symmetrical a_{ij} values will equal to 1.

$$a_{\mathrm{ij}} = \frac{w_{\mathrm{i}}}{w_{\mathrm{j}}}, \quad a_{\mathrm{ji}} = \frac{w_{\mathrm{j}}}{w_{\mathrm{i}}}; \qquad a_{\mathrm{ij}} \times a_{\mathrm{ji}} = \frac{w_{\mathrm{i}}}{w_{\mathrm{j}}} \times \frac{w_{\mathrm{j}}}{w_{\mathrm{i}}} = \frac{w_{\mathrm{i}}}{w_{\mathrm{i}}} = 1$$

$$PCM = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ {}^{1}/_{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ {}^{1}/_{a_{1n}} & {}^{1}/_{a_{2n}} & \cdots & 1 \end{pmatrix}$$

A value of a_{ij} can be derived using two other values of a_{ik} and a_{kj}. Such values are denoted as *indirect judgments*, and are used for consistency analysis.

$$a_{ik} = \frac{w_i}{w_k}, \quad a_{kj} = \frac{w_k}{w_j}$$
$$a_{ij} = a_{ik} \times a_{kj}$$
$$a_{ij} = \frac{w_i}{w_k} \times \frac{w_k}{w_j}$$
$$a_{ij} = \frac{w_i}{w_j}$$

As a result of reciprocal property of *PCM*s the neccessary and sufficient number of a_{ij} values is $\frac{n(n-1)}{2}$ to create a *complete PCM*. Therefor completing only upper or lower triangle of the first diagonal is enough to achieve a complete *PCM*. Graph representation of *Complete Pairwise Comparison Matrix (CPCM)* with the order of n is shown in Figure 2.1.

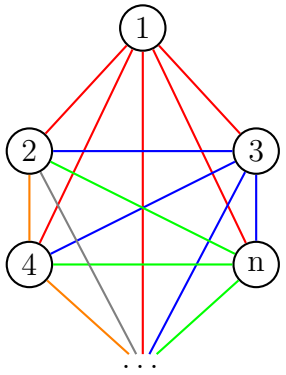


Figure 2.1: Graph representation of a *CPCM* with the order of n

A *CPCM* is a *pairwise comparison matrix* such that every specimen that belongs to the *PCM* is compared against each and every other specimen in the same set. In graph theory terms, every node is adjacent to each and every other node therefor letting a *CPCM* to form a complete graph. The completeness attribute (See Appx. A) proves that graph of a *CPCM* is also connected. Since a value a_{ij} can be derived through values a_{ik} and a_{kj}, and Harker's study [26] proving that *Incomplete Pairwise Comparison Matrices (ICPM)* can be used as the same manner as complete ones, as long as the graph of such matrix is connected.

An *IPCM* is a *pairwise comparison matrix* such that the first $(n-1)$ number of a_{ij} values are gathered from an observer, and remaining values are derived indirectly. Graph representation of an *IPCM* with the order of n is shown in Figure 2.2.

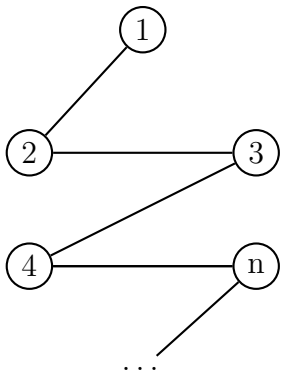


Figure 2.2: Graph representation of an *IPCM* with the order of n

An indirect comparison value can be derived by using two or more relevant comparison values:

$$\begin{aligned} a_{ij} &= a_{ik} \times a_{kj}, && i,j,k \in N && \text{1 step indirect judgment} \\ a_{ij} &= a_{ik} \times a_{kt} \times a_{tj}, && i,j,k,t \in N && \text{2 step indirect judgment} \\ &\vdots \end{aligned}$$

2.1.1 Numerical Examples

In Section 2.1 working principles of *pairwise comparison method* were given. In this section, to clarify these working principles the numerical example (the set of balls) given in Section 2.1 is further investigated.

The set of specimens were defined as a set of balls with different amounts of weight, and are exactly weighed as below:

$$B = \{i| \quad i \in \{b_1, b_2, b_3, b_4\}\}$$

$$\begin{aligned} w_1 &= 1kg \quad w_3 = 4kg \\ w_2 &= 2kg \quad w_4 = 8kg \end{aligned}$$

In order to create the *PCM* of the given example, first the specimens should be placed as row and column of the matrix:

$$PCM = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \end{array}$$

First diagonal of *PCM* is derived as follows:

$$a_{11} = \frac{w_1}{w_1}, \quad a_{11} = \frac{1kg}{1kg}, \quad a_{11} = 1$$

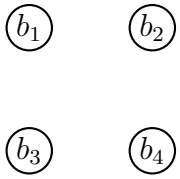


Figure 2.3: Initial PCM and graph representation

$$a_{22} = \frac{w_2}{w_2}, \quad a_{22} = \frac{2kg}{2kg}, \quad a_{22} = 1$$
$$a_{33} = \frac{w_3}{w_3}, \quad a_{33} = \frac{4kg}{4kg}, \quad a_{33} = 1$$
$$a_{44} = \frac{w_4}{w_4}, \quad a_{44} = \frac{8kg}{8kg}, \quad a_{44} = 1$$

These values are placed in the PCM as follows:

$$PCM = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{array}$$

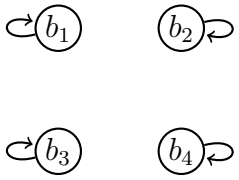


Figure 2.4: Placement of diagonal values

The first row of the PCM is derived as follows:

$$a_{12} = \frac{w_1}{w_2}, \quad a_{12} = \frac{1kg}{2kg}, \quad a_{12} = \frac{1}{2}$$
$$a_{13} = \frac{w_1}{w_3}, \quad a_{13} = \frac{1kg}{4kg}, \quad a_{13} = \frac{1}{4}$$
$$a_{14} = \frac{w_1}{w_4}, \quad a_{14} = \frac{1kg}{8kg}, \quad a_{14} = \frac{1}{8}$$

Derived a_{ij} values above constitutes the first row of the PCM. Due to reciprocal property of PCM the first column is constituted from the inversion derived a_{ij} values.

$$a_{21} = \frac{w_2}{w_1}, \quad a_{21} = \frac{2kg}{1kg}, \quad a_{21} = 2$$

$$a_{31} = \frac{w_3}{w_1}, \quad a_{31} = \frac{4kg}{1kg}, \quad a_{31} = 4$$
$$a_{41} = \frac{w_4}{w_1}, \quad a_{41} = \frac{8kg}{1kg}, \quad a_{41} = 8$$

These values are placed in the PCM as follows:

$$PCM = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 1 & 1/2 & 1/4 & 1/8 \\ 2 & 1 & & \\ 4 & & 1 & \\ 8 & & & 1 \end{pmatrix} \end{array}$$

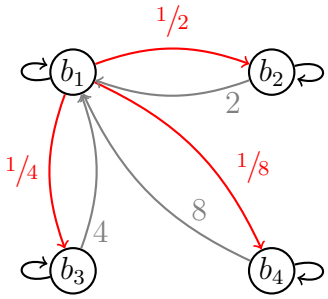


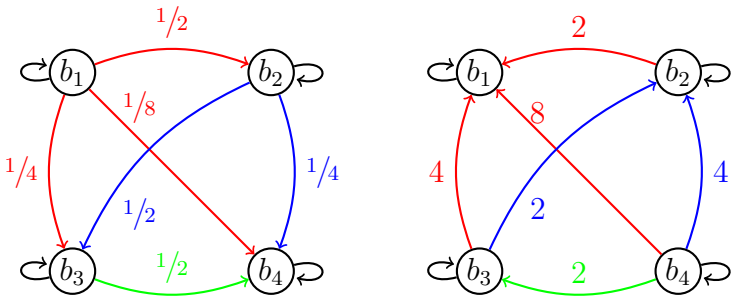
Figure 2.5: Graph representation of first row and columns of PCM

In order to complete the PCM remaining row and column values are derived smilarly, and shown in the below $CPCM$:

$$CPCM = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 1 & 1/2 & 1/4 & 1/8 \\ 2 & 1 & 1/2 & 1/4 \\ 4 & 2 & 1 & 1/2 \\ 8 & 4 & 2 & 1 \end{pmatrix} \end{array}$$

In Section 2.1, it was stated that $(n-1)$ pairwise comparison is enough to form a PCM. There are different first $(n-1)$ pairwise comparison can be chosen accordingly different rules. Wedley[27] and Ishizaka[28] states different rules on chosing $n-1$ *pairwise comparisons*, explained further in Section 1.1. In this study a number of *pairwise comparisons* are chosen randomly in order to constitude $IPCM$s.

In the latter part of this study, it is shown that $(n-1)$ PCs are redundant to reach a conclusive result. Therefor this study utilizes randomly chosen m number of direct PCs. Every PCM with less than $\frac{n(n-1)}{2}$ number of direct PCs are defined as *Incomplete Pairwise Comparison Matrix (IPCM)*. For illustrative reasons only the case of $m = (n-1)$ $IPCM$ is given in this section.



(a) Upper triangle (b) Lower triangle

Figure 2.6: Graph representations of the *CPCM*

The numerical example given earlier in this section is to demonsrate the constitution of *CPCM*s, in order to demonsrate how *IPCM*s are constituted the same set of balls are used. The initial *PCM* is created as the same manner as before by placing specimens in matrix's row and column, and the first diagonal entries are also placed as ones.

$$PCM = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{array}$$

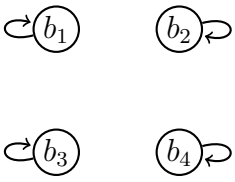
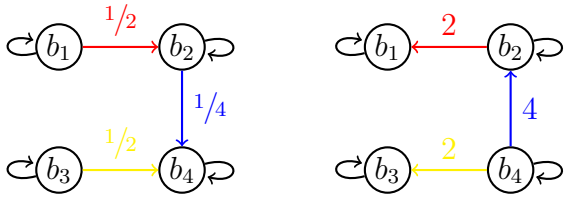


Figure 2.7: Initial PCM and graph representation

Let a_{12}, a_{34} and a_{24} be the first three *pairwise comparisons*, in this case $n-1$ equals to 3 since $n = 4$. Randomly chosen a_{ij} values and their symmetrical values $\left(a_{21}, a_{43}, a_{42}\right)$ are placed in the *PCM* in order to constitude *IPCM* as follows:

$$IPCM = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 1 & 1/2 & & \\ 2 & 1 & & 1/4 \\ & & 1 & 1/2 \\ & 4 & 2 & 1 \end{pmatrix} \end{array}$$

Due to reciprocality property of *PCM*s deriving the missing values in one triangle is enough.The missing *pairwise comparison* values in the upper triangle are $\left(a_{13}, a_{14}, a_{23}\right)$.



(a) Upper triangle (b) Lower triangle

Figure 2.8: The *IPCM* and graph representation

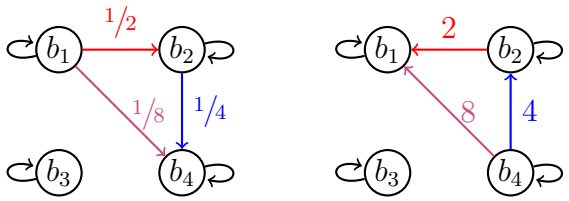
Calculation process for the missing values a_{14} and a_{41} are demonstrated as follows[3]:

$$a_{ij} = a_{ik} \times a_{kj}, \quad i, j, k \in N$$

$$a_{14} = a_{12} \times a_{24}$$

$$a_{14} = \frac{1}{2} \times \frac{1}{4}$$

$$a_{14} = \frac{1}{8}, \quad a_{41} = 8$$



(a) Upper triangle (b) Lower triangle

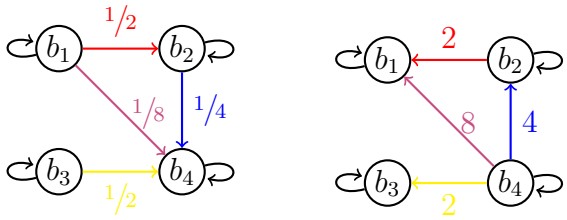
Figure 2.9: Graph representation of calculation process for a_{14} and a_{41}

These values are placed in the *IPCM* as follows:

$$IPCM = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 1 & 1/2 & & 1/4 \\ 2 & 1 & & 1/4 \\ & & 1 & 1/2 \\ 4 & 4 & 2 & 1 \end{pmatrix} \end{array}$$

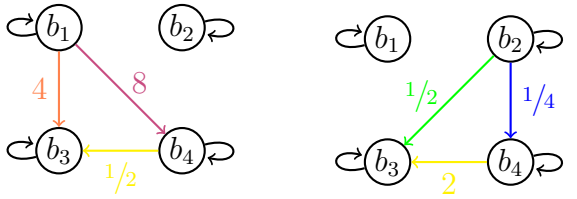
Remaining missing values are also calculated smilarly and their graph representations are shown in Figure 2.11. Some missing values can be derived by using both upper and lower triangle values, in this case for both a_{13} and a_{23}.

[3]A missing judgment is derived by the geometrical mean of every path possible. When m is equal to $(n-1)$ there is only one path for each missing judgment. While m increases so does the number of paths for each missing judgment.



(a) Upper triangle (b) Lower triangle

Figure 2.10: Graph of *IPCM*

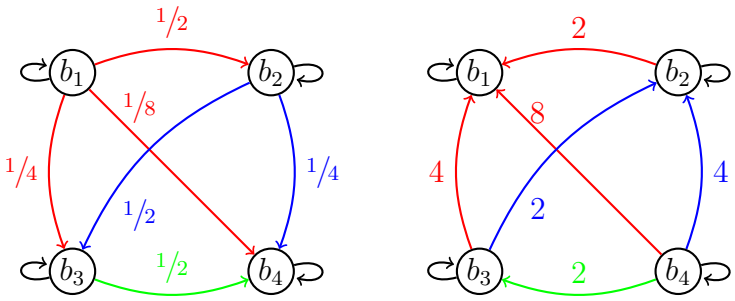


(a) Calculation of a_{13} (b) Calculation of a_{23}

Figure 2.11: Graph representation for calculations of missing values

When all the missing *pairwise comparisons* are derived, and placed in the *PCM* a *CPCM* is constituted with the help of indirect judments. The *CPCM* that is constituted by indirect judgments and the graph representation of final *PCM* are shown below.

$$CPCM = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{pmatrix} 1 & 1/2 & 1/4 & 1/8 \\ 2 & 1 & 1/2 & 1/4 \\ 4 & 2 & 1 & 1/2 \\ 8 & 4 & 2 & 1 \end{pmatrix} \end{array}$$



(a) Upper triangle (b) Lower triangle

Figure 2.12: Graph representations of the *CPCM*

Both *CPCM*s acquired in this section is identical because the exact weights of the balls were measurable. However this is not possible when the exact weights are immeasurable. In such cases both *CPCM* and *IPCM* may differentiate from the real *PCM*.

2.2 Consistency

Consistency is essential in order to be certain that evaluation process returns a valid result. Since the main objective of a decision aid tool is to guide decision makers to a realistic choice, consistency of the raw data is directly related to the validity of the result. AHP resorts to experts' opinion in order to gain insight into the subject. As a conclusion, judgment matrices are established for mathematical evaluation. However due to human factor in this process, judgment matrices are inclined to inconsistency.

In this section reasons of inconsistency, *measurement error* and *transivity violations*, are explored.

2.2.1 Measurement Error

In Section 2.1 it was stated that *pairwise comparisons* are used in order to estimate the weight of each specimen in a judgment matrix. A judgment matrix A is constituted of a_{ij} values which are weight ratio for two specimens i and j. It was also stated that the exact weight of a specimen is unknown, and is estimated by the use of a_{ij} values. An error in measurement is denoted as $\varepsilon_{\mathrm{ij}}$, and it defines the variation from the real weight real value.

$$a_{\mathrm{ij}} = \left(\frac{w_i}{w_j}\right)\varepsilon_{\mathrm{ij}}, \qquad 0 < \varepsilon < 1 \quad \forall i, j \in N$$

When $\varepsilon_{\mathrm{ij}}$ is equal to 1, it means the estimated ratio a_{ij} reflects the real ratio exactly. Some level of error is expected while gathering information.

The maximum eigenvalue of a *PCM* equals to its order (n). Any differentiation in error causes an increase in the maximum eigenvalue of the *PCM*, compromizing the consistency of judgment matrix.

2.2.2 Violations of Transitivity

Ali et al. [30] state that a violation in a ranking R of a tournament T occurs whenever for some pair of (i, j), i is ranked higher than j in R, while i is defeated by j in T. Similar to tournaments, a violation occurs in a *PCM* when at least three specimens i, j and k create a cycle in final relative ordering.

Let w_i, w_j and w_k be weights of specimens i, j and k respectively. When comparing three specimens, the first two *pairwise comparison* defines the nature of the third *pairwise comparison*. For example when $w_i > w_j$ and $w_j > w_k$ then w_i is expected to be larger than w_k. However when w_k is stated to be larger than w_i, a cycle is created.

Figure 2.13 illustrates interactions between specimens i, j and k in cases of transitivity and intransitivity. Transitivity violations causes logical disturbance in the *PCM* leading to inconsistency.



(a) Transitivity (b) Intransitivity

Figure 2.13: Graph representations for transitivity and intransitivity

2.3 Survey

AHP consults to expert's opinion in order to gain insight on the subject of the decision problem. Expert's opinion shapes the interactions between hierarchy. Since without information on the subject *AHP* does not have data to evaluate, gathering data from experts is the first step of the *AHP*.

In order to gather information from expertsa survey suitable for pairwise comparison method should be prepared. The survey should consists of questions for each pairwise comparison matrix in hierarchy, and each pair of specimen should be in the survey to eliminate any missing information. If missing information is acceptable the survey should be designed accordingly. A scale suitable for the decision problem should be identified.

In order to gather viable information experts should be notified how to fill out the survey. If there exists a specimen which can cause misunderstandings or misguided information it should be explained cleary to achieve consistent matrices.

2.4 Evaluation Process

Once every *PCM* that belongs to the decision hierarchy is constructed by the decision maker's judgments, evaluation process starts. In this part of the *AHP* weight vector of each *PCM* is derived, each *PCM*'s consistency is analyzed, and if the consistency of all the *PCM*s are within in the expected intervals each alternative's score is derived.

Weight vector of a given *PCM* is generally derived by *Eigenvector Method (EVM)*. This method utilizes right eigenvector of the given *PCM*. As a result of this step; first the weight vector is achieved, and second the maximum eigenvalue ($\lambda_{\max}$) is aquired. If the consistency analysis (Section 2.4.2) shows that the given *PCM* is acceptable, afore mentioned weight vector can be utilized. Derived weight vectors are used to calculate score points of alternatives, showing how much each alternative satisfies the goal.

2.4.1 Calculation of Weight Vector

Dong et al.'s [25] study states that different prioritization methods gives smilar results when the same scale is used. Therefor, this study utilizes *EVM* due to its ease of use.

In this section *EVM* will be explained as it is the only method utilized for calculation of the weight vectors.

EVM is a result of the right eigenvector problem and the calculation process is illustrated as below.

$$A = [a_{ij}] \qquad a_{ij} > 0 \qquad i, j \in N$$
$$A \times x = x \times \lambda_{max}$$

$$A = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{array} \begin{array}{c} \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \end{array} \\ \begin{pmatrix} 1 & 1/5 & 1 & 1 & 9 \\ 5 & 1 & 1/2 & 1/7 & 1/6 \\ 1 & 2 & 1 & 9 & 1/9 \\ 1 & 7 & 1/9 & 1 & 1 \\ 1/9 & 6 & 9 & 1 & 1 \end{pmatrix} \end{array}$$

$$x = \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{array} \begin{pmatrix} 0,2153 \\ 0,1089 \\ 0,2199 \\ 0,1820 \\ 0,2739 \end{pmatrix}$$

$$A \times x = \begin{pmatrix} 3,1042 \\ 1,3670 \\ 2,3217 \\ 1,4578 \\ 3,1120 \end{pmatrix}$$

$$\lambda_{max} = \begin{pmatrix} 14,4183 \\ 12,5555 \\ 10,5587 \\ 8,0083 \\ 11,3614 \end{pmatrix}$$

$$\lambda_{max} = \frac{14,4183 + 12,5555 + 10,5587 + 8,0083 + 11,3614}{5}$$
$$\lambda_{max} = 11,3804$$
$$CI = 1,5951$$

2.4.2 Consistency Analysis

A *PCM* is such a matrix that shows all the proporties of a *Positive Reciprocal Matrix (PRM)*. The maximum eigenvalue of a *PRM* is equal to order of the matrix, (n) incase of total consistency. The maximum eigenvalue will increase with the loss of consistency. Consistency of a *PRM* is measured as *Consistency Index*; the distance from the maximum eigenvalue $(\lambda_{\max})$ of the given *PRM* incase of total consistency.

$$CI = \frac{\lambda_{\max} - n}{n - 1} \tag{2.2}$$

CI of a given *PRM* equals to zero incase of total consistency.

$$\begin{aligned} \lambda_{\max} &= n \\ CI &= \frac{\lambda_{\max} - n}{n - 1} \\ &= \frac{n - n}{n - 1} \\ &= \frac{0}{n - 1} \\ CI &= 0 \end{aligned}$$

However *consistency index* alone is not enough to accept or reject a *PCM* as a consistent matrix. In order to accept or reject a *PCM*, *consistency index* of should be compared against a benchmark. There are two methods to do so as stated in Section 1.1.

Random Index (RI) is the expected value of *consistency index* for a sample of randomly created *PCM*s. A *random index* value is discriminately derived for each order of matrix. In deterministic approach *consistency index* of a given *PCM* is compared against the relative*random index* as in Equation 2.3.

$$CR = \frac{CI}{RI} \tag{2.3}$$

By comparing *consistency index* against *random index*, a value for evaluating the consistency defined as *consistency rate* is derived. Deterministic approach requires a threshold value in order to evaluate *consistency rate*. Ten percent (0.10) threshold is widely accepted but it can differ as regard to order of the matrix or the type of decision problem.

Distinctively from deterministic approach, probabilistic approach requires a *random index* distribution. Instead of comparing two specific value, *consistency index* is statistically hypothesis tested. When null hypothesis is not rejected given *PCM* is accepted as consistent.

$$h_0 : \mu_{\text{CI}} = \mu_{\text{RI}}$$

$$h_1 : \mu_{\text{CI}} \neq \mu_{\text{RI}}$$

The advantage of probabilistic approach is its ability to work under a confidence level that can be specified as regard to sensitivity of the decision problem. While deterministic approach can be modified as regard to sensitivity of the decision problem, it is more rigid than probabilistic approach.

CHAPTER 3

Methodology

In this section methodology of research is explained step by step. In order to test the hypothesis given in Section 1.3 and to establish random index distributions, a randomized data set (Figure 3.2) is generated with Python 2.7.6 [31, 32, 33, 34, 35].

Pairwise Comparison Matrices (*PCMs*) are utilized for gathering data in AHP, as mentioned in Section 2.1. Figure 3.1 shows a randomly generated *PCM* sample, with the order of 5. The first diagonal of a *PCM* is constituted of ones by convention. Since the opposite triangle entries are reciprocals of each other, random simulation of one chosen triangle is enough. In this study, the upper triangle entries are generated randomly, and lower triangle entries are derived accordingly. In Section **??** generation of different matrix types are given in detail.

$$A = \begin{bmatrix} 1 & 1/5 & 1 & 1 & 9 \\ 5 & 1 & 1/2 & 1/7 & 1/6 \\ 1 & 2 & 1 & 9 & 1/9 \\ 1 & 7 & 1/9 & 1 & 1 \\ 1/9 & 6 & 9 & 1 & 1 \end{bmatrix}$$

Figure 3.1: Pairwise comparison matrix A, ($n = 5$)

3.1 Random Index Study

The data used in statistical analysis is simulated. Simulated data sets can be summarized as Figure 3.2.

As in Figure 3.2, the data consists of two main types of matrices:

1. **Complete Pairwise Comparison Matrix (*CPCM*) Data Set:** A *CPCM* is a pairwise comparison matrix such that, every pairwise comparison is made by a decision maker. When simulated, every pairwise comparison is randomly generated.

2. **Incomplete Pairwise Comparison Matrix (*IPCM*) Data Set:** An *IPCM* is a pairwise comparison matrix such that, n being the order of the matrix m 3.1 number of *pairwise comparisons* are made by a decision maker. When simulated m number of *pairwise comparisons* are randomly generated, and missing *pairwise comparisons* are derived using existing ones.

$$\begin{aligned} (n-1) \leq & m < \frac{n(n-1)}{2} \\ & s(m) = \frac{n^2 - 3n + 4}{2} \end{aligned} \tag{3.1}$$

For each level of n, $s(m)$ set of 10000 matrices are generated randomly in 10 runs. Each *pairwise comparison* is generated randomly generated from the Saaty scale 3.2 [15]:

$$S = \left\{ \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 9 \right\} \tag{3.2}$$

For each simulated matrix, a *Consistency Index (CI)* value is calculated. Consistency index is calculated as in Section 2.4.2. In Section 3.1.1 generation process of random *PCM*s and *CI* calculation is given in detail.

In order to test the hypotheses given in Section 1.3 consistency index distrubitions are analyized with Python [31, 32, 33, 34, 35]. In Section 3.2, statistical analysis process are given in detail.

3.1.1 Generation of Pairwise Comparison Matrices

In this section the algorithm used for generation of *PCM*s is explained. The algorithm code is written and run in Python 2.7.6 [31, 32, 33, 34, 35].

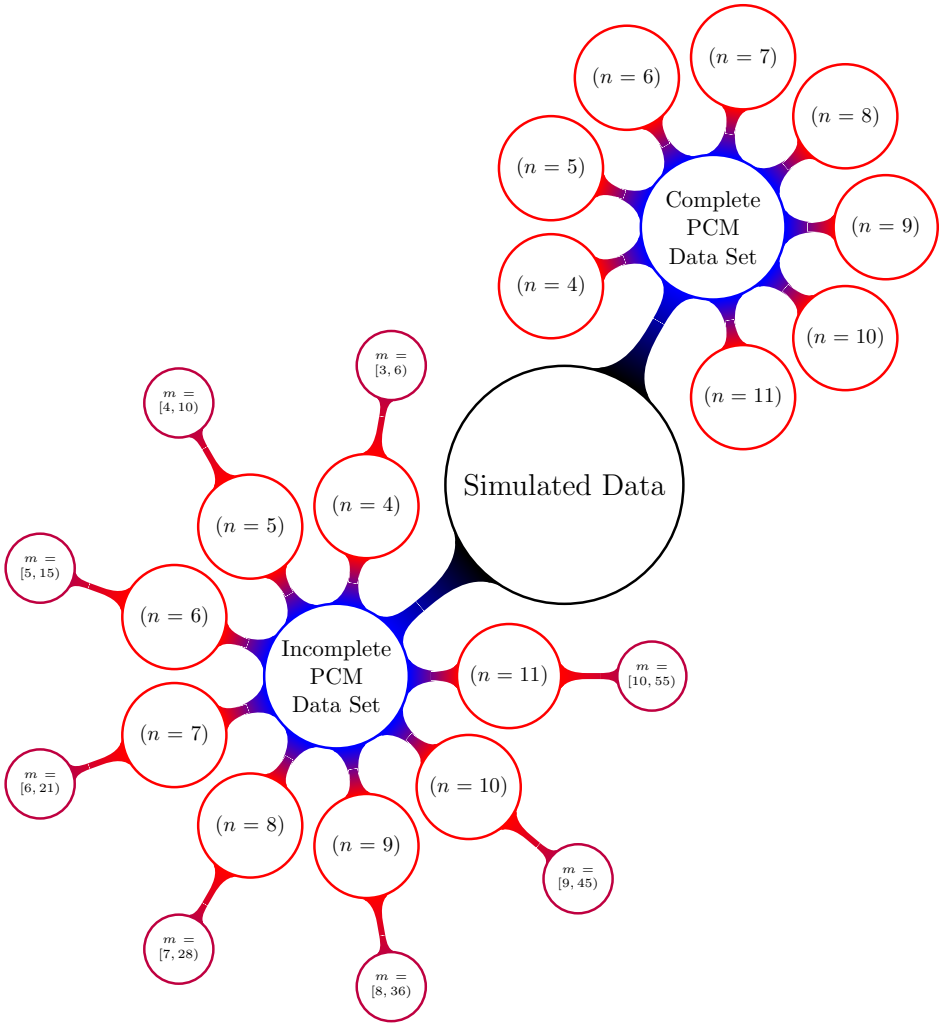


Figure 3.2: Simulated data sets friction, (n: order of matrix)

The algorithm consists of three variables 3.3:

- **r**: Number of matrices to be created,
- **n**: The order of matrices to be created and
- **m**: The number of direct *pairwise comparisons* in a matrix.

$$\begin{aligned} r \in R \quad & R = \{1, 2, \dots\} \\ n \in N \quad & N = \{4, 5, \dots, 11\} \\ m \in M \quad & M = [(n-1), \frac{n(n-1)}{2}) \end{aligned} \tag{3.3}$$

The algorithm is constituted of four parts, each marked by a counter:

- $counter_1$ is responsible for counting the number of matrices yet to be created,
- $counter_2$ is responsible for assigning the first $(n-1)$ direct *pairwise comparisons* to create a spanning tree,

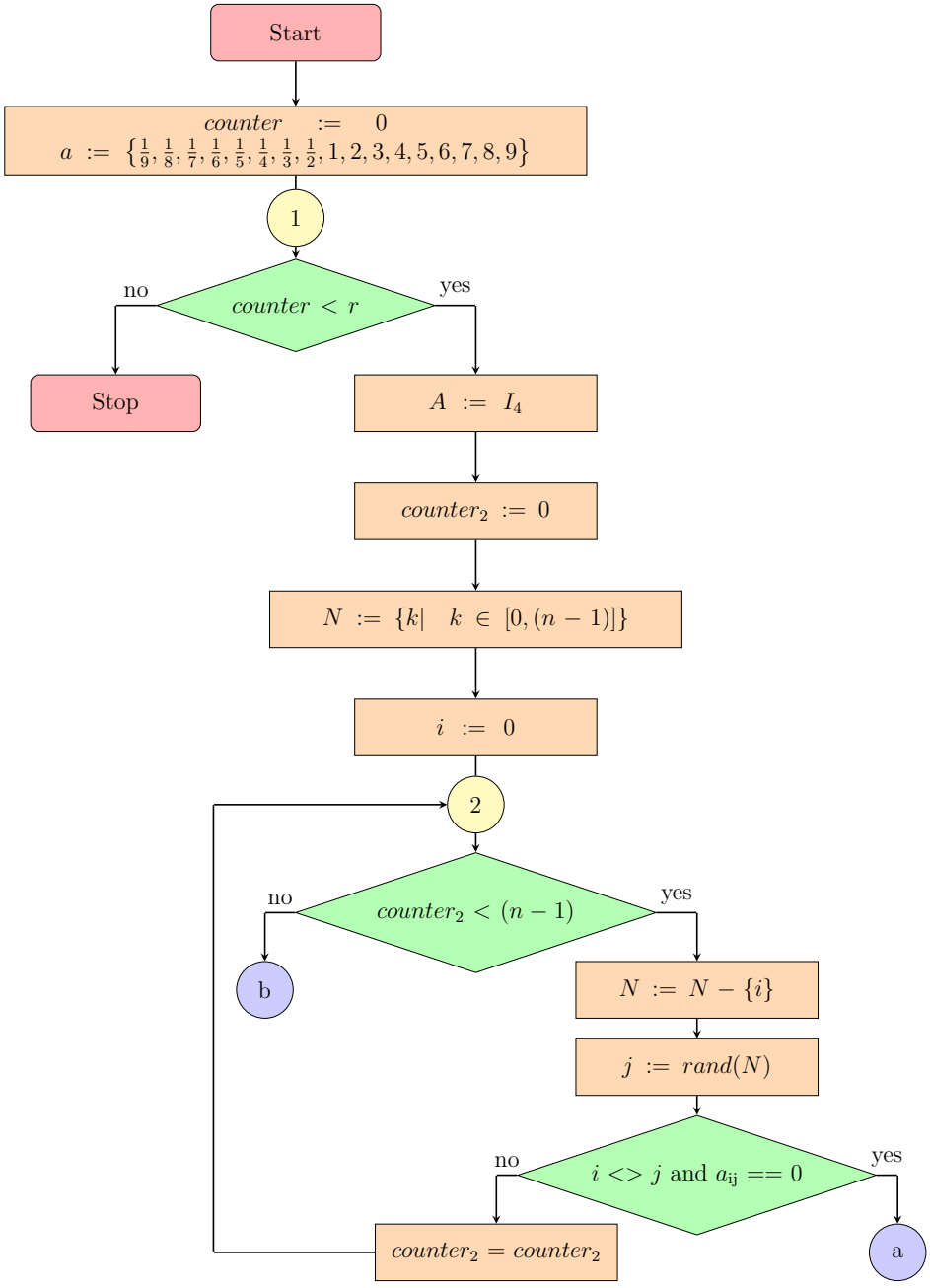


Figure 3.3: Algorithm for PCM simulation (1)

- $counter_3$ is responsible for assigning the remaining direct *pairwise comparisons* when m is larger than $(n-1)$, and
- $counter_4$ is responsible for deriving the missing judgments after m number of direct *pairwise comparison* is randomly assigned.

A is the *pairwise comparison matrix* created with the algorithm, and its rows and columns are defined as i and j [1] respectively. A is initially generated as an identity

[1]The first element of sets I and J are 0 because of counting rules of Python.

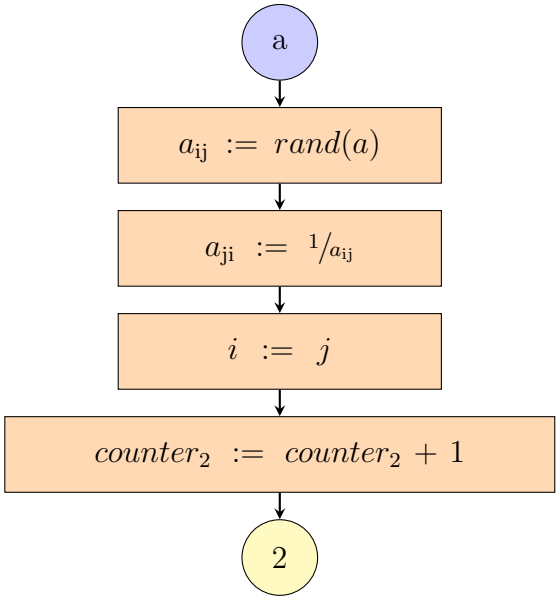


Figure 3.4: Algorithm for PCM simulation (2)

matrix with the order of n.

$$
\begin{aligned}
&A = [I]_{\text{(n-1)}} \\
&i \in I \quad I = \{0, 1, \ldots, (n-1)\} \\
&j \in J \quad J = \{0, 1, \ldots, (n-1)\}
\end{aligned}
$$

$$
A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ (n-1) \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & \ldots & (n-1) \end{array} \\
\begin{pmatrix}
1 & \ldots & \ldots & \ldots & \ldots \\
\ldots & 1 & \ldots & \ldots & \ldots \\
\ldots & \ldots & 1 & \ldots & \ldots \\
\ldots & \ldots & \ldots & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & 1
\end{pmatrix}
\end{array}
$$

The minimum necessary number of *direct pairwise comparison (m)* in order to create a *complete pairwise comparison matrix* is $(n-1)$, and the maximum number is $^{n(n-1)}/_2$ as stated in Equation 3.3. A *direct pairwise comparison* is such a comparison that is gathered from an expert. In order to simulate such a *pairwise comparison*, a random value from Saaty scale (Equation 3.2) is assigned to a randomly chosen *pairwise comparison* (a_{ij}).

The algorithm generates $(n-1)$ number of *direct pairwise comparisons* independently from variable m in order to create a spanning tree of resulting *PCM*. Generation of the spanning tree is described in Figures 3.3, 3.4, 3.5. The process can be summarized as below:

Step 1: Assign i (indice for row) as 0,

Step 2: Remove i from N (the set of row and column numbers),

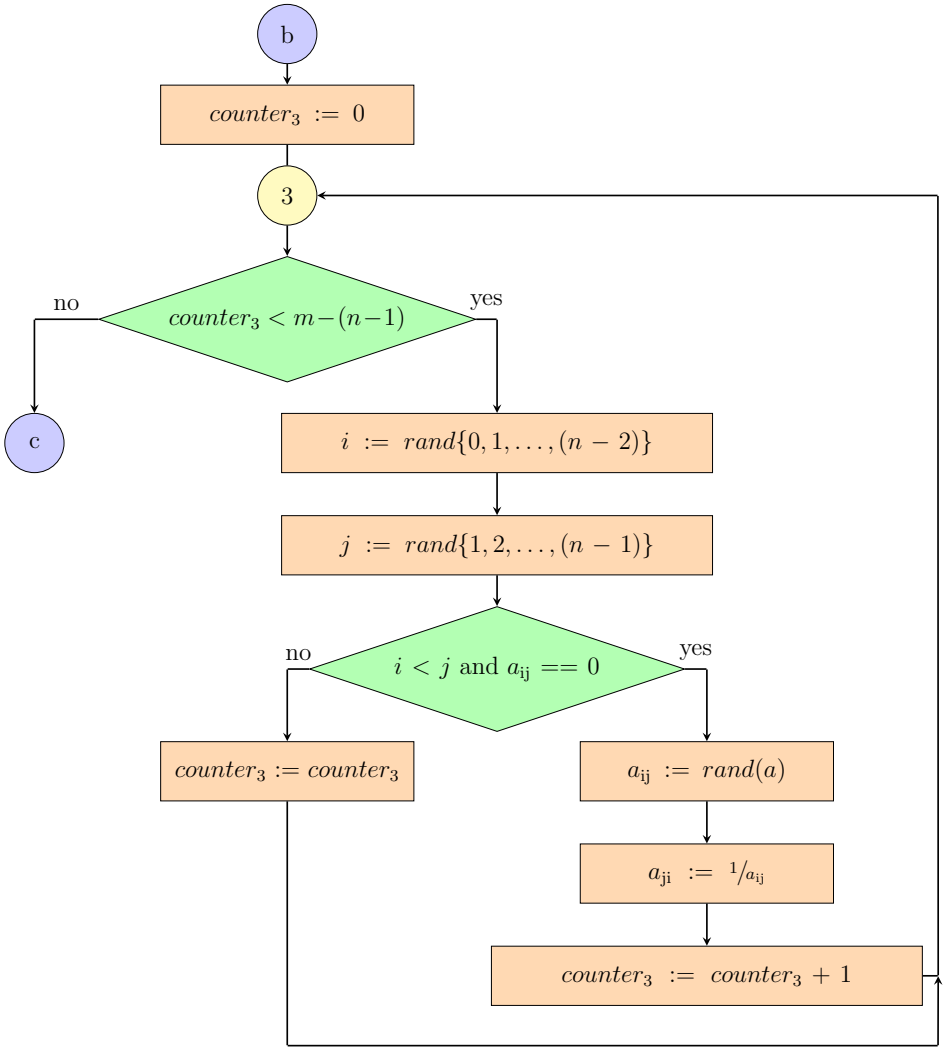


Figure 3.5: Algorithm for PCM simulation (3)

Step 3: Assign j (indice for column) from N randomly,

Step 4: While i is different from j and a_{ij} is different from 0 assign a_{ij} from a (the Saaty scale) randomly,

Step 5: Assign i as j,

Step 6: Increase $counter_2$ by 1, and return to step 2,

Step 7: Stop when $counter_2$ is larger than $(n-1)$.

After the spanning tree is generated, if m is larger than $(n-1)$, remainning *direct pairwise comparisons* are assigned smilarly. This process is described in Figure 3.5.

When m number of *direct pairwise comparisons* are randomly assigned, the algorithm proceeds to calculate the missing *pairwise comparisons*. This process is described in Figure 3.6, 3.7.

After assigning m number of *directpairwise comparisons* the algorithm proceeds to

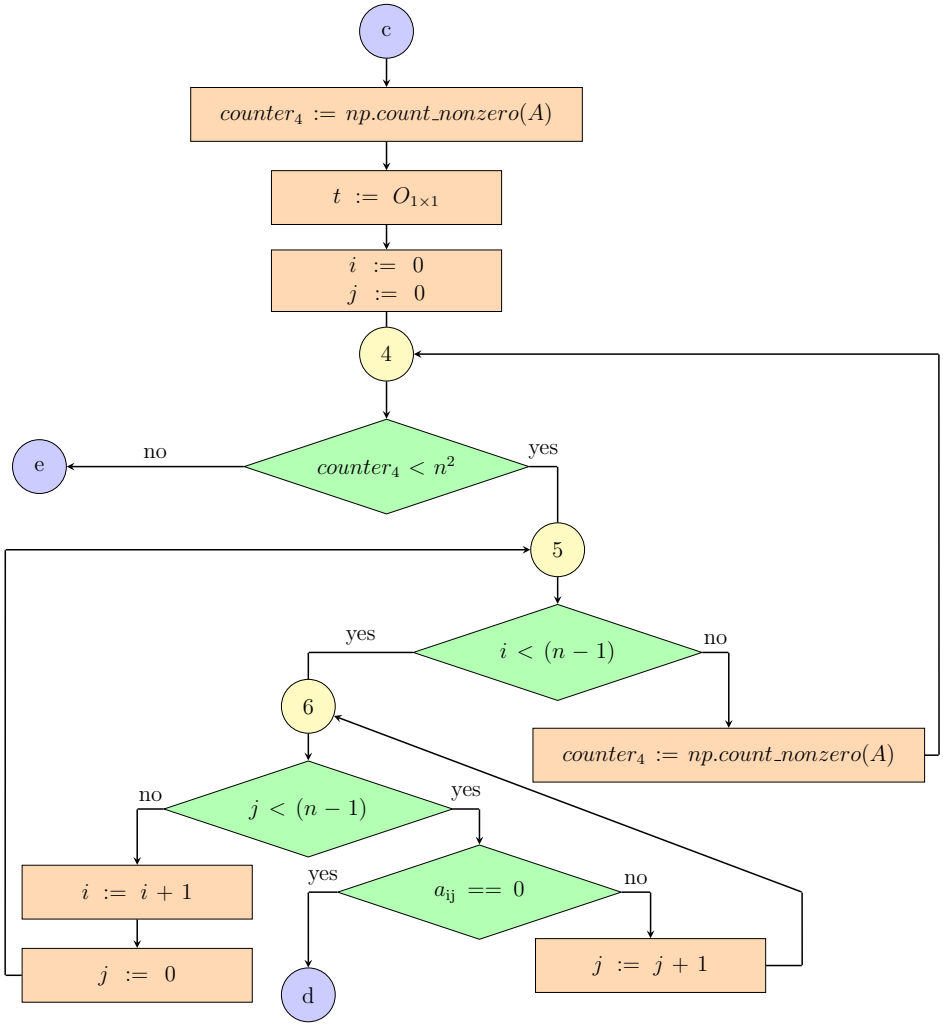


Figure 3.6: Algorithm for PCM simulation (4)

calculate missing *pairwise comparisons* if m is not equal to $^{n(n-1)}/_2$. In order to count the number of *indirect pairwise comparisons* to calculate, the $counter_4$ is assigned as the a_{ij} values not equal to zero (Equation 3.4).

$$counter_4 = np.count_nonzero(A) \tag{3.4}$$

Missing a_{ij} values (which are equal to zero) are searched by step by step within in the *PCM*. A third indice variable is (k) defined to calculate missing *pairwise comparisons*. When an a_{ij} value is changed, $counter_4$ increases by 2 because of the reciprocality. The process stops when Equation 3.4 returns 0, meaning none of the a_{ij} equals to zero. Calculation process of *indirect pairwise comparisons* is described in Figures 3.6, 3.7. The process can be summarized as follows:

Step 1: Assign $counter_4$ as the number of non-zero a_{ij} values 3.4 in the *PCM A*.

Step 2: Generate an empty array t to store *indirect pairwise comparison* results.

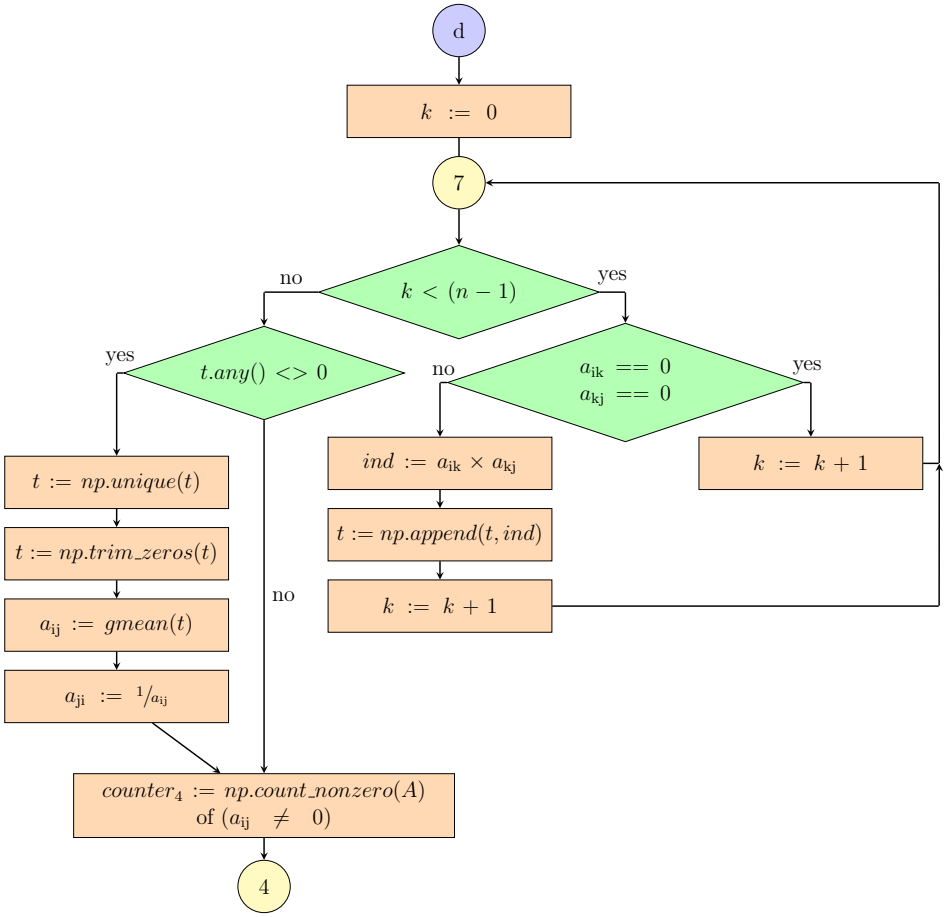


Figure 3.7: Algorithm for PCM simulation (5)

Step 3: Assign i and j as 0

Step 4: While $counter_4$ is less than n^2 continue to Step 5, otherwise proceed to calculate eigenvector.

Step 5: While i is less than $(n-1)$ if j is less than $(n-1)$ assign k as 0.

Step 6: If k is less than $(n-1)$ check values a_{ik} and a_{kj}. When k is larger than $(n-1)$ proceed to Step 10.

Step 7: If both a_{ik} and a_{kj} are equal to 0 assign variable ind as the product of a_{ik} and a_{kj} as in Equation 3.5:

$$ind := a_{ik} \times a_{kj} \tag{3.5}$$

Step 8: Append value of ind to the array t as in Equation 3.6

$$t := np.append(t, ind) \tag{3.6}$$

Step 9: Increase k by one and return to Step 6.

Step 10: If the array t consists of non-zero values proceed to Step 14.

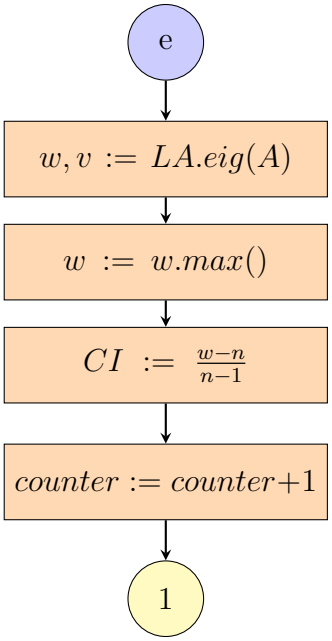


Figure 3.8: Algorithm for PCM simulation (6)

Step 11: Remove duplicate items in t, and trim zeros.

Step 12: Assign a_{ij} as the geometric mean of the items in array t.

Step 13: Assign a_{ji} as $1/a_{ij}$.

Step 14: Recalculate $counter_4$ as the number of non-zero elements in *PCM A*.

Step 15: Return to Step 4.

When all of the *pairwise comparisons* are derived, the eigenvector and the maximum eigenvalue of the *PCM* can be calculated. In order to calculate the maximum eigenvalue of the generated *PCM* *linalg* package of *NumPy* is utilized. *linalg.eig()* function calculates the eigenvector of a given *PCM* and corresponding eigenvalues. The maximum eigenvalue is stored as w to calculate *consistency index*. When the *consistency index* is calculated for the generated *PCM*, $counter_1$ is increased by one and the algorithm proceeds to generate a new *PCM*.

3.2 Analysis

In this section analysis process of randomly generated data is described for $n = 6$, for examplatory.

The data for the *PCM*s with the order of 6 is generated as given in Section 3.1.1. The analysis process of this study can be divided in five parts:

Step 1: Analysis of descriptive statistics (*Table 3.1*),

Step 2: Analysis of histograms,

Step 3: Analysis of normal probability density function,

Step 4: Calculation of confidence intervals for different α levels (*Table 3.2*),

Step 5: Analysis of normality,

Step 6: Comparison of descriptive statistics, *CI* distributions and confidence intervals between different levels of m.

As the first step of the analysis process descriptive statistics for each step of m is investigated. In Table 3.1 descriptive statistics for *PCM* with order of 6 is summarized. In Figure 3.9 simulated and normalized mean values and standard variation for *CI* are shown.

Table 3.1: Descriptive statistics for $n = 6$

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
6	0,1749	0,0000	0,5356	1,0021	0,0360	0,0997
7	0,4852	0,0215	0,8648	1,1314	0,0505	0,1064
8	0,7453	0,0860	1,0588	1,0777	0,0618	0,1174
9	0,9372	0,3797	1,1818	1,0186	0,0612	0,1200
10	1,0544	1,9836	1,2449	0,9134	0,0834	0,1423
11	1,1239	1,2975	1,2800	0,8035	0,0929	0,1588
12	1,2026	0,9238	1,3063	0,7021	0,1336	0,1928
13	1,2371	0,8621	1,3114	0,6168	0,1331	0,2102
14	1,2552	1,4976	1,2970	0,5364	0,2519	0,2648
15	1,2642	1,4030	1,2535	0,4080	0,3980	0,3427

For each level of m a histogram is drawn with the normal probability density function. Figures **??** and **??**[2] show such graphs for *PCM* with order of 6. In order to accept or reject a given *PCM* normal probability density functions can be used for probabilistic testing[3] while simulated distrubition means can be used as *Random Index* for deterministic testing. For each level of m a specific distribution and *Random Index* has been established. Therefor the normal probability distribution and the *Random Index* is chosen as regard to the number of *direct pairwise comparisons*. *CI* means and standart variance are summarized for both simulated and normalized distributions in Table 3.1.

For different levels of α, confidence intervals are investigated to determine which values of m supplies the widest confidence interval range. Three levels of α are investigated: $\alpha = 0.05, \quad \alpha = 0.10, \quad \alpha = 0.20$. Confidence intervals for each α level are summarized in Table 3.2. In order to calculate confidence intervals *IBM SPSS Statistics 20*[36] is utilized. The data is also tested for normality with Kurtosis and Skewness tests. Kurtosis and Skewness test results are summarized in Figure 3.10. In Figure 3.11 all of the normal probability density functions for $n = 6$ are shown together.

[2] $m = 5$ is omitted because every *CI* value equals to zero.

[3] *Consistency Index* calculated for a given *PCM* should be transferred accordingly to z distribution first.

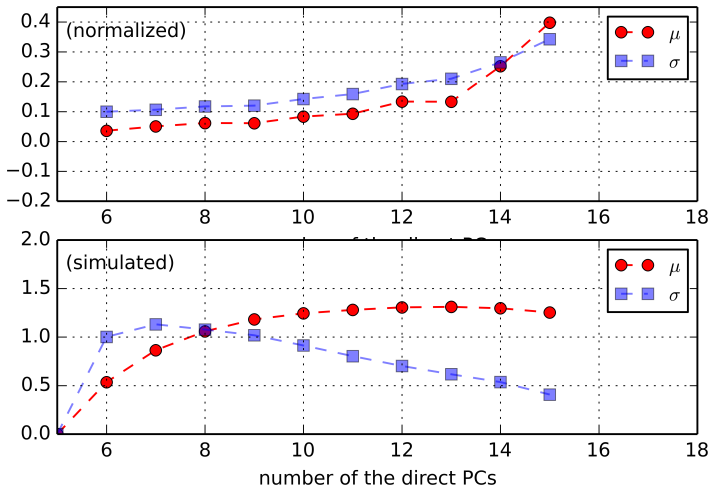


Figure 3.9: Simulated and normalized mean and standard variation for $n = 6$

Table 3.2: Confidence intervals for $n = 6$

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
6	0,516	0,5553	0,5191	0,5521	0,5228	0,5485
7	0,8426	0,887	0,8462	0,8834	0,8503	0,8793
8	1,0377	1,0799	1,0411	1,0765	1,045	1,0726
9	1,1618	1,2017	1,165	1,1985	1,1687	1,1948
10	1,227	1,2628	1,2298	1,2599	1,2332	1,2566
11	1,2643	1,2958	1,2668	1,2933	1,2697	1,2903
12	1,2925	1,32	1,2947	1,3178	1,2973	1,3153
13	1,2993	1,3235	1,3013	1,3216	1,3035	1,3193
14	1,2865	1,3075	1,2882	1,3058	1,2901	1,3039
15	1,2455	1,2615	1,2467	1,2602	1,2482	1,2587

The results of analysis process can be summarized as follows:

1. Mean *CI* value increases rapidly within the first four levels (m equals to 5 through 9) and starts to stabilize in the fifth level.
2. Mean *CI* value starts to decrease after nineth $m = 11$ level.
3. Mean *CI* value for m equals to 15 (*the complete pairwise comparison matrix* instance) converges to when m equals to 10.
4. The standart variation for the simulated data reaches its peak value for m equals 7. The following largest standart variations belongs to m equals to 8 and 9. The standart varitaion starts to decrese after third $m = 7$ level.
5. While the number of *direct pairwise comparisons* increases the corresponding distribution converges to a normal distribution (*Figure 3.10.*)

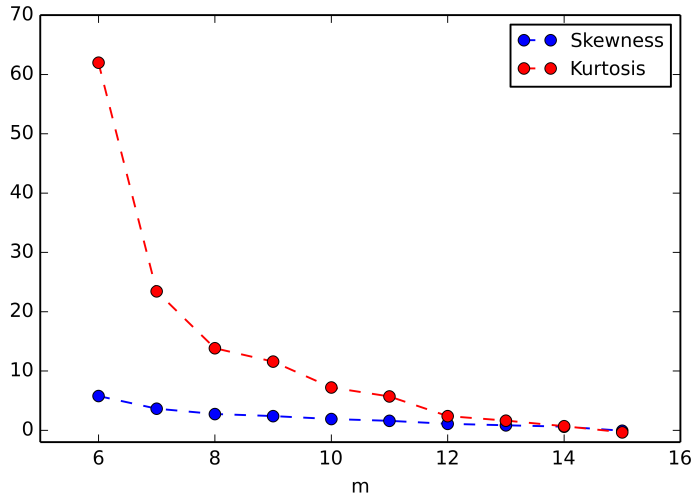


Figure 3.10: Kurtosis and Skewness test results for $n = 6$

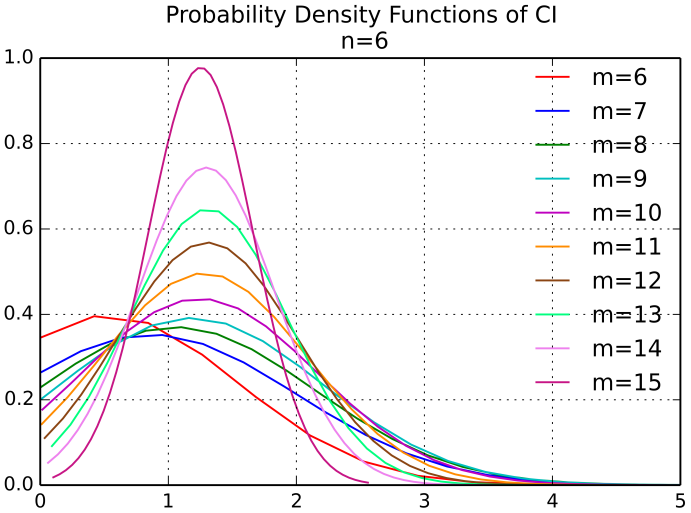


Figure 3.11: Normal probability density functions for $n = 6$

6. Confidence interval for the simulated data reaches its widest range for m equals 7 for each level of α. The second and third widest confidence interval ranges belong to m equals to 8 and 9. Confience interval range starts to decrease after the sixth ($m = 10$) level.

The analysis results suggests that it is more likely to create a consistent *PCM* when the number of *direct pairwise comparisons* is 7 through 10. Since the confidence interval is narrower when the number of *direct pairwise comparisons* is larger than 10 for *PCM*s with the order of 6. Therefor in order to create consistent *PCM*s with the order of 6, suggested number of *direct pairwise comparisons* is $7, 8$ and 9. More *direct pairwise comparisons* can be chosen however accepting such a *PCM* would be harder,

statistically.

In conclusion for *PCM*s with the order of 6, the number of *direct pairwise comparisons* is suggested to be $7, 8$ or 9. If necessary more *direct pairwise comparisons* can be utilized. The normal probability density distribution or the *Random Index* shoudl be chosen accordingly to be able to evaluate the consistency level of the given *PCM*, correctly.

3.3 Results

In this section analysis results are given for all simulated *PCM*s with the orders of 4 through 11. The preliminary results are given as below:

- Mean *consistency index* for each level of the direct pairwise comparisons are diifferent.
- The maximum mean *consistency index* is reached before the the number of direct pairwise comparisons reaches its maximum.
- The distribution of *consistency index* converges to a normal distribution while the number of direct pairwise comparisons increases.
- While the number of direct pairwise comparisons increase, *consistency index* of a given *PCM* converges to the mean *consistency index* of the distribution.
- A *confidence interval* distribution for a given *PCM* with the order of n and with the m number of direct pairwise comparisons, shows smilar confidence interval width for different levels of confidence levels.

Random Index Study is conducted for *PCM*s with the order (n) of 4 through 11. For each level of n, $\frac{(n-1)\times(n-1)}{2}$ number of *RPCM* sets are generated. *Random Index* values are given in Appendix C.

3.3.1 PCMs with the order of 4

Table 3.3 shows mean *consistency index* values for *PCM*s with the order of 4. Since there are only three levels of m[4] a limitation for m is not investigated. *PCM*s with the order of 4 are included in this study in order to derive *random index* values for *PCM*s with missing judgments.

Table 3.3: Descriptive statistics for $n = 4$

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
4	0,1944	0,0515	0,4933	0,7382	0,1238	0,1924
5	0,4816	0,1495	0,7595	0,7746	0,1561	0,1941
6	0,7391	0,2906	0,8784	0,6256	0,3052	0,2290

[4]The number of direct pairwise comparisons

Figure 3.12 shows mean *consistency index* and its standart variation for each level of m. Table 3.4 shows confidence intervals for each level of m for $\alpha = 0.05$, $\alpha = 0.1$ and $\alpha = 0.2$.

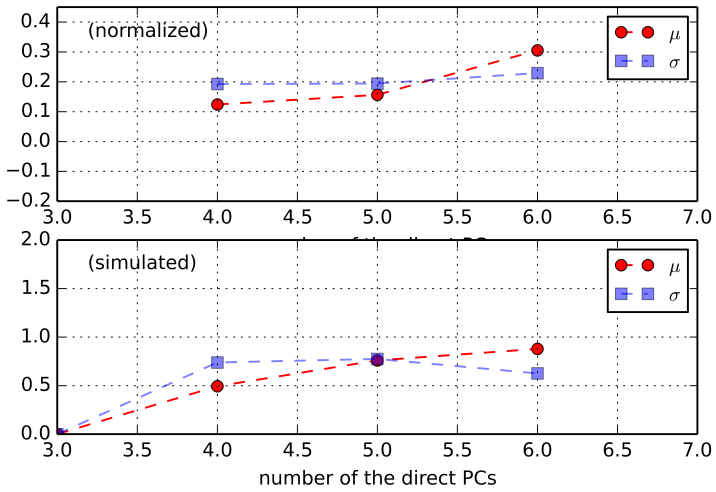


Figure 3.12: Simulated and normalized mean and standard variation for $n = 4$

Table 3.4: Confidence intervals for $n = 4$

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
4	0,4788	0,5077	0,4811	0,5054	0,4838	0,5017
5	0,7444	0,7747	0,7468	0,7723	0,7496	0,7695
6	0,8661	0,8907	0,8681	0,8887	0,8704	0,8864

Figure 3.13 shows Skewness and Kurtosis test results for each level of m. While the number of direct pairwise comparisons increase the distribution converges to a normal distribution.

Figure 3.14 shows probability density functions for each level of m. Results of skewness and kurtosis tests are supported by probability density function curves.

3.3.2 PCMs with the order of 5

Table 3.5 shows descriptive statistics for *PCM*s with the order of 5 for each level of m. Mean consistency index value increase rapidly within the first three level of m, and starts to stabilize. Mean consistency index reaches its maximum value for m equals 9. Mean consistency index value for m equals 10 converges to when m equals to 9. The standard variation for mean consistency index reaches its maximum when m equals to 6, and reaches its minimum value when m equals to 10. Figure 3.15 summarizes how consistency index behaves between different levels of m.

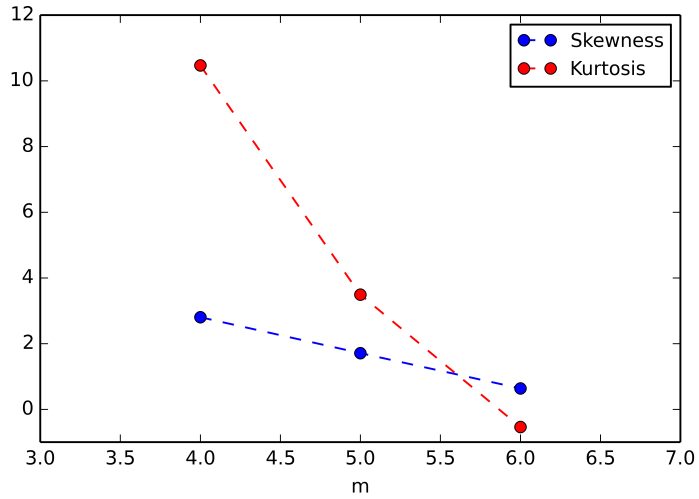


Figure 3.13: Kurtosis and Skewness test results for $n = 4$

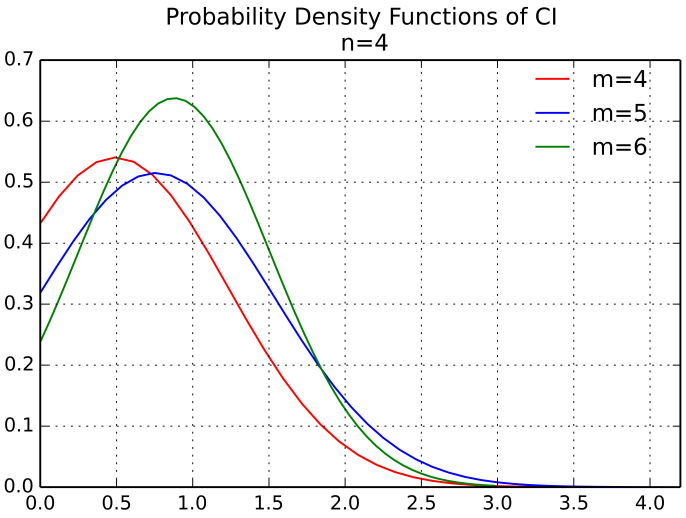


Figure 3.14: Normal Probability Density Functions of CI $n = 4$

Table 3.5: Descriptive statistics for $n = 5$

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
5	0,1827	0,0000	0,5101	0,8616	0,0675	0,1411
6	0,4742	0,0377	0,7828	0,9046	0,0930	0,1523
7	0,7188	0,1117	0,9633	0,8635	0,1021	0,1598
8	0,9053	0,6935	1,0609	0,7669	0,1410	0,1869
9	1,0365	0,9659	1,1184	0,6612	0,2196	0,2209
10	1,1117	0,8331	1,1119	0,5065	0,3615	0,2708

Table 3.6: Confidence intervals for $n = 5$

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
5	0,4932	0,5270	0,4959	0,5243	0,4991	0,5212
6	0,7650	0,8005	0,7679	0,7977	0,7712	0,7944
7	0,9464	0,9803	0,9491	0,9775	0,9523	0,9744
8	1,0459	1,0759	1,0483	1,0735	1,0511	1,0707
9	1,1054	1,1313	1,1075	1,1292	1,1099	1,1268
10	1,1020	1,1218	1,1036	1,1202	1,1054	1,1184

Table 3.6 shows confidence intervals for each level of m for $\alpha = 0.05$, $\alpha = 0.1$ and $\alpha = 0.2$. Confidence intervals for each level of α reaches its maximum width for m equals 6. The second and third maximum width occurs for m equals to 7 and 5 respectively.

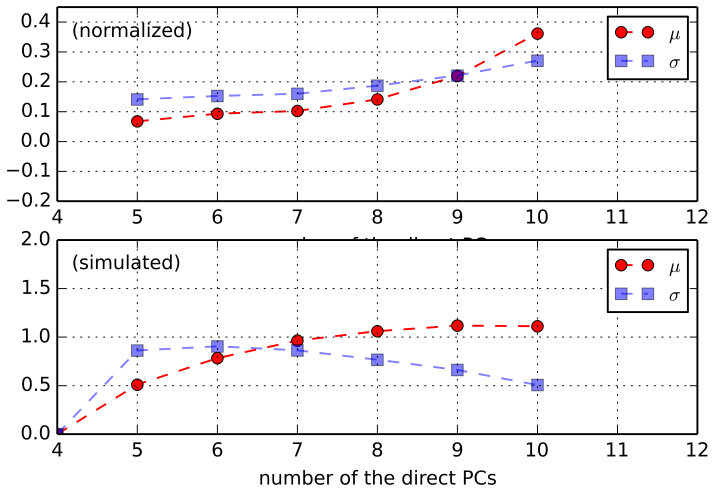


Figure 3.15: Simulated and normalized mean and standard variation for $n = 5$

Figure 3.16 shows skewness and kurtosis test results of consistency index distributions for each level of m. Distributions converges to a normal distribution while m increases.

Figure 3.17 shows probability density function curves for each level of m. Skewness and kurtosis test results are supported by probability density function curves.

As a result of analysis, suggested number of direct pairwise comparisons for *PCM*s with the order of 5 is $5, 6$ or 7. Corresponding *random index* values for each level of m is given in Appendix C in Table C.2.

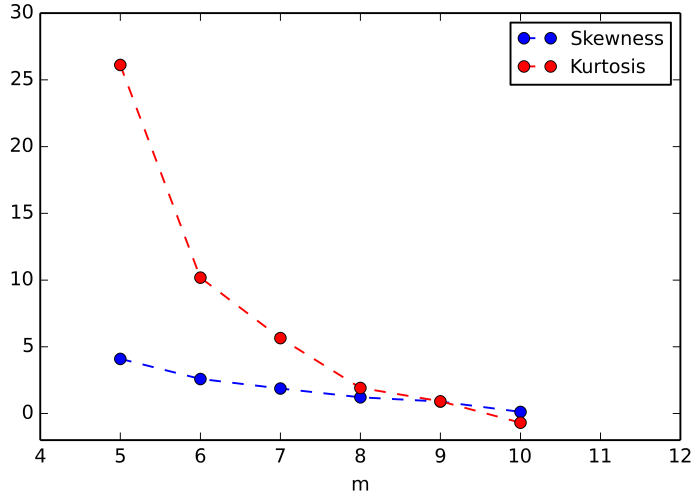


Figure 3.16: Kurtosis and Skewness test results for $n = 5$

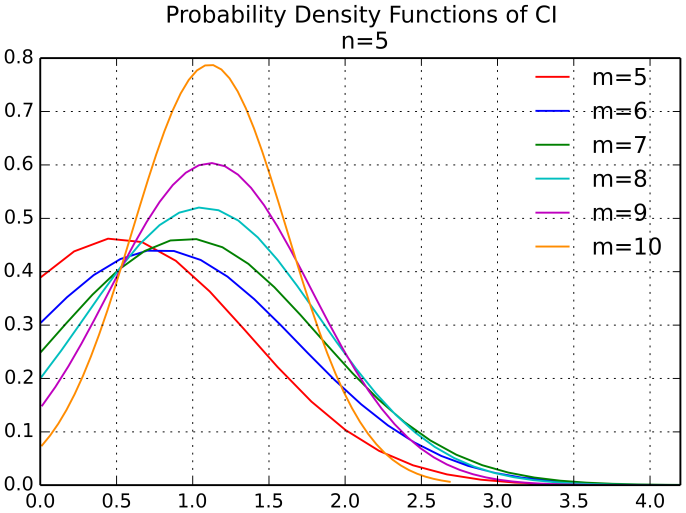


Figure 3.17: Normal Probability Density Functions of CI $n = 5$

3.3.3 PCMs with the order of 7

Table 3.7 shows descriptive statistics for PCMs with the order of 7 for each level of m. Mean consistency index value increase rapidly within the first six level of m, and starts to stabilize. Mean consistency index reaches its maximum value for m equals 14. Mean consistency index value for m equals 21 converges to when m equals to 10. The standard variation for mean consistency index reaches its maximum when m equals to 9, and reaches its minimum value when m equals to 21. Figure 3.18 summarizes how consistency index behaves between different levels of m.

Table 3.8 shows confidence intervals for each level of m for $\alpha = 0.05$, $\alpha = 0.1$ and

Table 3.7: Descriptive statistics for $n = 7$

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
7	0,173	0,0000	0,5710	1,1342	0,0335	0,0901
8	0,4969	0,0342	0,9408	1,3306	0,0405	0,0886
9	0,7511	0,3487	1,1723	1,4019	0,0442	0,0884
10	0,9544	0,8878	1,3171	1,3963	0,0294	0,0747
11	1,1078	0,3870	1,4289	1,2752	0,0400	0,0888
12	1,1862	1,2436	1,4275	1,0772	0,0596	0,1138
13	1,2801	0,7564	1,4878	0,9810	0,0860	0,1362
14	1,3307	1,0507	1,5001	0,8835	0,0931	0,1488
15	1,3424	1,5262	1,4791	0,7826	0,1144	0,1712
16	1,3704	1,3989	1,4891	0,7314	0,0970	0,1697
17	1,3923	1,0835	1,4872	0,6485	0,1441	0,2068
18	1,3967	1,3144	1,4637	0,5654	0,1871	0,2432
19	1,3888	1,1237	1,4386	0,5075	0,1859	0,2633
20	1,3743	1,7676	1,4029	0,4291	0,3144	0,3298
21	1,3502	1,7658	1,3381	0,3320	0,4495	0,4245

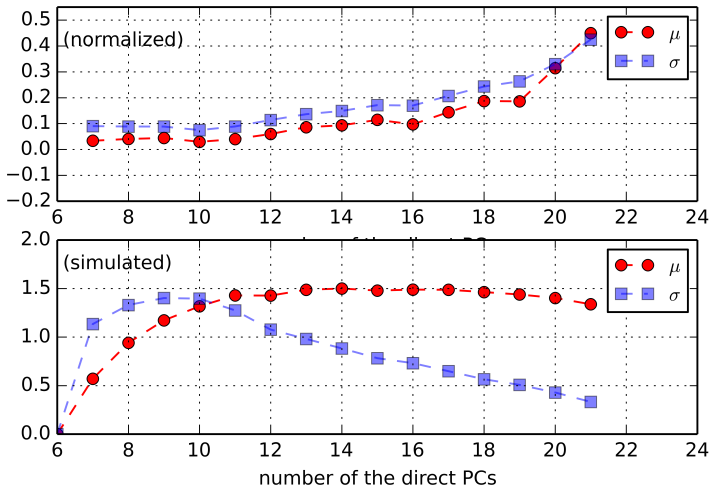


Figure 3.18: Simulated and normalized mean and standard variation for $n = 7$

$\alpha = 0.2$. Confidence intervals for each level of α reaches its maximum width for m equals 9. Following maximum confidence interval width occurs for m equals to 10, 8 and 11 respectively.

Figure 3.19 shows skewness and kurtosis test results of consistency index distributions for each level of m. Distributions converges to a normal distribution while m increases.

Figure 3.20 shows probability density function curves for each level of m. Skewness and kurtosis test results are supported by probability density function curves.

Table 3.8: Confidence intervals for $n = 7$

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
7	0,5488	0,5932	0,5523	0,5897	0,5565	0,5855
8	0,9147	0,9669	0,9189	0,9627	0,9237	0,9578
9	1,1448	1,1998	1,1492	1,1954	1,1543	1,1903
10	1,2897	1,3444	1,2941	1,3400	1,2992	1,335
11	1,4039	1,4539	1,408	1,4499	1,4126	1,4453
12	1,4064	1,4486	1,4098	1,4452	1,4137	1,4413
13	1,4686	1,5071	1,4717	1,504	1,4753	1,5004
14	1,4828	1,5174	1,4856	1,5146	1,4888	1,5114
15	1,4638	1,4945	1,4662	1,492	1,4691	1,4891
16	1,4748	1,5034	1,4771	1,5011	1,4797	1,4985
17	1,4745	1,4999	1,4765	1,4979	1,4789	1,4955
18	1,4526	1,4747	1,4544	1,473	1,4564	1,4709
19	1,4286	1,4485	1,4302	1,4469	1,4321	1,4451
20	1,3945	1,4113	1,3959	1,41	1,3974	1,4084
21	1,3316	1,3446	1,3326	1,3436	1,3338	1,3423

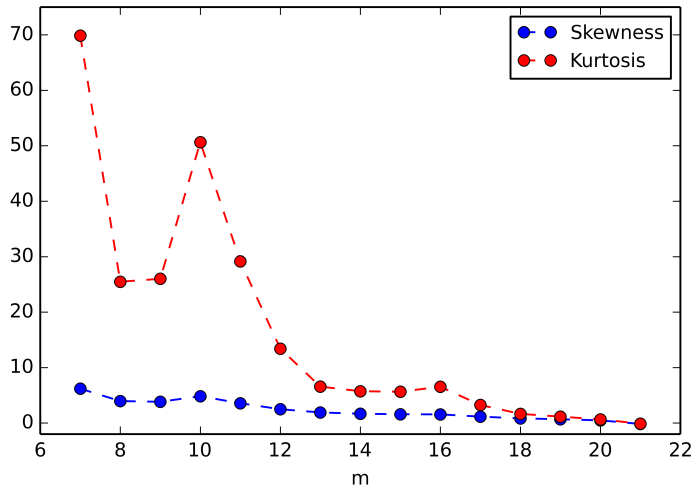


Figure 3.19: Kurtosis and Skewness test results for $n = 7$

As a result of analysis, suggested number of direct pairwise comparisons for *PCM*s with the order of 7 is $8, 9, 10$ or 11. Corresponding *random index* values for each level of m is given in Appendix C in Table C.4.

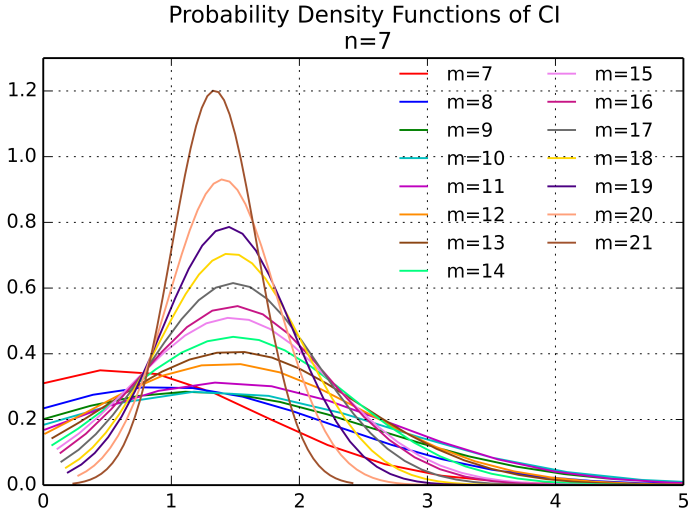


Figure 3.20: Normal Probability Density Functions of CI $n = 7$

3.3.4 PCMs with the order of 8

Table 3.9 shows descriptive statistics for PCMs with the order of 8 for each level of m. Mean consistency index value increase rapidly within the first five level of m, and starts to stabilize. Mean consistency index reaches its maximum value for m equals 14. Mean consistency index value for m equals 28 converges to when m is between 10 and 11. The standard variation for mean consistency index reaches its maximum when m equals to 9, and reaches its minimum value when m equals to 28. Figure 3.21 summarizes how consistency index behaves between different levels of m.

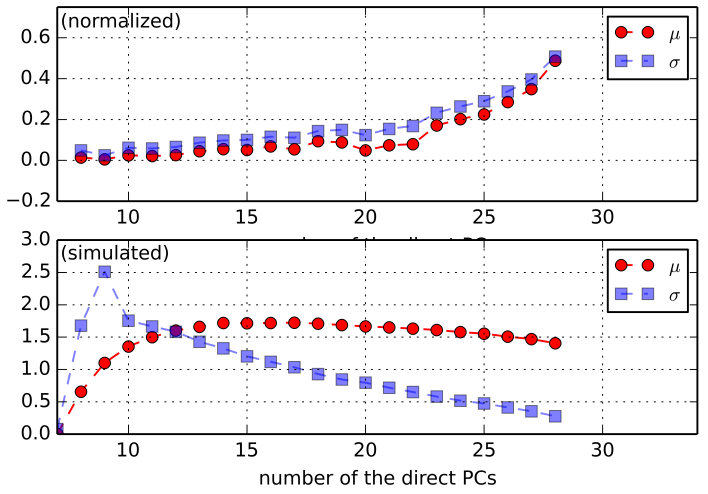


Figure 3.21: Simulated and normalized mean and standard variation for $n = 8$

Table 3.10 shows confidence intervals for each level of m for $\alpha = 0.05$, $\alpha = 0.1$ and

Table 3.9: Descriptive statistics for $n = 8$

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
8	0,1735	0,0000	0,6549	1,6777	0,0136	0,0485
9	0,5344	0,0464	1,1004	2,5092	0,0051	0,0245
10	0,8235	1,4475	1,3545	1,7553	0,0248	0,0613
11	1,0294	0,5242	1,4992	1,6653	0,0210	0,0584
12	1,1815	1,1279	1,6020	1,5828	0,0257	0,0654
13	1,2923	1,0590	1,6579	1,4286	0,0448	0,0864
14	1,3889	1,5674	1,7185	1,3254	0,0551	0,0971
15	1,4437	0,5439	1,7145	1,2023	0,0508	0,0998
16	1,4724	0,6350	1,7176	1,1183	0,0683	0,1156
17	1,5145	1,6534	1,7233	1,0339	0,0541	0,1111
18	1,5233	1,0601	1,7076	0,9287	0,0939	0,1433
19	1,5378	2,0849	1,6866	0,8449	0,0877	0,1494
20	1,5557	1,8820	1,6640	0,7953	0,0492	0,1237
21	1,5449	1,7343	1,6493	0,7177	0,0735	0,1543
22	1,5463	1,4919	1,6308	0,6512	0,0793	0,1684
23	1,5396	1,4528	1,6103	0,5806	0,1707	0,2332
24	1,5134	1,7320	1,5773	0,5184	0,2020	0,2637
25	1,5086	1,0474	1,5531	0,4740	0,2250	0,2895
26	1,4703	1,1169	1,5072	0,4128	0,2852	0,3374
27	1,4498	1,8962	1,4684	0,3536	0,3491	0,3960
28	1,4144	1,1907	1,4058	0,2775	0,4879	0,5080

$\alpha = 0.2$. Confidence intervals for each level of α reaches its maximum width for m equals 9. Following maximum confidence interval width occurs for m equals to 10, 8, 11 and 12 respectively.

Figure 3.22 shows skewness and kurtosis test results of consistency index distributions for each level of m. Distributions converges to a normal distribution while m increases.

Figure 3.23 shows probability density function curves for each level of m. Skewness and kurtosis test results are supported by probability density function curves.

As a result of analysis, suggested number of direct pairwise comparisons for *PCM*s with the order of 8 is $9, 10, 11$ or 12. Corresponding *random index* values for each level of m is given in Appendix C in Table C.5.

3.3.5 PCMs with the order of 9

Table 3.11 shows descriptive statistics for *PCM*s with the order of 9 for each level of m. Mean consistency index value increase rapidly within the first four level of m, and starts to stabilize. Mean consistency index reaches its maximum value for m equals 17. Mean consistency index value for m equals 36 converges to when m is between

Table 3.10: Confidence intervals for $n = 8$

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
8	0,622	0,6878	0,6273	0,6825	0,6334	0,6764
9	1,0513	1,1496	1,0592	1,1417	1,0683	1,1326
10	1,3201	1,3889	1,3256	1,3834	1,332	1,377
11	1,4666	1,5319	1,4718	1,5266	1,4779	1,5206
12	1,5709	1,633	1,5759	1,628	1,5817	1,6222
13	1,6299	1,6859	1,6344	1,6814	1,6396	1,6762
14	1,6925	1,7445	1,6967	1,7403	1,7015	1,7355
15	1,6909	1,7381	1,6947	1,7343	1,6991	1,7299
16	1,6957	1,7395	1,6992	1,736	1,7033	1,7319
17	1,703	1,7436	1,7063	1,7403	1,7100	1,7365
18	1,6894	1,7258	1,6923	1,7229	1,6957	1,7195
19	1,6701	1,7032	1,6727	1,7005	1,6758	1,6974
20	1,6484	1,6796	1,6509	1,6771	1,6538	1,6742
21	1,6352	1,6634	1,6375	1,6611	1,6401	1,6585
22	1,618	1,6435	1,6201	1,6415	1,6224	1,6391
23	1,5989	1,6217	1,6007	1,6198	1,6028	1,6177
24	1,5671	1,5875	1,5688	1,5858	1,5707	1,5839
25	1,5438	1,5624	1,5453	1,5609	1,547	1,5592
26	1,4991	1,5153	1,5004	1,514	1,5019	1,5125
27	1,4615	1,4753	1,4626	1,4742	1,4639	1,4729
28	1,4003	1,4112	1,4012	1,4103	1,4022	1,4093

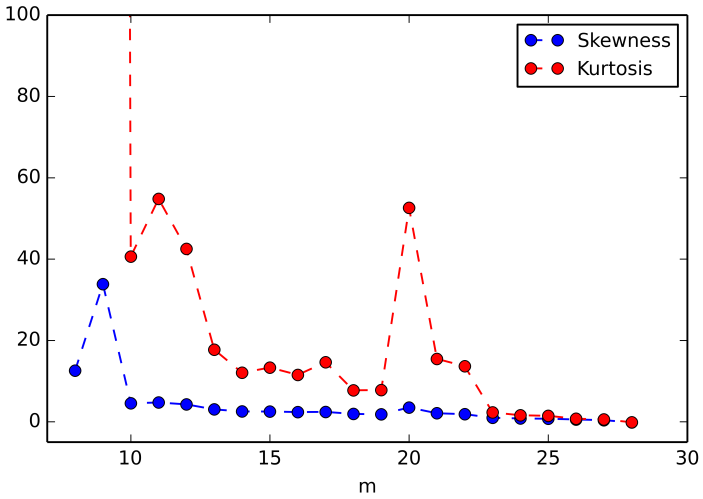


Figure 3.22: Kurtosis and Skewness test results for $n = 8$

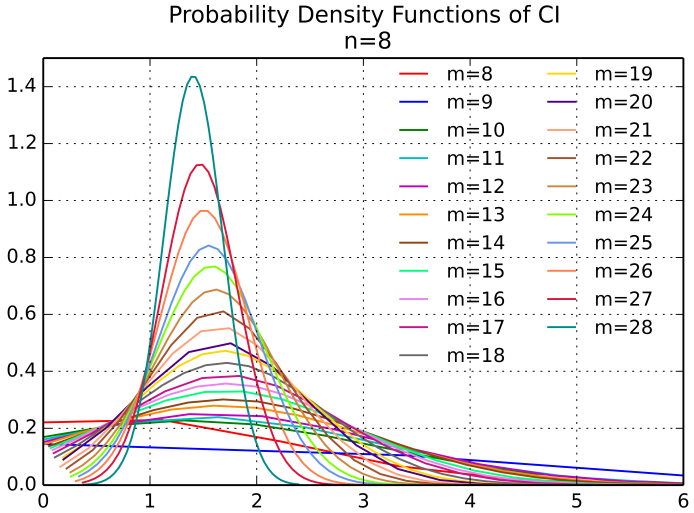


Figure 3.23: Normal Probability Density Functions of CI $n = 8$

14 and 15. The standard variation for mean consistency index reaches its maximum when m equals to 11, and reaches its minimum value when m equals to 36. Figure 3.24 summarizes how consistency index behaves between different levels of m.

Table 3.11: Descriptive statistics for $n = 9$

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
9	0,1903	0,0293	0,7475	1,8772	0,0138	0,0459
10	0,5483	0,362	1,2634	2,4404	0,0155	0,0418
11	0,8326	1,2831	1,5098	2,4859	0,0109	0,0354

Table 3.11: Continued

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
12	1,0731	1,1501	1,7155	2,3629	0,0112	0,0366
13	1,2572	1,2292	1,8661	2,1397	0,0209	0,0507
14	1,4158	1,3512	1,9598	2,0167	0,0139	0,0436
15	1,491	1,1768	1,978	1,8658	0,0258	0,0595
16	1,5707	2,2725	1,991	1,6699	0,0282	0,0655
17	1,6304	1,3307	1,9997	1,4821	0,0497	0,0868
18	1,6552	1,6733	1,977	1,4024	0,0344	0,078
19	1,6944	1,9804	1,9906	1,3231	0,0428	0,0878
20	1,7115	1,5703	1,9425	1,17	0,0272	0,0775
21	1,7063	1,9811	1,9255	1,0822	0,0496	0,104
22	1,7169	1,5006	1,8928	0,973	0,057	0,117
23	1,7081	1,2958	1,884	0,9131	0,0833	0,1393
24	1,6964	1,7232	1,8375	0,8376	0,0735	0,1406
25	1,6968	1,3824	1,8245	0,8028	0,0585	0,1319
26	1,6954	1,2445	1,8019	0,7147	0,0995	0,1726
27	1,6802	1,4764	1,7751	0,6692	0,0973	0,1786
28	1,6704	1,5186	1,7499	0,6095	0,1223	0,2052
29	1,6447	1,5552	1,717	0,5593	0,1118	0,2101
30	1,626	1,5552	1,6877	0,5049	0,1734	0,2593
31	1,607	1,5384	1,6655	0,4742	0,1994	0,2815
32	1,5798	1,7548	1,6252	0,4327	0,2411	0,3151
33	1,5462	1,2815	1,5861	0,3925	0,2372	0,3381
34	1,5222	1,5101	1,5536	0,348	0,3103	0,3943
35	1,486	1,8612	1,5051	0,3019	0,3907	0,461
36	1,4576	1,4405	1,4499	0,2393	0,526	0,586

Table 3.12 shows confidence intervals for each level of m for $\alpha = 0.05$, $\alpha = 0.1$ and $\alpha = 0.2$. Confidence intervals for each level of α reaches its maximum width for m equals 11. Following maximum confidence interval width occurs for m equals to 10, 12, 13 and 14 respectively.

Table 3.12: Confidence intervals for $n = 9$

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
9	0,7107	0,7843	0,7166	0,7783	0,7234	0,7715
10	1,2155	1,3112	1,2232	1,3035	1,2321	1,2947
11	1,461	1,5585	1,4689	1,5507	1,4779	1,5416
12	1,6692	1,7618	1,6766	1,7544	1,6852	1,7458
13	1,8242	1,9081	1,8309	1,9013	1,8387	1,8935
14	1,9203	1,9993	1,9266	1,993	1,9339	1,9856

Table 3.12: Continued

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
15	1,9414	2,0145	1,9473	2,0087	1,954	2,0019
16	1,9582	2,0237	1,9635	2,0184	1,9696	2,0124
17	1,9707	2,0288	1,9753	2,0241	1,9807	2,0187
18	1,9495	2,0045	1,9539	2,0001	1,959	1,995
19	1,9646	2,0165	1,9688	2,0123	1,9736	2,0075
20	1,9195	1,9654	1,9232	1,9617	1,9275	1,9574
21	1,9042	1,9467	1,9077	1,9433	1,9116	1,9393
22	1,8737	1,9119	1,8768	1,9088	1,8803	1,9053
23	1,8661	1,9019	1,8689	1,899	1,8723	1,8957
24	1,8211	1,8539	1,8237	1,8513	1,8268	1,8482
25	1,8088	1,8403	1,8113	1,8377	1,8142	1,8348
26	1,7879	1,8159	1,7902	1,8137	1,7928	1,8111
27	1,7619	1,7882	1,7641	1,7861	1,7665	1,7836
28	1,738	1,7619	1,7399	1,7599	1,7421	1,7577
29	1,706	1,728	1,7078	1,7262	1,7098	1,7242
30	1,6778	1,6976	1,6794	1,696	1,6812	1,6941
31	1,6562	1,6748	1,6577	1,6733	1,6594	1,6716
32	1,6168	1,6337	1,6181	1,6324	1,6197	1,6308
33	1,5785	1,5938	1,5797	1,5926	1,5811	1,5912
34	1,5468	1,5604	1,5479	1,5593	1,5491	1,5581
35	1,4992	1,5111	1,5002	1,5101	1,5013	1,509
36	1,4453	1,4546	1,446	1,4539	1,4469	1,453

Figure 3.25 shows skewness and kurtosis test results of consistency index distributions for each level of m. Distributions converges to a normal distribution while m increases.

Figure 3.26 shows probability density function curves for each level of m. Skewness and kurtosis test results are supported by probability density function curves.

As a result of analysis, suggested number of direct pairwise comparisons for PCMs with the order of 9 is $10, 11, 12$ or 13. Corresponding *random index* values for each level of m is given in Appendix C in Table C.6.

3.3.6 PCMs with the order of 10

Table 3.13 shows descriptive statistics for PCMs with the order of 10 for each level of m. Mean consistency index value increase rapidly within the first five level of m, and starts to stabilize. Mean consistency index reaches its maximum value for m equals 19. Mean consistency index value for m equals 45 converges to when m is between 11 and 12. The standard variation for mean consistency index reaches its maximum when m equals to 14, and reaches its minimum value when m equals to 45. Figure 3.27 summarizes how consistency index behaves between different levels of m.

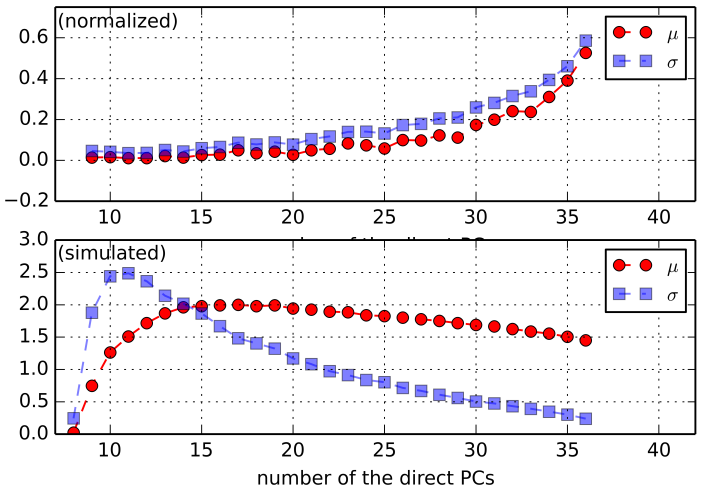


Figure 3.24: Simulated and normalized mean and standard variation for $n = 9$

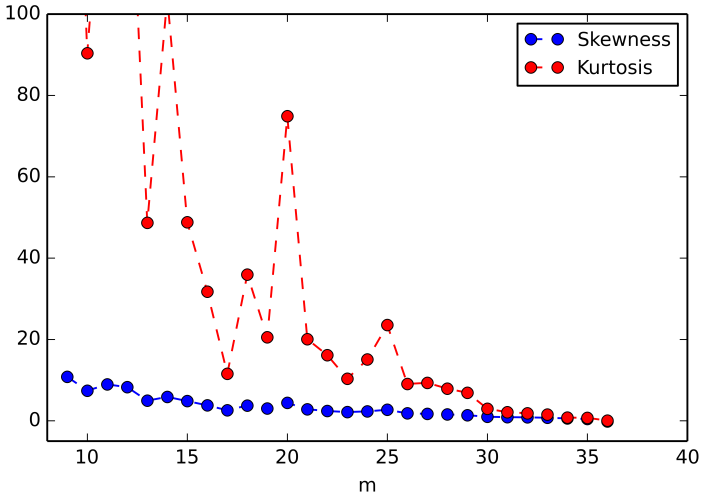


Figure 3.25: Kurtosis and Skewness test results for $n = 9$

Table 3.13: Descriptive statistics for $n = 10$

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
10	0,1938	0	0,8504	2,9009	0,0057	0,0242
11	0,5648	0	1,389	3,3269	0,0048	0,0208
12	0,9176	0,6515	1,7603	3,0448	0,0102	0,0307
13	1,1484	1,5485	1,9318	2,6701	0,0137	0,0375
14	1,3723	1,1741	2,2269	3,9341	0,0052	0,0196
15	1,5191	1,5869	2,2565	2,7718	0,0111	0,0333

Table 3.13: Continued

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
16	1,6342	1,6363	2,3253	2,9431	0,0056	0,0236
17	1,7259	1,1477	2,298	2,078	0,0332	0,0615
18	1,7762	1,1951	2,3242	2,1327	0,0179	0,0473
19	1,8515	1,0601	2,3406	1,9665	0,0178	0,0492
20	1,8557	1,2886	2,3003	1,7814	0,0231	0,0582
21	1,8669	1,293	2,2527	1,653	0,0186	0,0549
22	1,8959	1,6884	2,2461	1,5474	0,0273	0,0673
23	1,89	2,0073	2,2098	1,3495	0,0538	0,0944
24	1,8778	1,4196	2,1568	1,3103	0,0362	0,0828
25	1,8896	2,0378	2,1238	1,1362	0,062	0,1101
26	1,8595	2,0467	2,0711	1,075	0,0462	0,1017
27	1,8714	2,5074	2,0748	1,0617	0,0492	0,1054
28	1,8515	1,4495	2,0293	0,9376	0,0626	0,1238
29	1,8557	1,9133	2,0188	0,9543	0,0385	0,1006
30	1,8412	1,7136	1,9766	0,7972	0,0775	0,1479
31	1,838	2,9555	1,9645	0,8156	0,034	0,104
32	1,8148	1,2934	1,9189	0,7317	0,0566	0,1373
33	1,7928	1,569	1,8832	0,6699	0,0458	0,1318
34	1,7717	1,5339	1,8543	0,6118	0,0892	0,1829
35	1,7605	1,3603	1,8373	0,5768	0,132	0,2186
36	1,7386	1,7363	1,8052	0,5317	0,1529	0,241
37	1,7193	1,3222	1,7784	0,4919	0,1358	0,2442
38	1,6925	1,4642	1,7404	0,4532	0,2163	0,297
39	1,6779	2,1209	1,7247	0,4285	0,2117	0,3083
40	1,64	1,6557	1,6829	0,3946	0,2158	0,3285
41	1,6204	1,4988	1,6562	0,367	0,2581	0,3634
42	1,5909	1,9744	1,6194	0,3347	0,2431	0,3823
43	1,5544	1,6584	1,5814	0,3012	0,4515	0,4684
44	1,5209	1,6639	1,5346	0,2555	0,4937	0,5491
45	1,4904	1,5233	1,4833	0,2088	0,6523	0,6758

Table 3.14 shows confidence intervals for each level of m for $\alpha = 0.05$, $\alpha = 0.1$ and $\alpha = 0.2$. Confidence intervals for each level of α reaches its maximum width for m equals 14. Following maximum confidence interval width occurs for m equals to 11, 12, 16 and 15 respectively.

Table 3.14: Confidence intervals for $n = 10$

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
10	0,7935	0,9072	0,8026	0,8981	0,8132	0,8875

Table 3.14: Continued

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
11	1,3238	1,4542	1,3343	1,4437	1,3464	1,4317
12	1,7006	1,82	1,7102	1,8104	1,7213	1,7993
13	1,8795	1,9841	1,8879	1,9757	1,8976	1,966
14	2,1497	2,304	2,1621	2,2916	2,1764	2,2773
15	2,2021	2,3108	2,2109	2,3021	2,2209	2,292
16	2,2676	2,3829	2,2768	2,3737	2,2875	2,363
17	2,2573	2,3388	2,2638	2,3322	2,2714	2,3247
18	2,2824	2,366	2,2891	2,3593	2,2968	2,3515
19	2,3021	2,3792	2,3083	2,373	2,3154	2,3658
20	2,2653	2,3352	2,271	2,3296	2,2774	2,3231
21	2,2203	2,2851	2,2255	2,2799	2,2315	2,2739
22	2,2158	2,2764	2,2206	2,2715	2,2263	2,2659
23	2,1834	2,2363	2,1876	2,232	2,1925	2,2271
24	2,1311	2,1825	2,1352	2,1784	2,14	2,1736
25	2,1015	2,146	2,1051	2,1424	2,1092	2,1383
26	2,0501	2,0922	2,0534	2,0888	2,0573	2,0849
27	2,054	2,0956	2,0573	2,0923	2,0612	2,0884
28	2,0109	2,0477	2,0139	2,0447	2,0173	2,0413
29	2,0001	2,0375	2,0031	2,0345	2,0066	2,031
30	1,961	1,9923	1,9635	1,9897	1,9664	1,9868
31	1,9485	1,9805	1,9511	1,9779	1,954	1,9749
32	1,9046	1,9333	1,9069	1,931	1,9095	1,9283
33	1,8701	1,8963	1,8722	1,8942	1,8746	1,8918
34	1,8423	1,8662	1,8442	1,8643	1,8464	1,8621
35	1,826	1,8486	1,8278	1,8468	1,8299	1,8447
36	1,7947	1,8156	1,7964	1,8139	1,7984	1,812
37	1,7688	1,7881	1,7704	1,7865	1,7721	1,7847
38	1,7315	1,7493	1,7329	1,7479	1,7346	1,7462
39	1,7163	1,7331	1,7176	1,7317	1,7192	1,7302
40	1,6752	1,6907	1,6764	1,6894	1,6779	1,688
41	1,649	1,6634	1,6502	1,6623	1,6515	1,6609
42	1,6128	1,626	1,6139	1,6249	1,6151	1,6237
43	1,5755	1,5873	1,5764	1,5863	1,5775	1,5852
44	1,5296	1,5396	1,5304	1,5388	1,5313	1,5379
45	1,4792	1,4874	1,4799	1,4868	1,4807	1,486

Figure 3.28 shows skewness and kurtosis test results of consistency index distributions for each level of m. Distributions converges to a normal distribution while m increases.

Figure 3.29 shows probability density function curves for each level of m. Skewness and kurtosis test results are supported by probability density function curves.

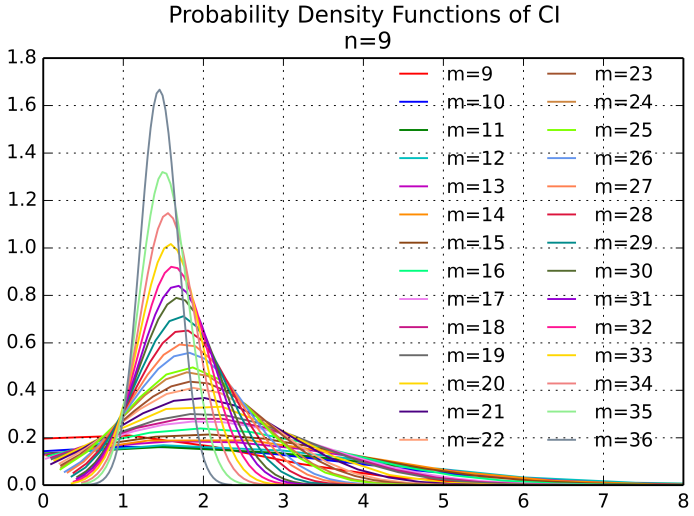


Figure 3.26: Normal Probability Density Functions of CI $n = 9$

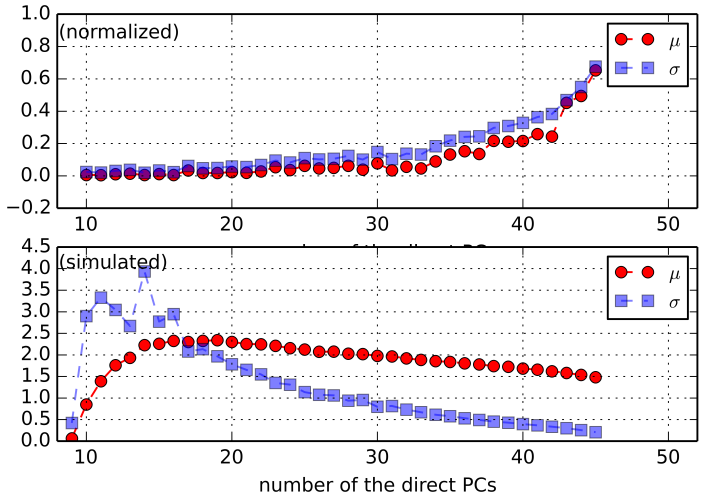


Figure 3.27: Simulated and normalized mean and standard variation for $n = 10$

As a result of analysis, suggested number of direct pairwise comparisons for PCMs with the order of 10 is $14, 15$ or 16. Corresponding *random index* values for each level of m is given in Appendix C in Table C.7.

3.3.7 PCMs with the order of 11

Table 3.15 shows descriptive statistics for PCMs with the order of 11 for each level of m. Mean consistency index value increase rapidly within the first four level of m, and starts to stabilize. Mean consistency index reaches its maximum value for m equals 18. Mean consistency index value for m equals 55 converges to when m is between

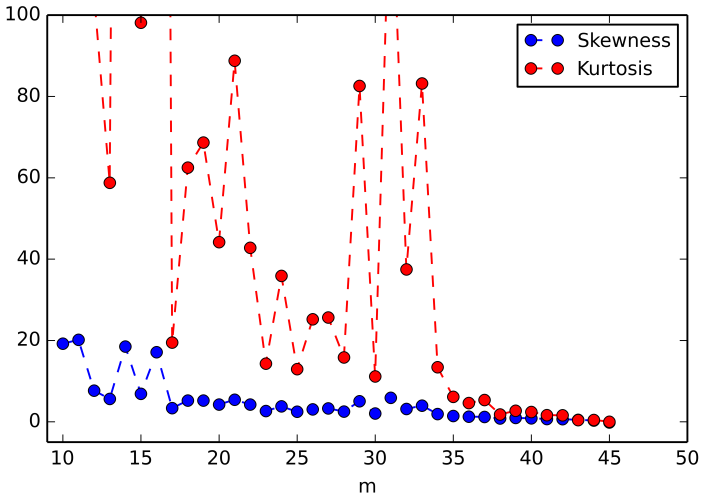


Figure 3.28: Kurtosis and Skewness test results for $n = 10$

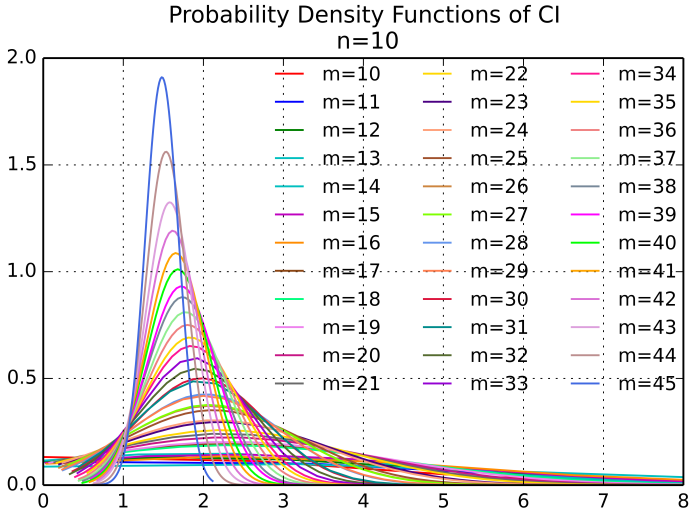


Figure 3.29: Normal Probability Density Functions of CI $n = 10$

11 and 12. The standard variation for mean consistency index reaches its maximum when m equals to 13, and reaches its minimum value when m equals to 55. Figure 3.30 summarizes how consistency index behaves between different levels of m.

Table 3.15: Descriptive statistics for $n = 11$

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
11	0,9666	1,1534	0,9817	1,1384	0,999	1,1211
12	1,5589	1,7568	1,5748	1,7409	1,5931	1,7225
13	1,9993	2,2872	2,0224	2,2641	2,0491	2,2374

Table 3.15: Continued

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
14	2,2199	2,4182	2,2358	2,4023	2,2542	2,3839
15	2,5305	2,7608	2,549	2,7423	2,5703	2,7209
16	2,6026	2,797	2,6182	2,7814	2,6363	2,7634
17	2,6489	2,8158	2,6623	2,8023	2,6778	2,7869
18	2,7105	2,8832	2,7244	2,8693	2,7404	2,8533
19	2,6498	2,7632	2,6589	2,7541	2,6694	2,7436
20	2,6427	2,7414	2,6506	2,7335	2,6598	2,7243
21	2,5926	2,6906	2,6005	2,6827	2,6096	2,6736
22	2,6246	2,7303	2,6331	2,7218	2,6429	2,712
23	2,5344	2,6265	2,5418	2,6191	2,5503	2,6105
24	2,5051	2,5778	2,5109	2,572	2,5177	2,5652
25	2,4718	2,5389	2,4772	2,5335	2,4834	2,5273
26	2,4119	2,4748	2,417	2,4697	2,4228	2,4639
27	2,4056	2,4698	2,4107	2,4647	2,4167	2,4587
28	2,3556	2,4125	2,3602	2,408	2,3654	2,4027
29	2,2991	2,3526	2,3034	2,3483	2,3084	2,3434
30	2,2645	2,3124	2,2683	2,3085	2,2728	2,3041
31	2,2339	2,2805	2,2376	2,2768	2,2419	2,2725
32	2,1949	2,2367	2,1983	2,2333	2,2022	2,2295
33	2,1759	2,2147	2,179	2,2116	2,1826	2,208
34	2,1609	2,2007	2,1641	2,1975	2,1678	2,1938
35	2,1136	2,148	2,1164	2,1452	2,1196	2,142
36	2,0752	2,1068	2,0777	2,1042	2,0806	2,1013
37	2,0529	2,0851	2,0555	2,0825	2,0585	2,0795
38	2,0208	2,0487	2,0231	2,0465	2,0257	2,0439
39	1,9838	2,0101	1,9859	2,008	1,9883	2,0056
40	1,9628	1,9877	1,9648	1,9857	1,9671	1,9834
41	1,9419	1,966	1,9438	1,964	1,946	1,9618
42	1,9101	1,9326	1,9119	1,9308	1,914	1,9287
43	1,8819	1,9029	1,8836	1,9012	1,8855	1,8992
44	1,8646	1,8841	1,8662	1,8825	1,868	1,8807
45	1,8331	1,8511	1,8345	1,8496	1,8362	1,848
46	1,8071	1,8249	1,8085	1,8235	1,8102	1,8218
47	1,7878	1,8044	1,7891	1,803	1,7906	1,8015
48	1,7578	1,7731	1,759	1,7719	1,7604	1,7705
49	1,7202	1,7344	1,7213	1,7333	1,7227	1,732
50	1,6971	1,7105	1,6982	1,7094	1,6994	1,7082
51	1,6626	1,6751	1,6636	1,6741	1,6648	1,6729
52	1,6324	1,6438	1,6333	1,6429	1,6344	1,6418
53	1,5929	1,6031	1,5937	1,6023	1,5946	1,6013
54	1,554	1,563	1,5548	1,5623	1,5556	1,5615

Table 3.15: Continued

			Simulated		Normalized	
m	**median**	**mode**	**mean**	**std dev.**	**mean**	**std dev.**
55	1,5114	1,5186	1,512	1,518	1,5126	1,5174

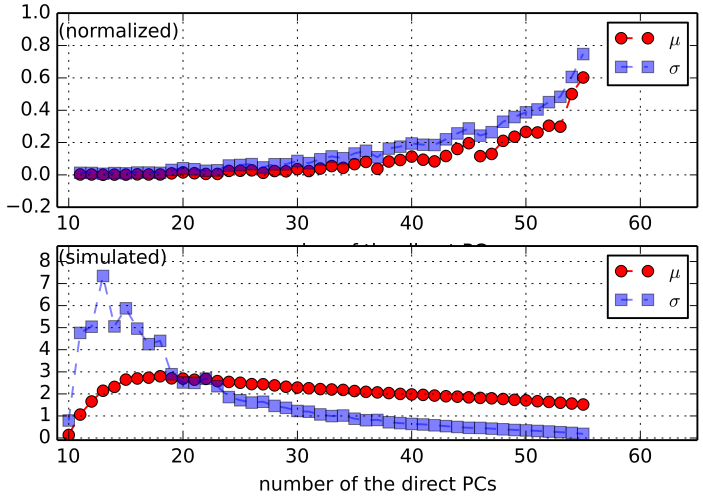


Figure 3.30: Simulated and normalized mean and standard variation for $n = 9$

Table 3.16 shows confidence intervals for each level of m for $\alpha = 0.05$, $\alpha = 0.1$ and $\alpha = 0.2$. Confidence intervals for each level of α reaches its maximum width for m equals 13. Following maximum confidence interval width occurs for m equals to 15, 14, 12 and 16 respectively.

Table 3.16: Confidence intervals for $n = 11$

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
11	0,9666	1,1534	0,9817	1,1384	0,999	1,1211
12	1,5589	1,7568	1,5748	1,7409	1,5931	1,7225
13	1,9993	2,2872	2,0224	2,2641	2,0491	2,2374
14	2,2199	2,4182	2,2358	2,4023	2,2542	2,3839
15	2,5305	2,7608	2,549	2,7423	2,5703	2,7209
16	2,6026	2,797	2,6182	2,7814	2,6363	2,7634
17	2,6489	2,8158	2,6623	2,8023	2,6778	2,7869
18	2,7105	2,8832	2,7244	2,8693	2,7404	2,8533
19	2,6498	2,7632	2,6589	2,7541	2,6694	2,7436
20	2,6427	2,7414	2,6506	2,7335	2,6598	2,7243
21	2,5926	2,6906	2,6005	2,6827	2,6096	2,6736
22	2,6246	2,7303	2,6331	2,7218	2,6429	2,712
23	2,5344	2,6265	2,5418	2,6191	2,5503	2,6105

Table 3.16: Continued

	$\alpha = 0.05$		$\alpha = 0.1$		$\alpha = 0.2$	
m	**lower**	**upper**	**lower**	**upper**	**lower**	**upper**
24	2,5051	2,5778	2,5109	2,572	2,5177	2,5652
25	2,4718	2,5389	2,4772	2,5335	2,4834	2,5273
26	2,4119	2,4748	2,417	2,4697	2,4228	2,4639
27	2,4056	2,4698	2,4107	2,4647	2,4167	2,4587
28	2,3556	2,4125	2,3602	2,408	2,3654	2,4027
29	2,2991	2,3526	2,3034	2,3483	2,3084	2,3434
30	2,2645	2,3124	2,2683	2,3085	2,2728	2,3041
31	2,2339	2,2805	2,2376	2,2768	2,2419	2,2725
32	2,1949	2,2367	2,1983	2,2333	2,2022	2,2295
33	2,1759	2,2147	2,179	2,2116	2,1826	2,208
34	2,1609	2,2007	2,1641	2,1975	2,1678	2,1938
35	2,1136	2,148	2,1164	2,1452	2,1196	2,142
36	2,0752	2,1068	2,0777	2,1042	2,0806	2,1013
37	2,0529	2,0851	2,0555	2,0825	2,0585	2,0795
38	2,0208	2,0487	2,0231	2,0465	2,0257	2,0439
39	1,9838	2,0101	1,9859	2,008	1,9883	2,0056
40	1,9628	1,9877	1,9648	1,9857	1,9671	1,9834
41	1,9419	1,966	1,9438	1,964	1,946	1,9618
42	1,9101	1,9326	1,9119	1,9308	1,914	1,9287
43	1,8819	1,9029	1,8836	1,9012	1,8855	1,8992
44	1,8646	1,8841	1,8662	1,8825	1,868	1,8807
45	1,8331	1,8511	1,8345	1,8496	1,8362	1,848
46	1,8071	1,8249	1,8085	1,8235	1,8102	1,8218
47	1,7878	1,8044	1,7891	1,803	1,7906	1,8015
48	1,7578	1,7731	1,759	1,7719	1,7604	1,7705
49	1,7202	1,7344	1,7213	1,7333	1,7227	1,732
50	1,6971	1,7105	1,6982	1,7094	1,6994	1,7082
51	1,6626	1,6751	1,6636	1,6741	1,6648	1,6729
52	1,6324	1,6438	1,6333	1,6429	1,6344	1,6418
53	1,5929	1,6031	1,5937	1,6023	1,5946	1,6013
54	1,554	1,563	1,5548	1,5623	1,5556	1,5615
55	1,5114	1,5186	1,512	1,518	1,5126	1,5174

Figure 3.31 shows skewness and kurtosis test results of consistency index distributions for each level of m. Distributions converges to a normal distribution while m increases.

Figure 3.32 shows probability density function curves for each level of m. Skewness and kurtosis test results are supported by probability density function curves.

As a result of analysis, suggested number of direct pairwise comparisons for PCMs with the order of 11 is $13, 14, 15$ or 16. Corresponding *random index* values for each level of m is given in Appendix C in Table C.8.

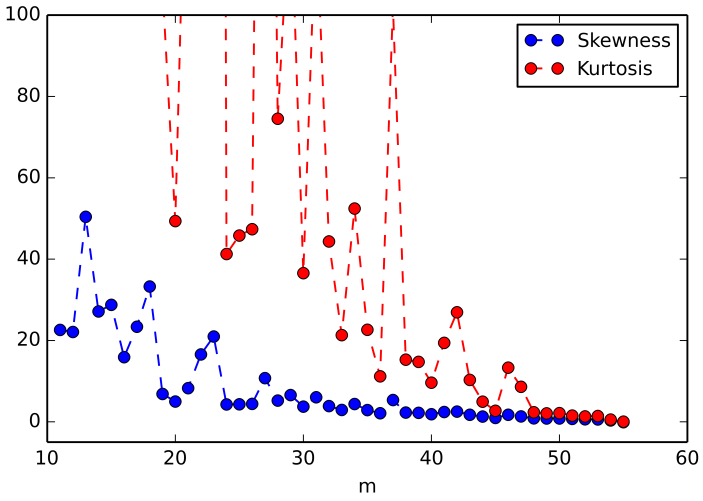


Figure 3.31: Kurtosis and Skewness test results for $n = 11$

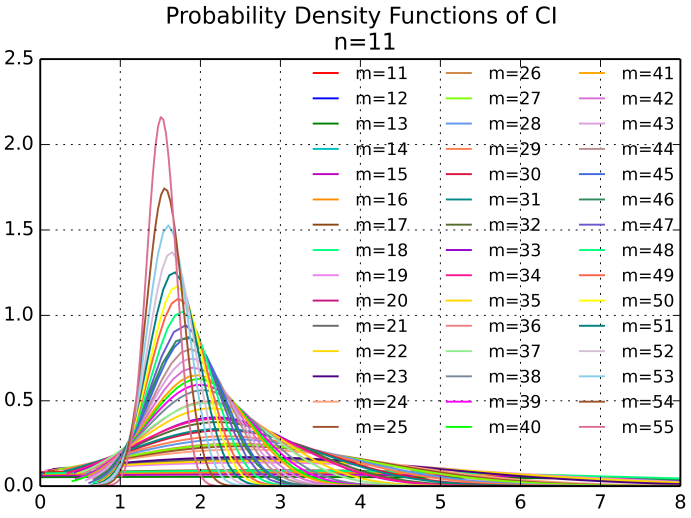


Figure 3.32: Normal Probability Density Functions of CI $n = 11$

CHAPTER 4

Proposed Method

In order to achieve a highly consistent *Pairwise Comparison Matrix*, the sufficient number of direct *pairwise comparisons* is analyzed. However the number of direct *pairwise comparisons* alone is not enough to achieve high levels of consistency. Causes of inconsistency can be eliminated up to a level by limiting the number of direct *pairwise comparisons* but a level of inconsistency remains indefinite. This uncertainty is to be expected since the only resource for gathering information is the human factor.

The literature shows that expert systems can be used to eliminate uncertainty [28]. Expert systems allows to dynamic checking of *consistency index* and transitivity violations. Spotting the *pairwise comparison* that causes the inconsistency is important for its reviewal without delay.

The uncertainty can be caused by the psychological dependency of *pairwise comparisons*. Psychological dependency can be eliminated by the random selection of *pairwise comparisons* to be asked to decision maker.

This study proposes such a method that will limit the number of *direct pairwise comparisons*, eliminate psychological dependency and identify if a *pairwise comparison* causes mathematical error or transitivity violations. Figure 4.2 and Figure 4.3 describes the algorithm for proposed method.

An interval derived from *indirect pairwise comparisons* is used to determine an acceptable value for a *direct pairwise comparison*. A *direct pairwise comparison* is accepted if it is within the derived interval, rejected otherwise. If a *pairwise comparison* is not within the acceptable interval DM is required to review his/her judgment.

In order to calculate an interval first the *indirect pairwise comparison* should be calculated. The *indirect pairwise comparison* for an a_{ij} is the geometric mean of all possible paths between specimens i and j. An acceptable value for an a_{ij} is decided as regard to the sensitivity of the decision problem. Table 4.1 shows examples for four *missing pairwise comparisons* with different sensitivity levels.

The rules below can be used as a guide for how to determine an acceptable interval:

Table 4.1: Acceptable intervals for sample missing pairwise comparisons

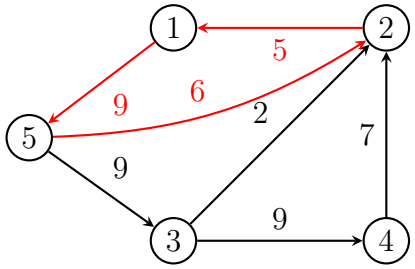
Missing	Calc.	Low Sens.		High Sens.	
		min	max	min	max
a_{16}	0.4	¼	1	⅓	½
a_{62}	4	2	6	3	5
a_{35}	8	6	9	7	9
a_{54}	56	7	9	9	9

- The decision problems sensitivity determines the length of interval
- The direction of *pairwise comparison* should be considered
- When a mathematilly derived *missing pairwise comparison* shows extreme values, acceptable interval should be chosen from the extreme values of the scale.

With each new *direct pairwise comparison* the number of paths between specimens i and j can increase. Therefor after a new *pairwise comparison* is entered to matrix intervals for missing *pairwise comparisons* are updated.

$$A = \begin{bmatrix} 1 & 1/5 & 1 & 1 & 9 \\ 5 & 1 & 1/2 & 1/7 & 1/6 \\ 1 & 2 & 1 & 9 & 1/9 \\ 1 & 7 & 1/9 & 1 & 1 \\ 1/9 & 6 & 9 & 1 & 1 \end{bmatrix}$$

(a) Random PCM



(b) Transitivity graph

Figure 4.1: Transitivity violation example

In order to eliminate any intransitivity occurance, a smilar control process is utilized. It was earlier stated that a violation of transitivity occurs when graph representation of the *PCM* consists of cycles. If a graph of *PCM* is drawn which will only consists of *pairwise comparisons* with values greater than 1 in the Saaty scale. Such a graph will only show interactions between specimens in magnitude. For a given random *PCM* (A) such a graph is shown in Figure 4.1.

In Figure 4.1 edges e_{15}, e_{52} and e_{21} creates a cycle, meaning there is a transitivity violation. Proposed method is designed to eliminate transitivty violations by utilizing acceptable intervals. However transitivity violations may occur while generating the

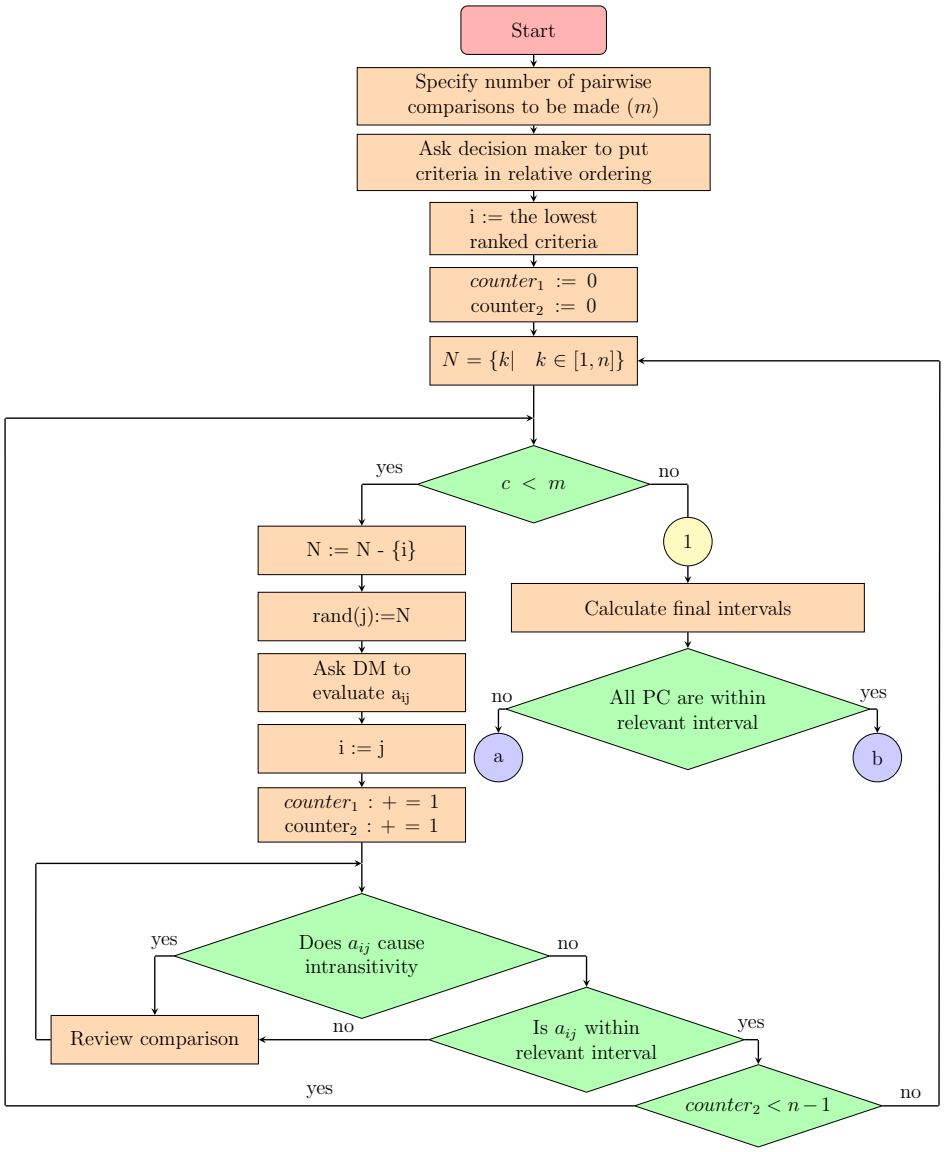


Figure 4.2: Algorithm of proposed method (1)

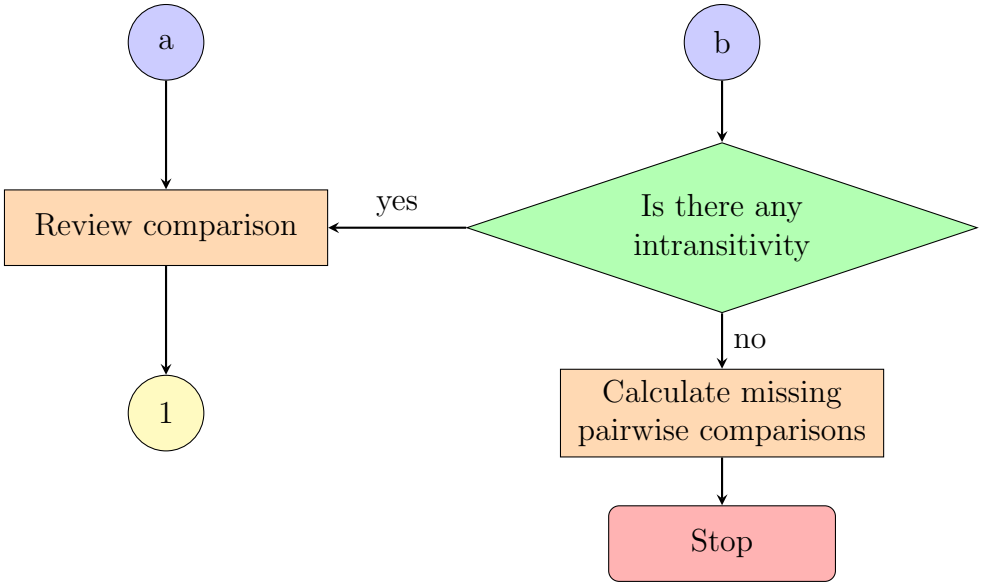


Figure 4.3: Algorithm of proposed method (2)

spanning tree. Therefor it is important to check for transitivity violations and ask DM to review his/her judgments before moving forward with remaining questions. In order to being through proposed method checks transitivity violations after each *pairwise comparison* is entered to matrix.

CHAPTER 5

Results and Discussion

In Section 1.2 objectives of this study are given. The main objective is to design a new data elicitation process in order to gather more reliable and consistent information from decision makers. In order to design such an elicitation process, first the causes of inconsistency are investigated. The causes of inconsistency are classified as *mathematical* and *non-mathematical*. *Mathematical* causes are idetified as *mathematical error* and *transitivity violations* (further explained in Section 2.2), and cause an increase in the maximum eigenvalue of a given pairwise comparison matrix. *Non-Mathematical* causes are identified as but not limited to *large number of pairwise comparisons*, *time needed for gathering data* and *pyschological dependency of pairwise comparisons*. *Non-Mathematical* causes may not cause an increase in the maximum eigenvalue of the given pairwise comparison matrix but may put the reliability of the data at risk. In order to construct highly consistent *PCM*s while factoring reliability in, a new data elicitation process is designed.

Proposed data elicitation process utilizes specific solutions for each cause of inconsistency, and combines them as one method. The solutions and their addressed problems are given below:

- **Limited number of direct pairwise comparisons**: In order to decrease the total number of pairwise comparisons, which are needed for an *Analytic Hierarchy Process*, and the time to make these pairwise comparisons, the number of direct pairwise comparisons are limited. The limitation is made as regard to both the order of *PCM* and the sensitivity of the decision problem. The determination of number of direct pairwise comparisons are explained in Section 3.2.
- **Complete randomization of pairwise comparisons**: In order to ensure psychological independency of pairwise comparisons, two step randomization is utilized. First the pairwise comparisons to be asked are randomly selected then asking order of these pairwise comparisons are randomly assigned.
- **Elimination of transitivity violations**: In order to prevent transitivity violations, the DM is required to put specimens in relative ordering. If a pairwise

Table 5.1: Suggested number of direct PCs

n	m (min)	**RI**	**m (max)**	**RI**
4	4	0,49	6	0,88
5	5	0,51	7	0,96
6	7	0,86	9	1,18
7	8	0,94	10	1,32
8	8	0,65	10	1,35
9	10	1,26	12	1,72
10	14	2,23	16	2,33
11	13	2,14	15	2,65

comparison contradicts the intial relative ordering, the DM is required to review the evaluation.

- **Acceptable measurement error level**: Some level of consistency distruption is needed to evaluate how consistent a given *PCM* against a randomly generated *PCM*. After the first two pairwise comparisons with an adjacent vertex on the graph, it is possible to derive missing pairwise comparisons mathematically. When a randomly selected pairwise comparison coincides with such a pairwise comparison, it is possible to compare these two values in order to make a judgment about the evaluation. Acceptable measurement error level is further discussed in Chapter 4.

Table 5.1 shows suggested number of direct pairwise comparisons for *PCM*s with the order of n and their corresponding *Random Index* values. The number of direct pairwise comparisons are suggested as regard to such a decision matrix with normal sensitivity. If the decision problem at hand holds a high sensitivity level or if seen necessary, the number of direct pairwise comparisons can be increased.

In order to investigate the level consistency for a given *PCM* first, its *consistency index* is derived then compared against a *Random Index*. A *Random Index Study* is conducted in order to derive *Random Index* values for *PCM*s with the order of n. A large mass of *Random Pairwise Comparison Matrices (RPCMs)* is generated for each PCM with the order of n. *Consistency Index* values for each *RPCM* is calculated. The mean *consistency index* is defined as *Random Index*, and used as benchmark for *consistency ratio* for *PCM*s.

In order to accept or reject a given *PCM* as regard to its consistency level, its *consistency index* is compared against the relevant *random index* value independently from the number of direct pairwise comparisons. Whether there are missing pairwise comparisons or not, the consistency index of a given *PCM* is compared against the *random index* derived from *RPCM*s which consists of all pairwise comparisons. However the distribution of *consistency index* is dependent on the number of missing pairwise comparisons. Therefor treating an *Incomplete PCM* as a *Complete PCM* will lead misguided results.

A *Random Index Study* is conducted as a part of this study. Divergently from existing

Table 5.2: Random Index values for PCMs with order of 6

# of PCs	Random Index	σ_{RI}
6	0,0360	0,0997
7	0,0505	0,1064
8	0,0618	0,1174
9	0,0612	0,1200
10	0,0834	0,1423
11	0,0929	0,1588
12	0,1336	0,1928
13	0,1331	0,2102
14	0,2519	0,2648
15	0,3980	0,3427

random index studies, for each level of n 10.000 matrices are randomly generated in 10 runs for each level m. In Section 3.1, *Random Index Study* is further explained. As a result for a *PCM* with the order of n, $\frac{(n-1)(n-2)}{2} \times 10.000$ number of matrices are generated, and a *Random Index* for each level of m is derived. In Table 5.2, *Random Index* for *PCM*s with the order of 6 is given. In Section C *Random Index* values for *PCM*s with order of 4 through 11 are given.

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APPENDIX A

Graph Theory

Graph theory[37] is the study of graphs, which are mathematical structures used to model pairwise relations between objects.

Let V be a finite set, and denote by

$$E\big(V\big) = \{\{u, v\} | u, v \in V, u \neq v\}$$

Definition 1 *A pair* $G = \big(V, E\big)$ *with* $E \subseteq E\big(V\big)$ *is called a graph on V. The elements of V are* vertices *of G, and those of E are the* edges *of G. The vertex set of a graph G is denoted by* V_G *and its edge set by* E_G*. Therefore* $G = \big(V_G, E_G\big)$.

A pair $\{u, v\}$ is usually written simply as uv. Notice t hat $uv = vu$. In order to simplify notations, it is also written as $v \in G$ and $e \in G$ instead of $v \in V_{\mathrm{G}}$ and $e \in E_{\mathrm{G}}$.

Definition 2 *For a graph G, it is denoted* $v_G = |V_G|$ *and* $\varepsilon_G = |E_G|$.

The number v_{G} of the vertices is called the **order** of G, and ε_{G} is the **size** of G. For an edge $e = uv \in G$, the vertices u and v are its **ends**. Vertices u and v are **adjacent** or **neighbours**, if $uv \in G$. Two edges uv and uw are **adjacent** if they have common end.

A graph G can be represented as a plane figure by drawing a line or a curve between the points u and v (representing vertices) if $e = uv$ is an edge of G. Figure A.1 is a geometric representation of the graph G with $V_{\mathrm{G}} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E_{\mathrm{G}} = \{v_1v_2, v_1v_3, v_2v_4, v_2v_3, v_5v_6\}$.

Definition 3 *A graph is said to be a* ***directed graph*** *or a* ***digraph*** $\big(D = (V, E)\big)$ *if its edges have direction, that is, the edges are ordered:* $E \subseteq V \times V$*. In this case,* $uv \neq vu$.

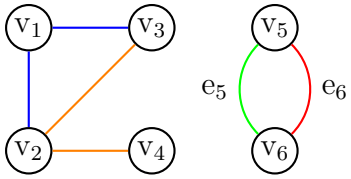


Figure A.1: Graph G

Directed graph edges are drawn as arrows, where the arrow point shows the ending vertex. A directed graph can contain edges uv and vu of *opposite directions*. Directed edges with swaped starting and ending vertices creates **loops**; such edges are also **parallel** if not directed. In Figure A.1 and Figure A.2, edges e_5 and e_6 are parallel, while edge e_4 in Figure A.2 creates a loop.

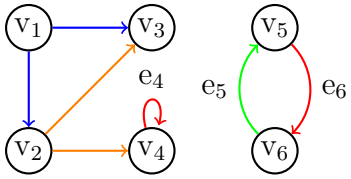


Figure A.2: Directed Graph D

Graphs can also be represented as matrices for mathematical programming purposes. Such a matrix is shown in Figure A.3.

Definition 4 *Let $V_G = \{v_1, \ldots, v_n\}$ be ordered. The **adjacency matrix** of G is the $n \times n$-matrix M with entries $M_{ij} = 1$ or $M_{ij} = 0$ according to whether $v_i v_j \in G$ or $v_i v_j \notin G$.*

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

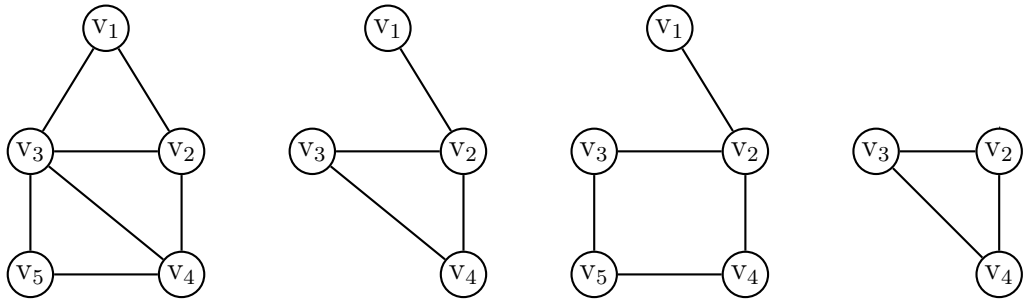
Figure A.3: An adjacency matrix

Definition 5 *Let $v \in G$ be a vertex of graph G. The **neighbourhood** of v is the set*

$$N_G(v) = \{u \in G | vu \in G\}$$

The **degree** of v is the number of its neighbours:

$$d_G(v) = |N_G(v)|$$



(a) G (b) a subgraph of G (c) spanning subgraph of G (d) induced subgraph of G

Figure A.4: Subgraphs of G

If $d_{\mathrm{G}}(v) = 0$, then v is said to be **isolated in** G. If $d_{\mathrm{G}}(v) = 1$, then v is said to be **a leaf of** G.

Definition 6 *A graph H is a **subgraph** of G, denoted by $H \subseteq G$, if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.*

Any of the edges or vertices can be removed to obtain subgraphs. A subgraph $H \subseteq G$ **spans** G (and H is a **spanning subgraph** of G), if every vertex of G is in H, i.e., $V_{\mathrm{H}} = V_{\mathrm{G}}$. To each subset $F \subseteq E_{\mathrm{G}}$ of edges, there corresponds a unique spanning subgraph of G.

$$G[F] = (V_{\mathrm{G}}, F)$$

A subgraph H is an **induced subgraph**, if $E_{\mathrm{H}} = E_{\mathrm{G}} \cap E(V_{\mathrm{H}})$. In this case, H is **induced** by its set of V_{H} vertices. In an induced subgraph $H \subseteq G$, the set E_{H} of edges consists of all $e \in E_{\mathrm{G}}$ such that $e \in E(V_{\mathrm{H}})$.

A graph G is $(G = K_{\mathrm{V}})$ is a **complete graph** on V, if every two vertices are adjacent:

$$E = E(V)$$

All complete graphs of order n are isomorphic with each other.

The **complement** of graph G is the graph $\bar{G}$ on V_{G}, where

$$E_{\bar{G}} = \{e \in E(V) | e \notin E_{\mathrm{G}}\}$$

A graph is said to be **regular**, if every vertex of G has the same degree. If this degree is equal to r, then G is **r-regular** or **regular of degree r**. A **discrete graph** is $0 - regular$, and a complete graph is $K_{\mathrm{n}} - regular$. In particular, $\varepsilon_{\mathrm{K_n}} = \frac{n(n-1)}{2}$, and

therefore $\varepsilon_G \leq \frac{n(n-1)}{2}$ for all graphs G that have order n.

Definition 7 *A graph is called* ***bipartite****, if V_G has a partition of two subsets X and Y such that each edge $uv \in G$ connect a vertex of X to a vertex of Y. In this case, (X, Y) is a* ***bipartition*** *of G, and G is* ***(X, Y)-bipartite****.*

A bipartite graph G is **complete (m,k)-bipartite**, if $|X| = m, |Y| = k$ and $uv \in G$ for all $u \in X$ and $v \in Y$. A graph G is bipartite if and only if V_G has a partition to two stable subsets. A subset $X \subseteq V_G$ is **stable** if $G(x)$ is a **discrete graph**. A graph G is bipartite if and only if G has no odd cycles (as a subgraph).

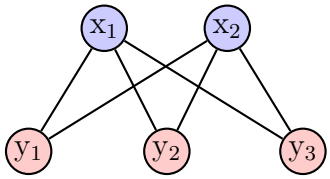


Figure A.5: A (2,3)-bipartite graph

Definition 8 *Let $e_i = u_i u_{i+1} \in G$ be edges of G for $i \in [1, k]$. The sequence*

$$W = e_1 e_2 \dots k$$

is a ***walk of length k from u_1 to u_{k+1}****.*

Edges e_i and e_{i+1} are adjacent for all $i \in [1, k]$.

A walk of some length from u to v is denoted by $W : u \to v$ The length of a walk W is denoted by $|W|$.

Definition 9 *Let $W = e_1 e_2 \dots k$, $(e_i = u_i u_{i+1})$ be a walk.*

- *W is* ***closed****, if $u_1 = u_{k+1}$*
- *W is a* ***path****, if $u_i \neq u_j$ for all $i \neq j$*
- *W is a* ***cycle****, if it is closed, and $u_i \neq u_j$ for all $i \neq j$ except that $u_1 = u_{k+1}$*
- *W is a* ***trivial path****, if its length is 0. A trivial path has no edges.*
- *For a walk $W : u_1 \to u_2 \to \dots u_{k+1} = v$, also $W^{-1} : v = u_{k+1} \to \dots \to u_2 \to u_1$ is a walk in G, called a* ***inverse walk****.*

A vertex u is an **end** of path P, if P starts or ends in u. Paths P and Q are **disjoint**, if they have no vertices in common, and they are **independent**, if they only share their ends. A (sub)graph, which is a path (cycle) of length $(k-1)$ $(k, respectively)$ having

k vertices is denoted by P_{k} (C_{k}, respectively). If k is **even (odd)**, then the path or cycle is **even (odd)**.

Definition 10 *If there exists a walk (and hence a path) from u to v in G, let*

$$d_G(u,v) = min\{k|u \to v\}$$

*be the **distance** between u and v. If there are no walks $u \to v$, let $d_G(u,v) = \infty$ by convention. A graph G is **connected**, if $d_G(u,v) < \infty$ for all $u, v \in G$; otherwise it is **disconnected**. The number of connected components of G is denoted as $c(G)$. If $c(G) = 1$, then G is connected.*

Definition 11 *An edge $e \in G$ is a **bridge** of the graph G, if $G - e$ has more connected components than G, that is*

$$c(G - e) > c(G)$$

If $e = uv$ is a bridge, than u and v are in different connected components of $(G - e)$. An edge e is a bridge if and only if e is not in any cycle of G.

In Figure A.6 (a), edges e_1 and e_2 are bridges. With the removal of these edges, the number of connected components of a graph G can be increased. Figure A.6 (b) shows the connected components of graph G, incase of removal of edge e_1. Figure A.6(a) represents a graph that consists of a single connected component while Figure A.6 (b) represents a graph that consists of two seperate connected components with removal of edge e_1.

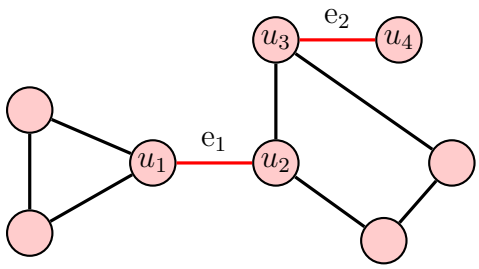
Definition 12 *A graph is called **acyclic (a forest)**, if it has no cycles. A **tree** is a connected acyclic graph, therefor a connected graph is a tree if and only if all its edges are bridges.*

Theorem 1 *Each connected graph has a **spanning tree**, that is, a spanning graph that is a tree.*

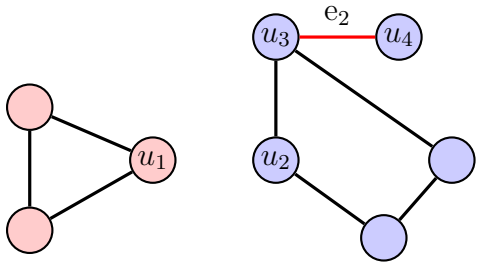
Proof 1 *Let $T \subseteq G$ be a maximum order subtree of G (i.e., subgraph that is a tree). If $V_T \neq V_G$, there exists an edge $uv \notin E_G$ such that $u \in T$ and $v \notin T$. But then T is not maximal; a contradiction.*

For each connected graoh G, $\varepsilon_{\mathrm{G}} \geq v_{\mathrm{G}} - 1$. Moreover, a connected graph is a tree if and only if $\varepsilon_{\mathrm{G}} = v_{\mathrm{G}} - 1$.

In Figure A.7 (a) shows a complete and connected graph G, with $v_{\mathrm{G}} = 4$ and $\varepsilon_{\mathrm{G}} = 6$; and (b) shows a connected graph (and a spanning tree of graph G), with $v_{\mathrm{G}} = 4$ and



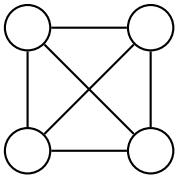
(a) c(G) = 1



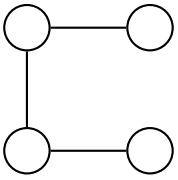
(b) c(G-e_1) = 2

Figure A.6: Connected components of a graph

$\varepsilon_G = 3$. If these edges should be weighted, missing edge weights can be derived from existing weights.



(a) Completely connected graph



(b) Connected graph

Figure A.7: Difference between connected and complete graphs

APPENDIX B

Histograms of Simulated Consistency Index Values

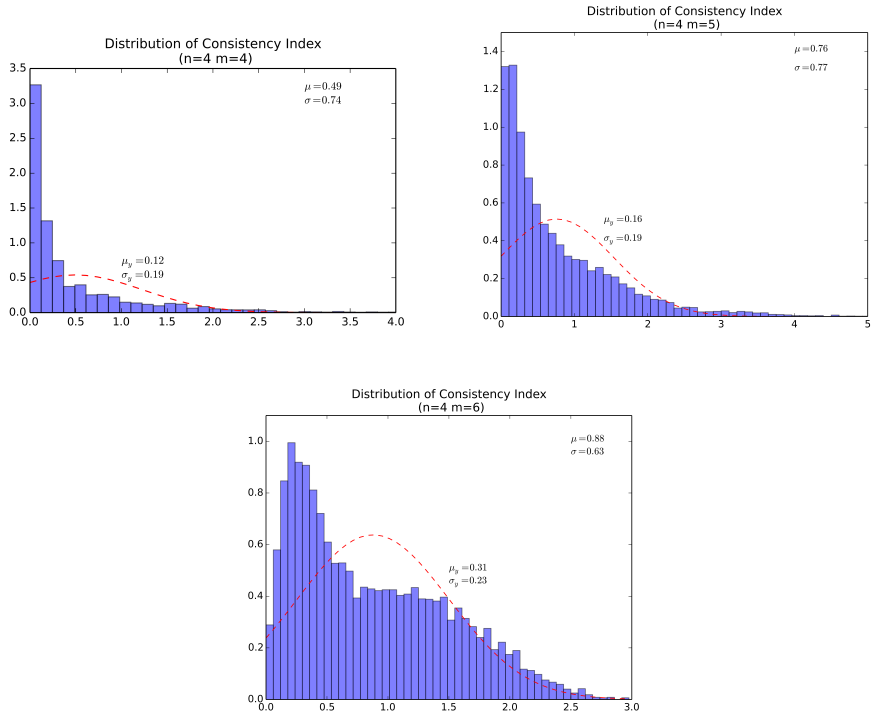


Figure B.1: Consistency index histograms for $n = 4$

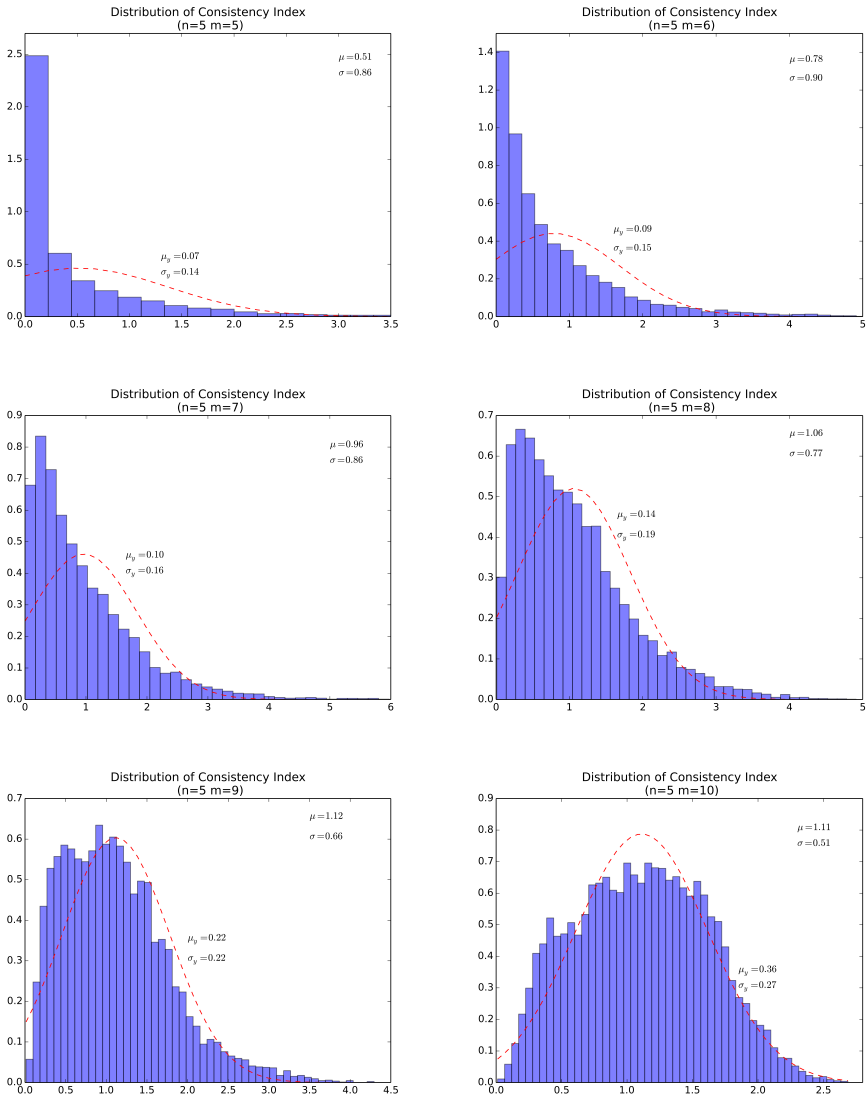


Figure B.2: Consistency index histograms for $n = 5$

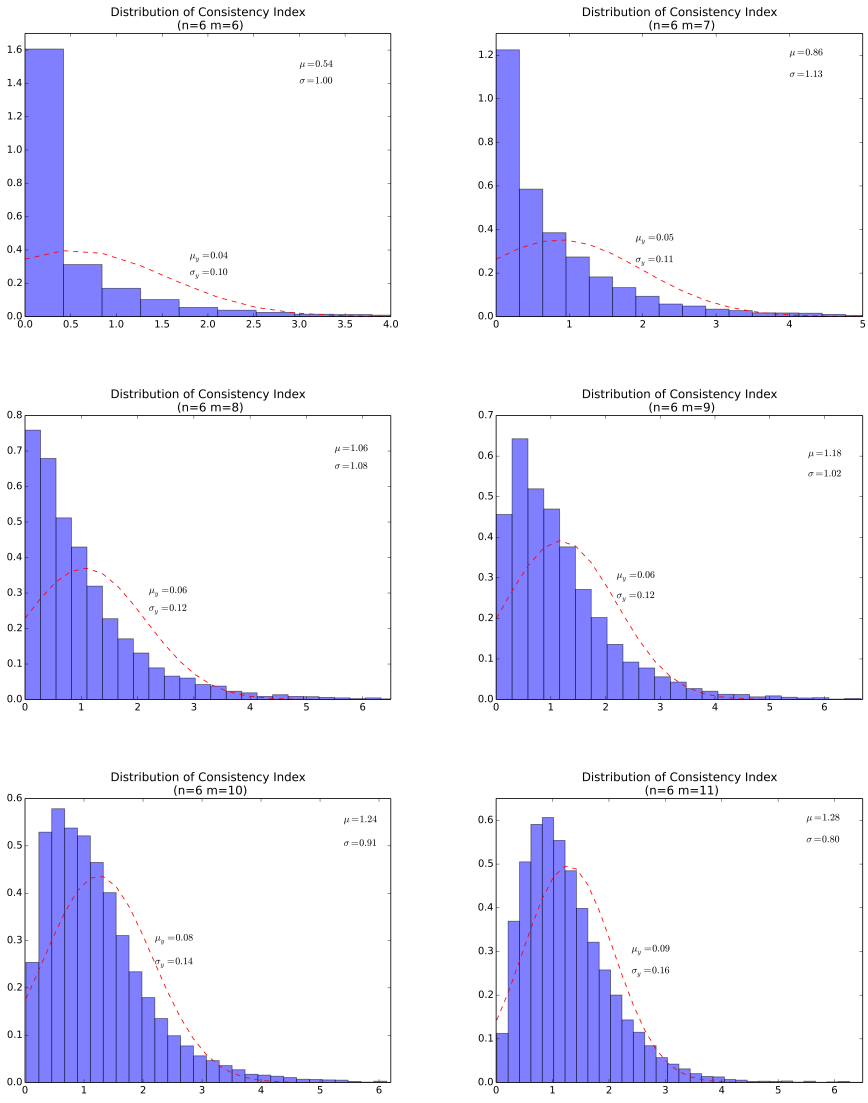


Figure B.3: Consistency index histograms for $n = 6$ (1)

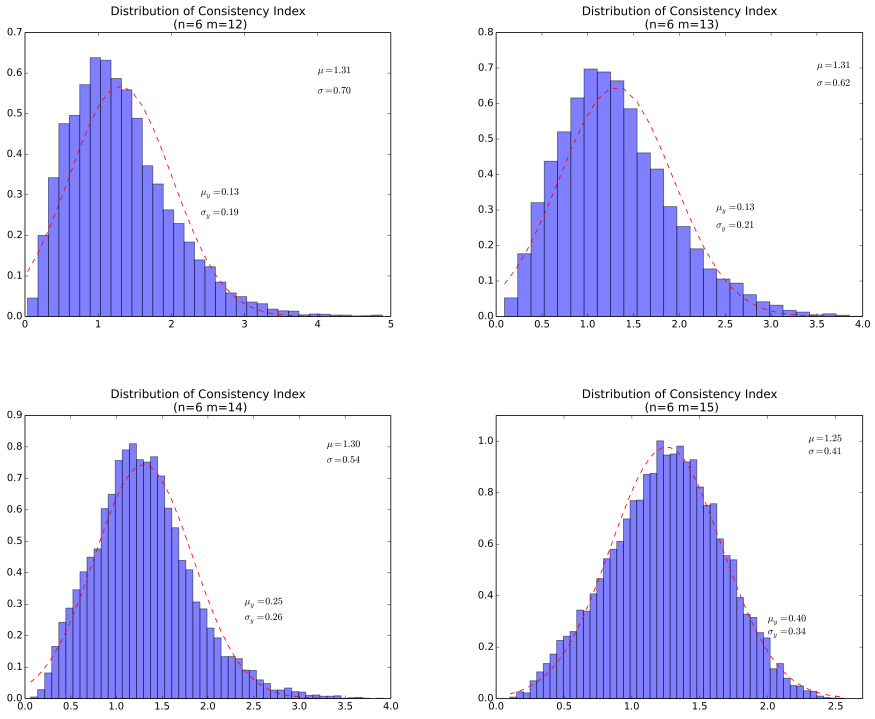


Figure B.4: Consistency index histograms for $n = 6$ (2)

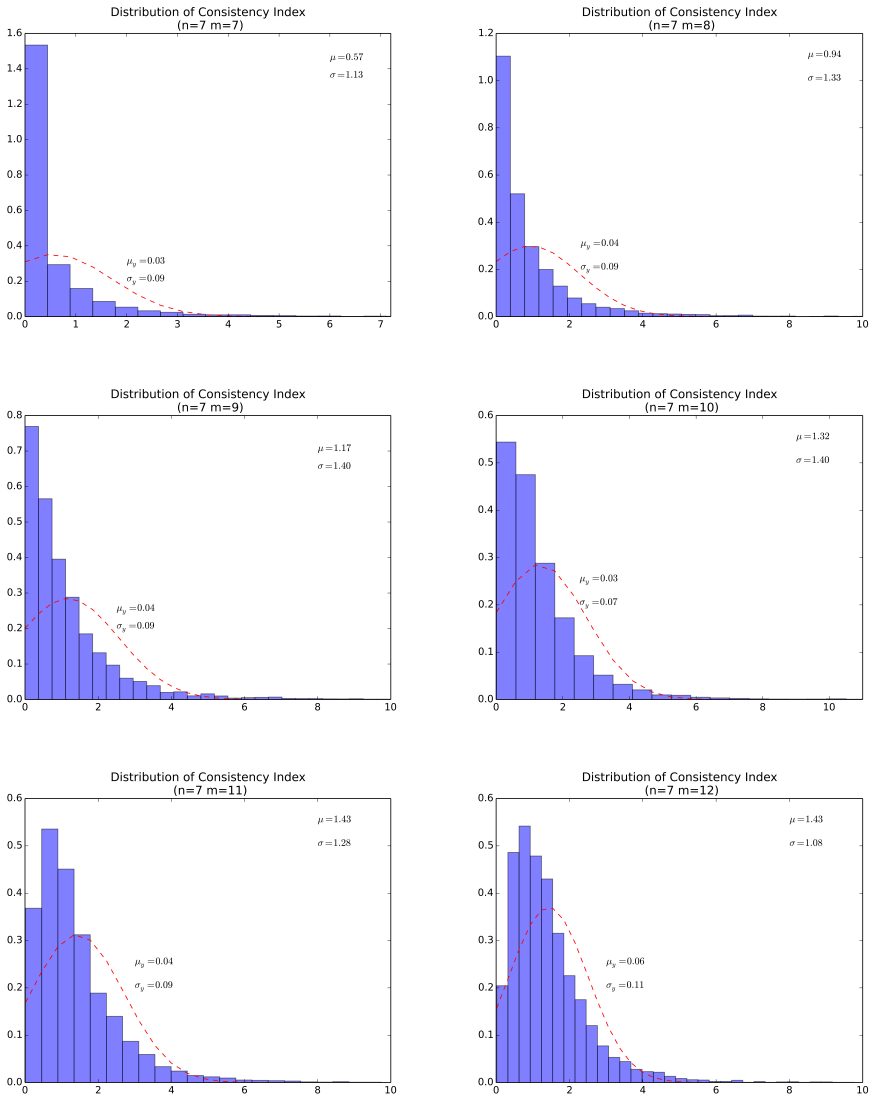


Figure B.5: Consistency index histograms for $n = 7$ (1)

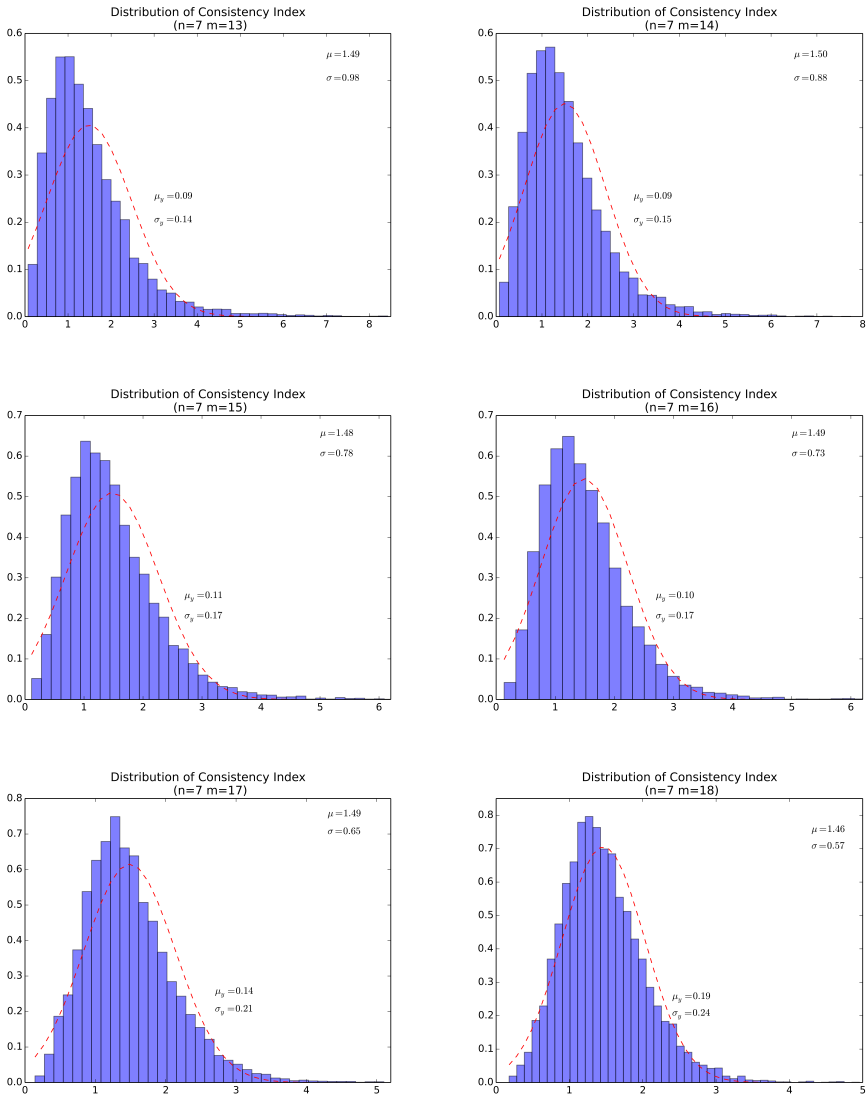


Figure B.6: Consistency index histograms for $n = 7$ (2)

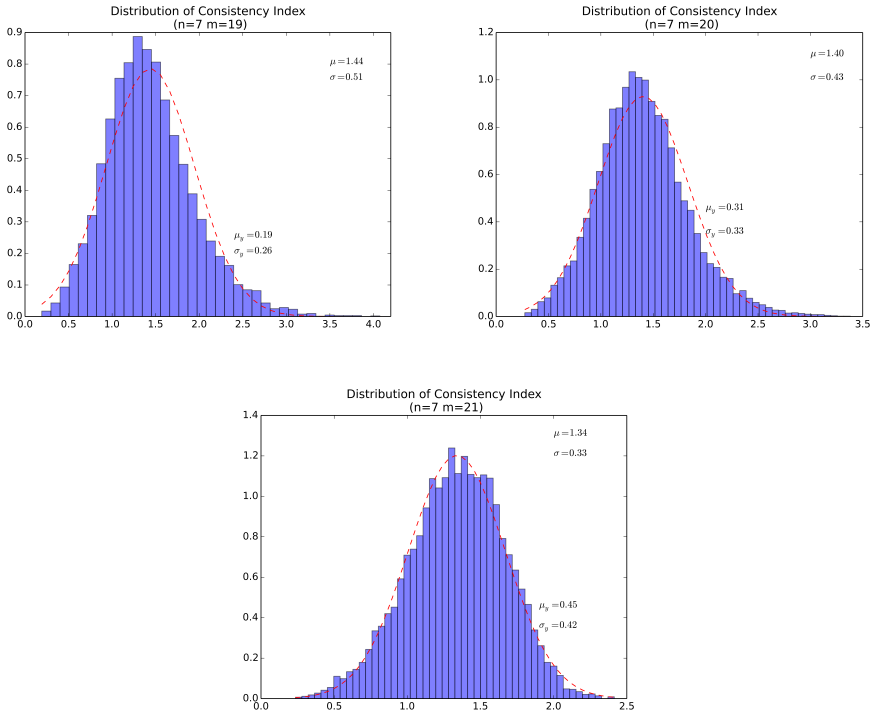


Figure B.7: Consistency index histograms for $n = 7$ (3)

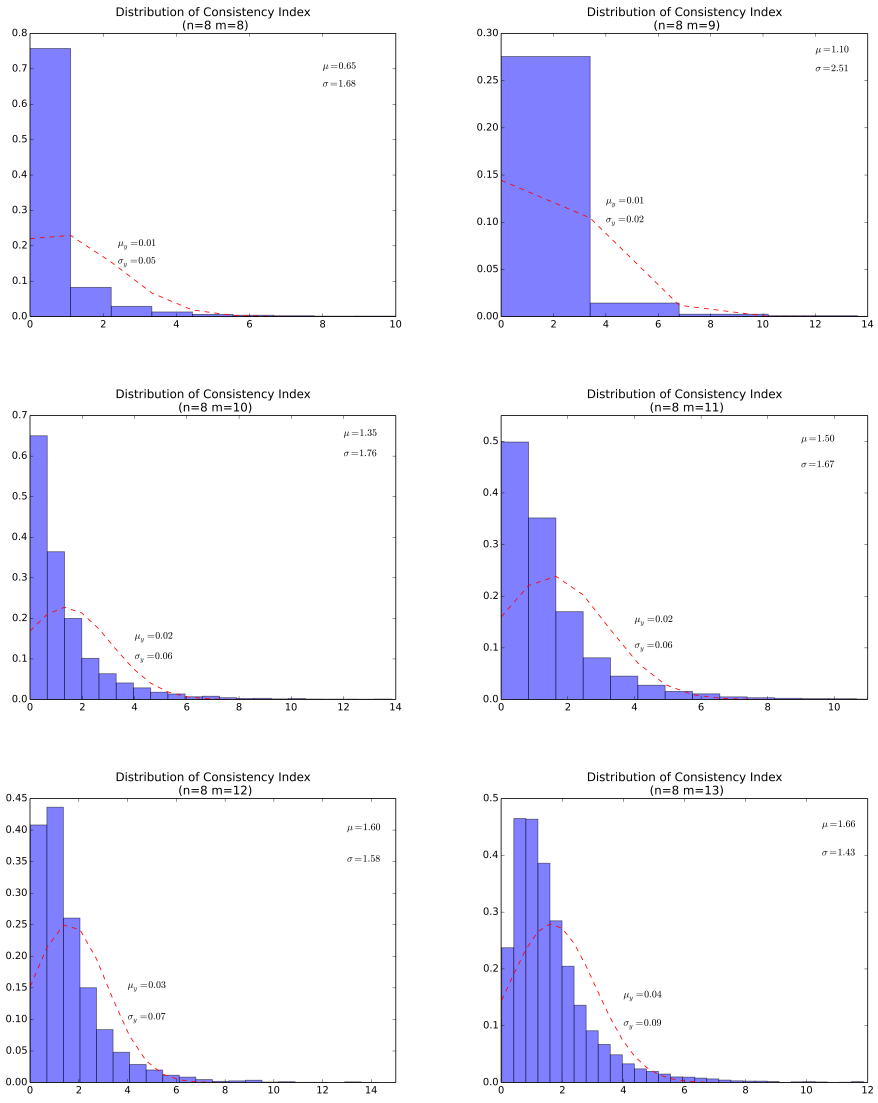


Figure B.8: Consistency index histograms for $n = 8$ (1)

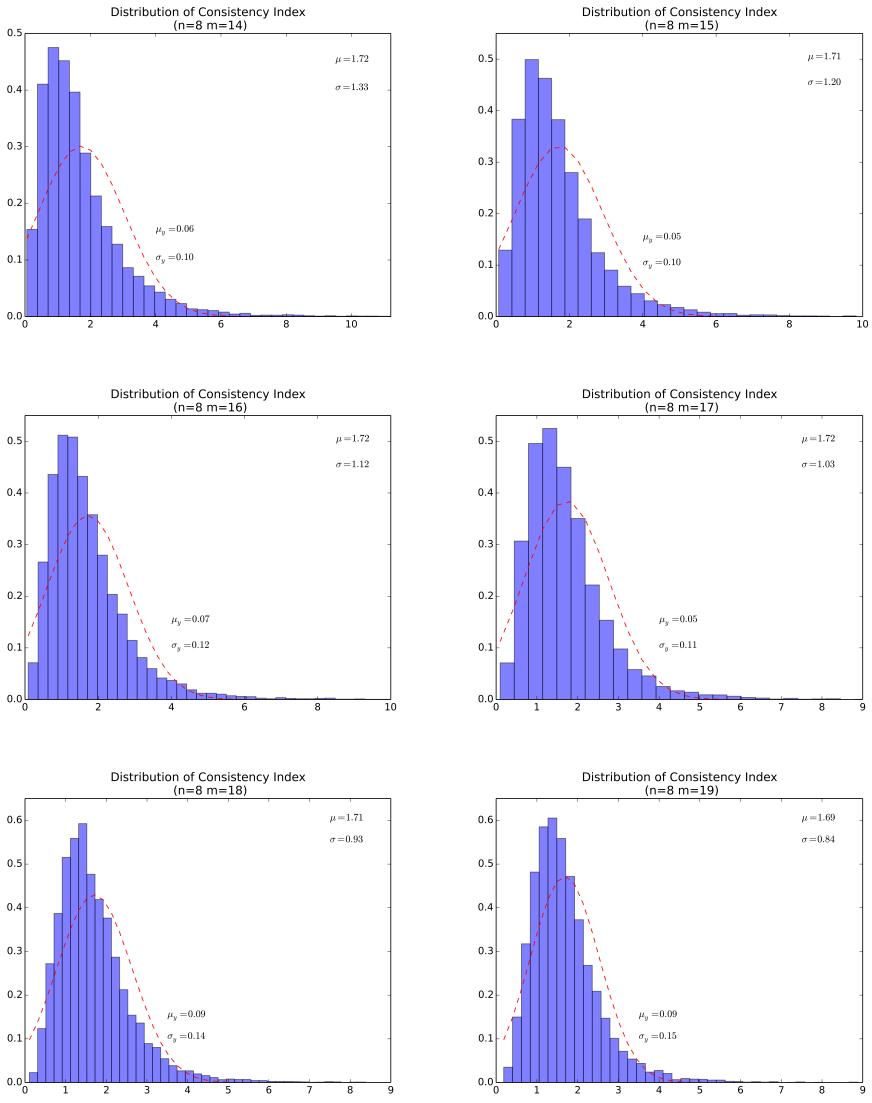


Figure B.9: Consistency index histograms for $n = 8$ (2)

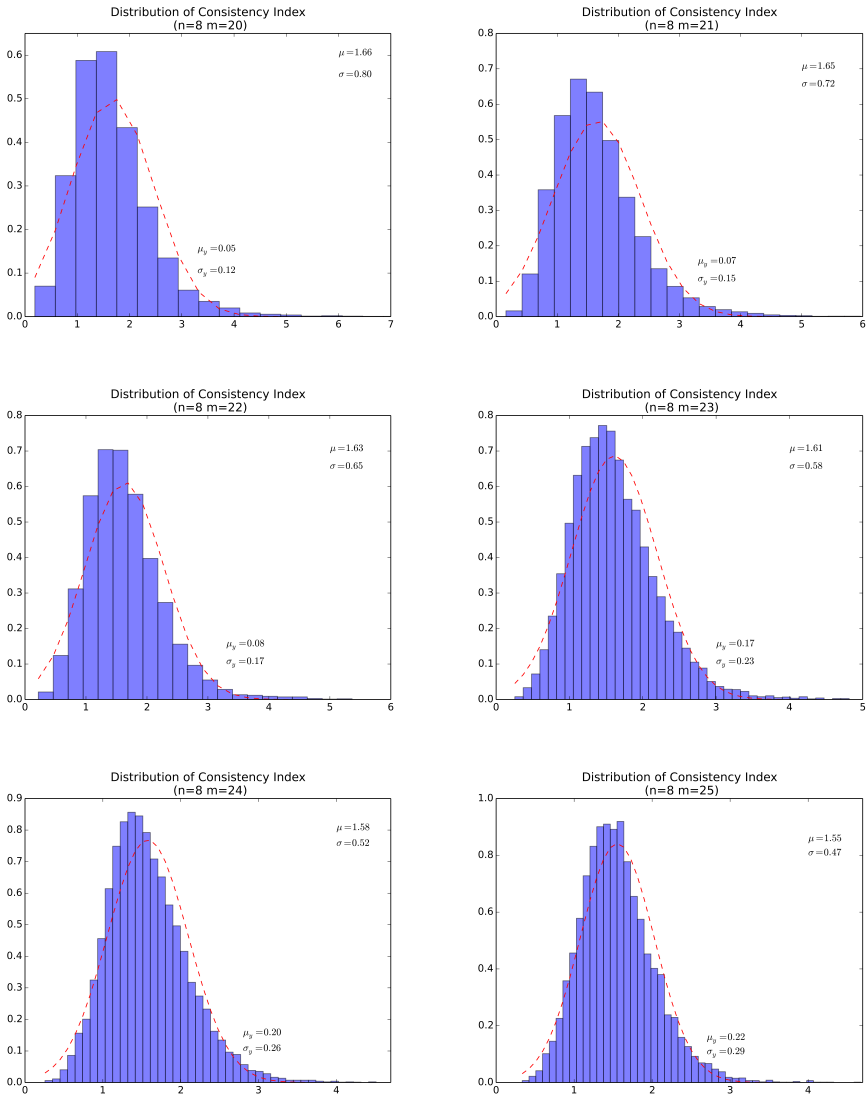


Figure B.10: Consistency index histograms for $n = 8$ (3)

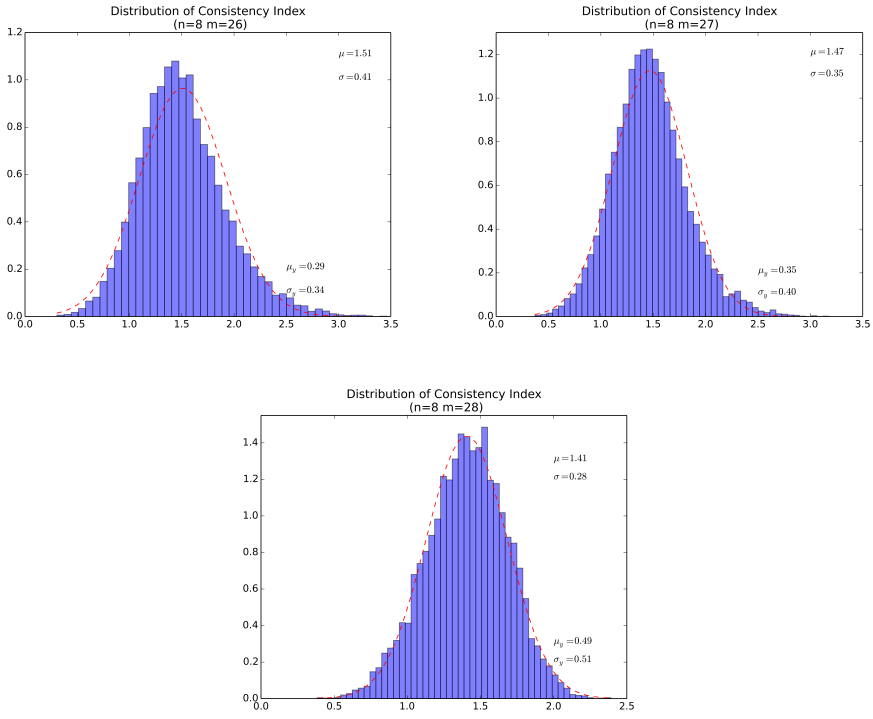


Figure B.11: Consistency index histograms for $n = 8$ (4)

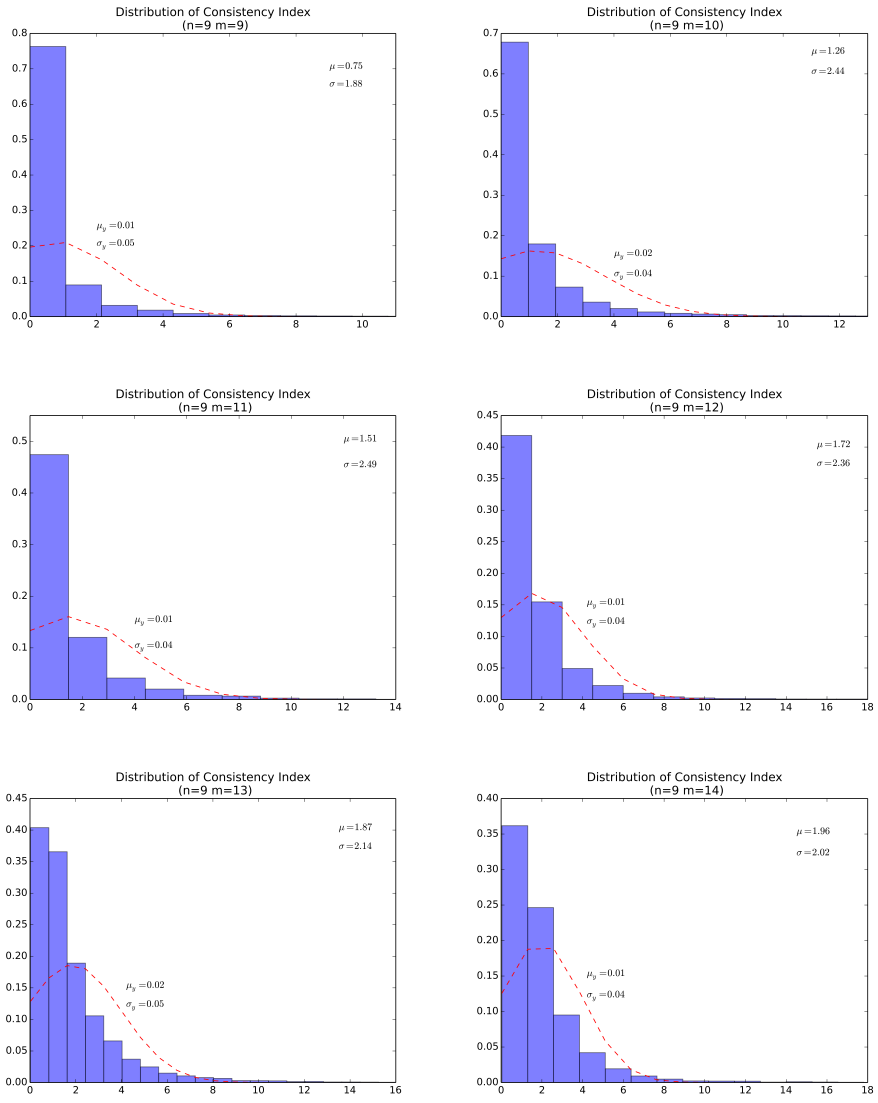


Figure B.12: Consistency index histograms for $n = 9$ (1)

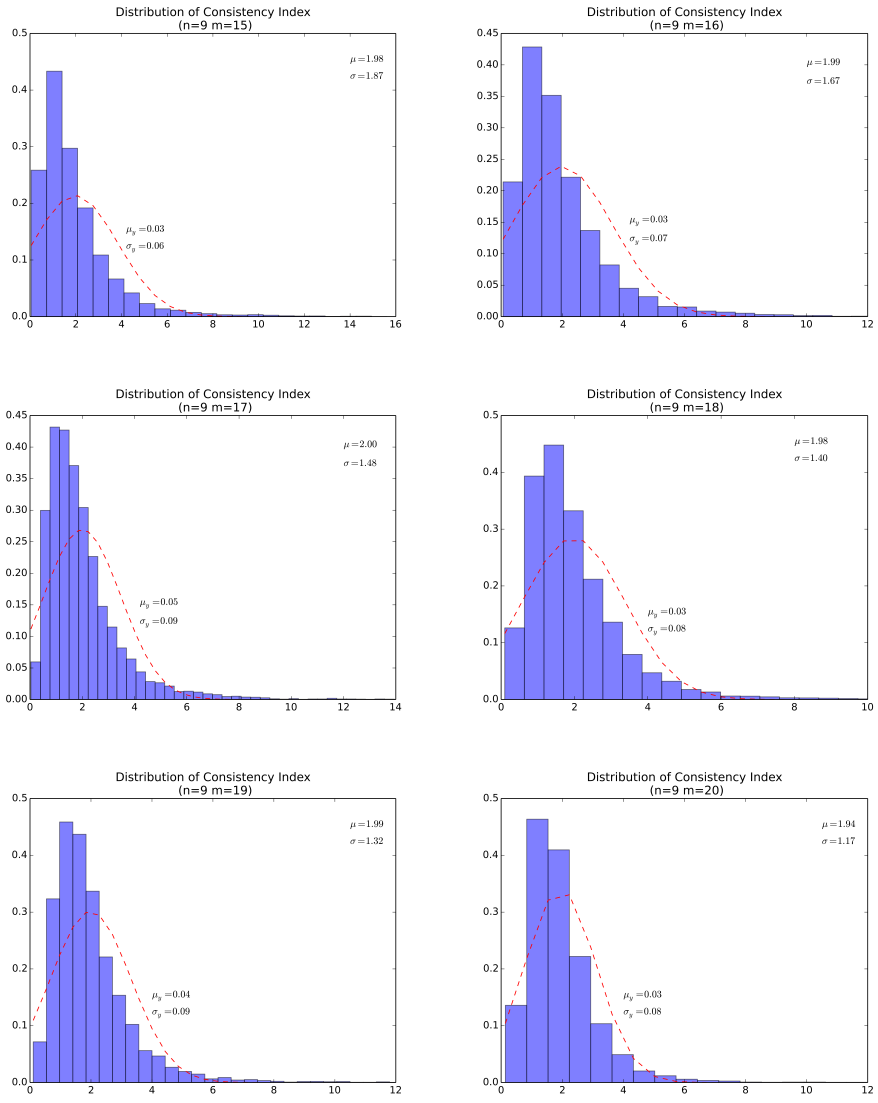


Figure B.13: Consistency index histograms for $n = 9$ (2)

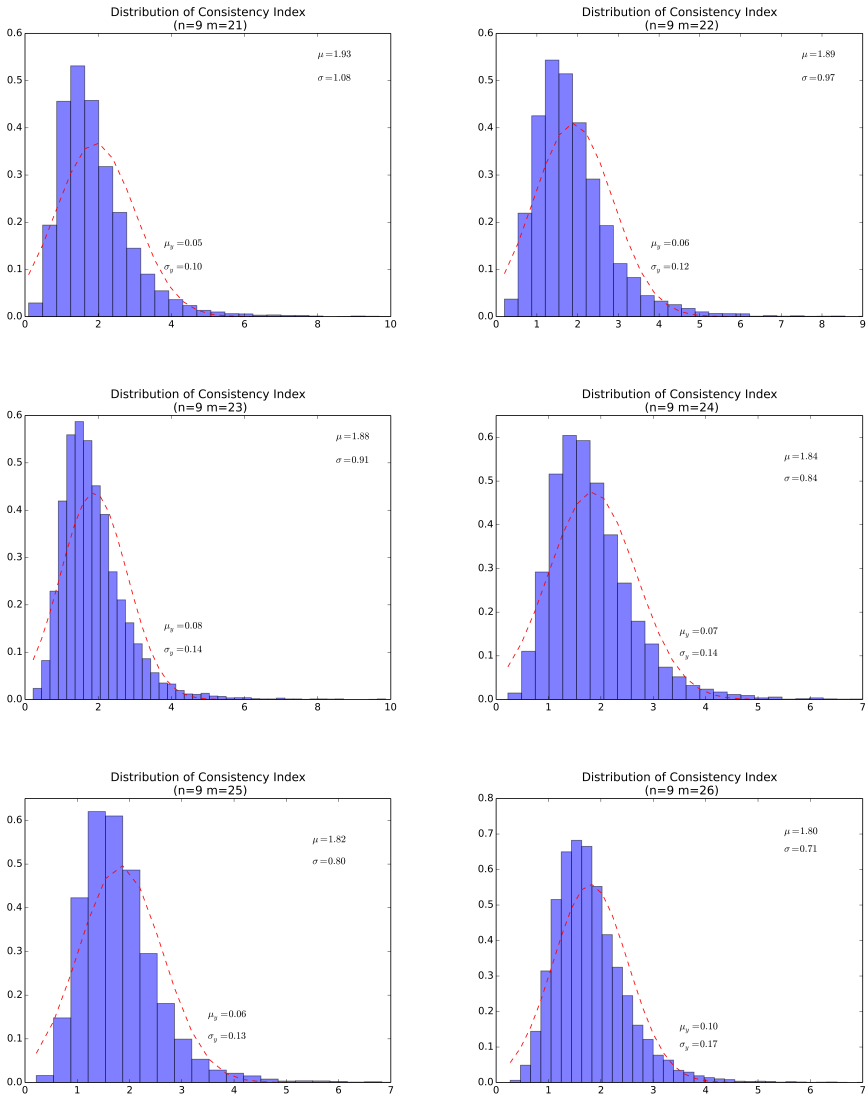


Figure B.14: Consistency index histograms for $n = 9$ (3)

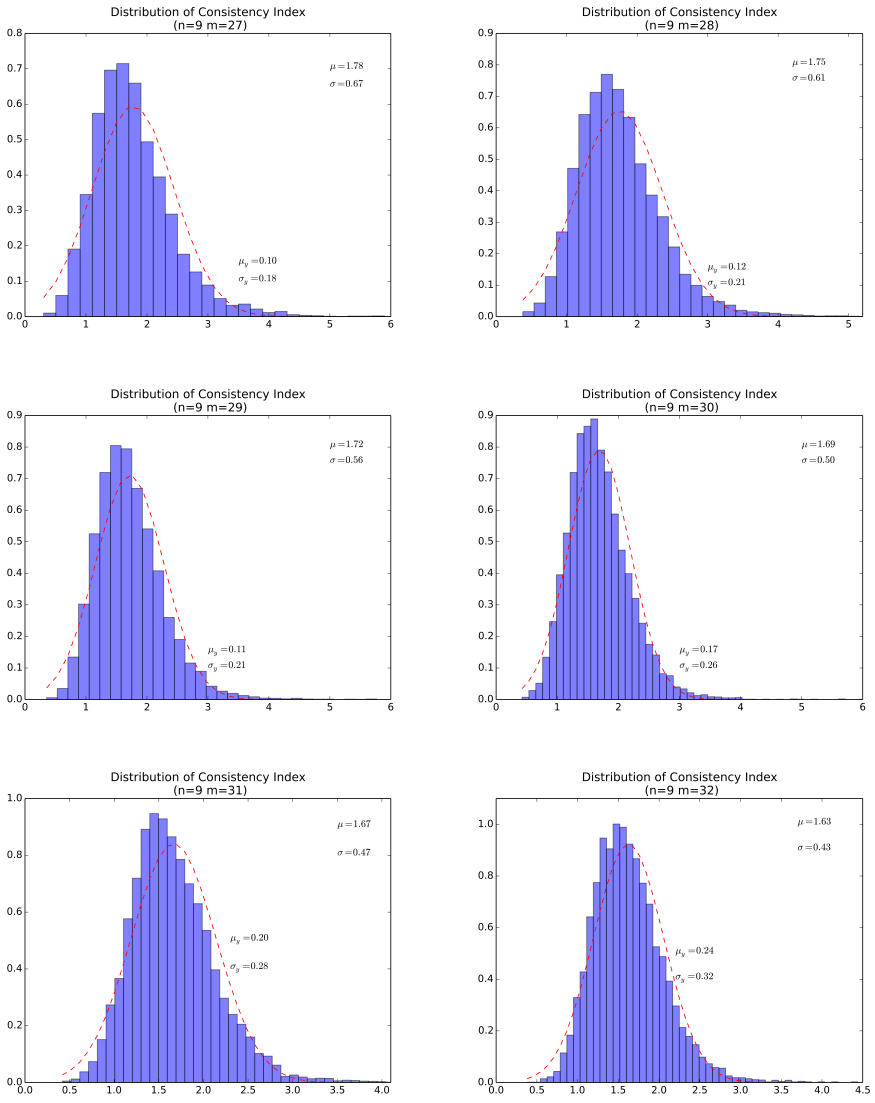


Figure B.15: Consistency index histograms for $n = 9$ (4)

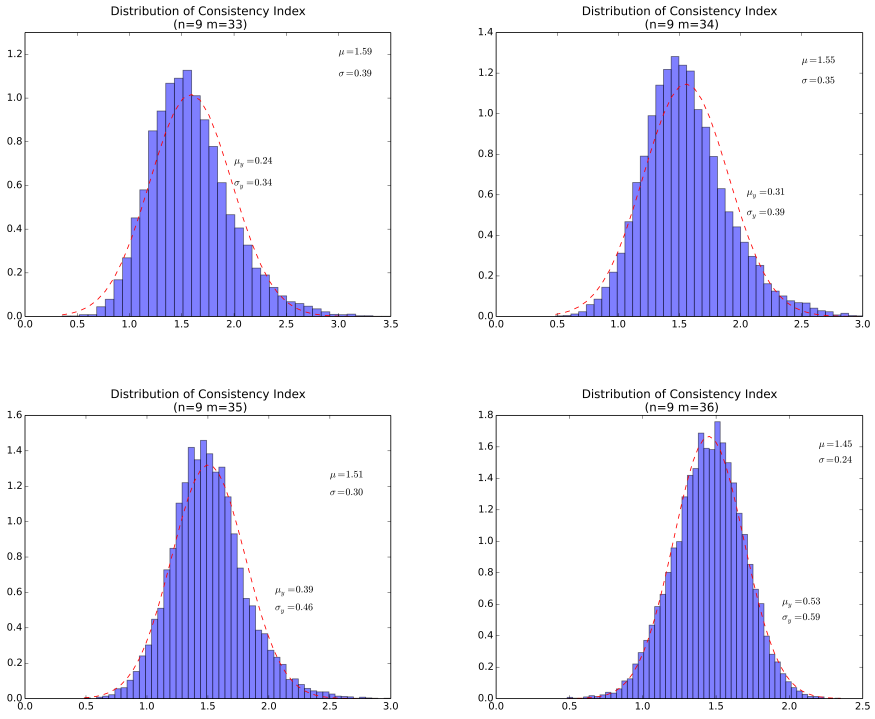


Figure B.16: Consistency index histograms for $n = 9$ (5)

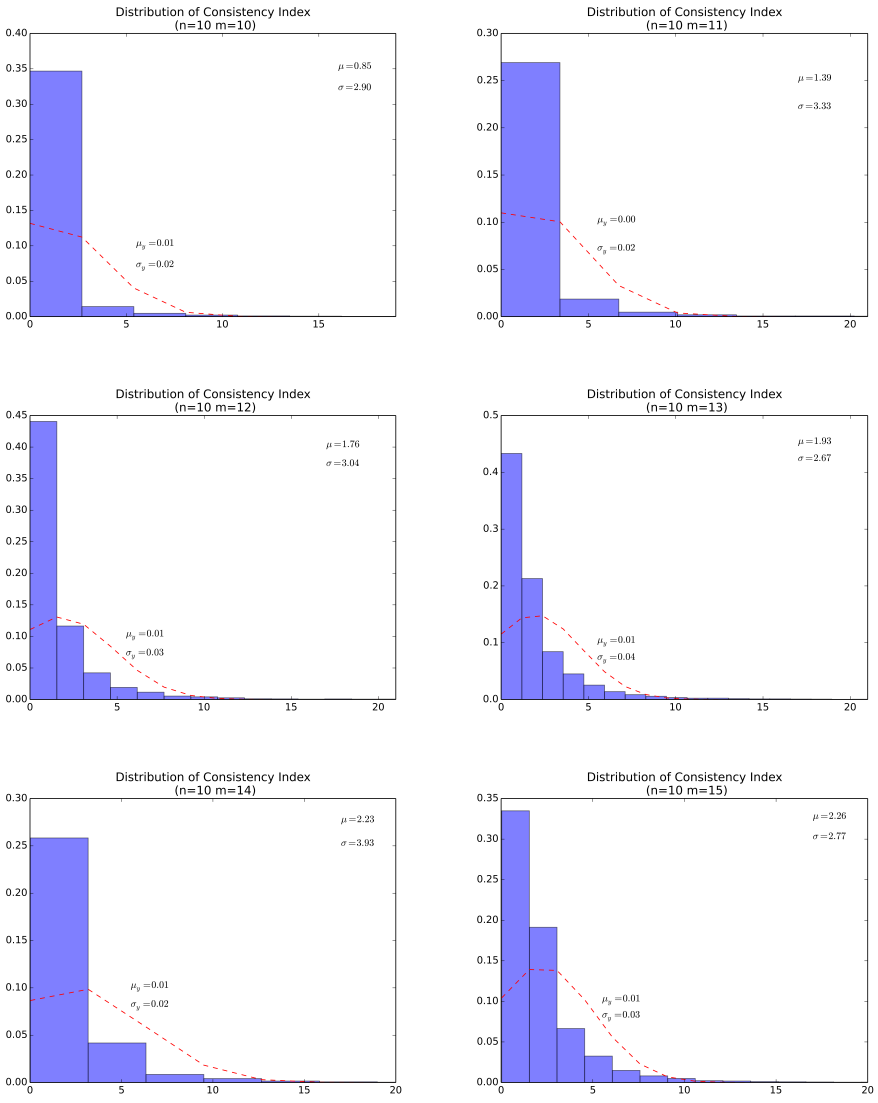


Figure B.17: Consistency index histograms for $n = 10$ (1)

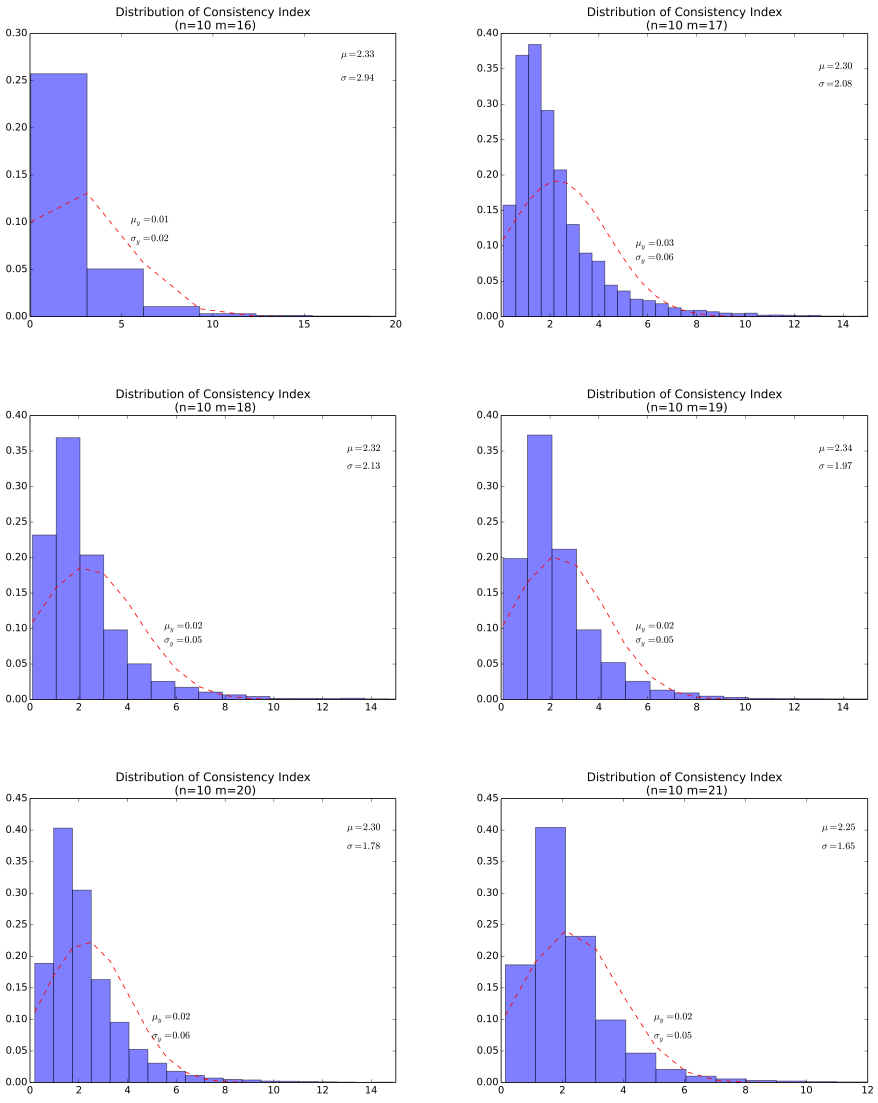


Figure B.18: Consistency index histograms for $n = 10$ (2)

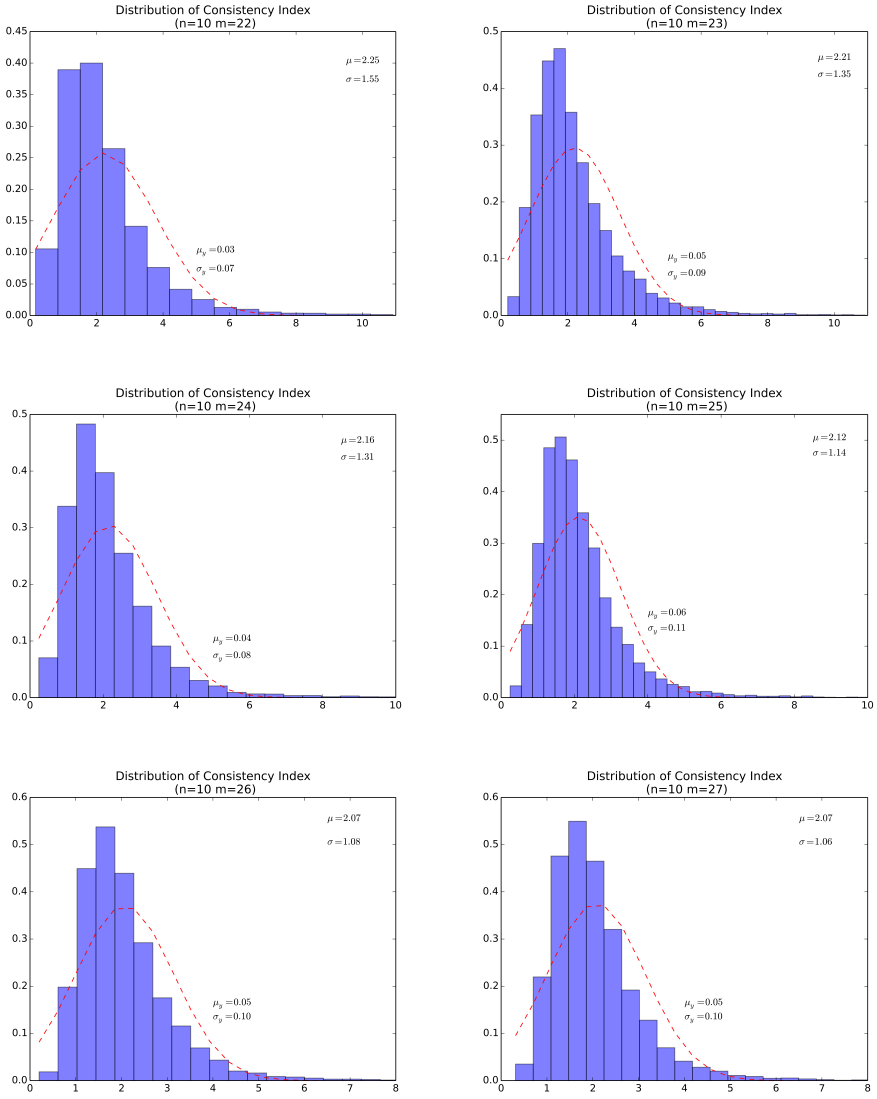


Figure B.19: Consistency index histograms for $n = 10$ (3)

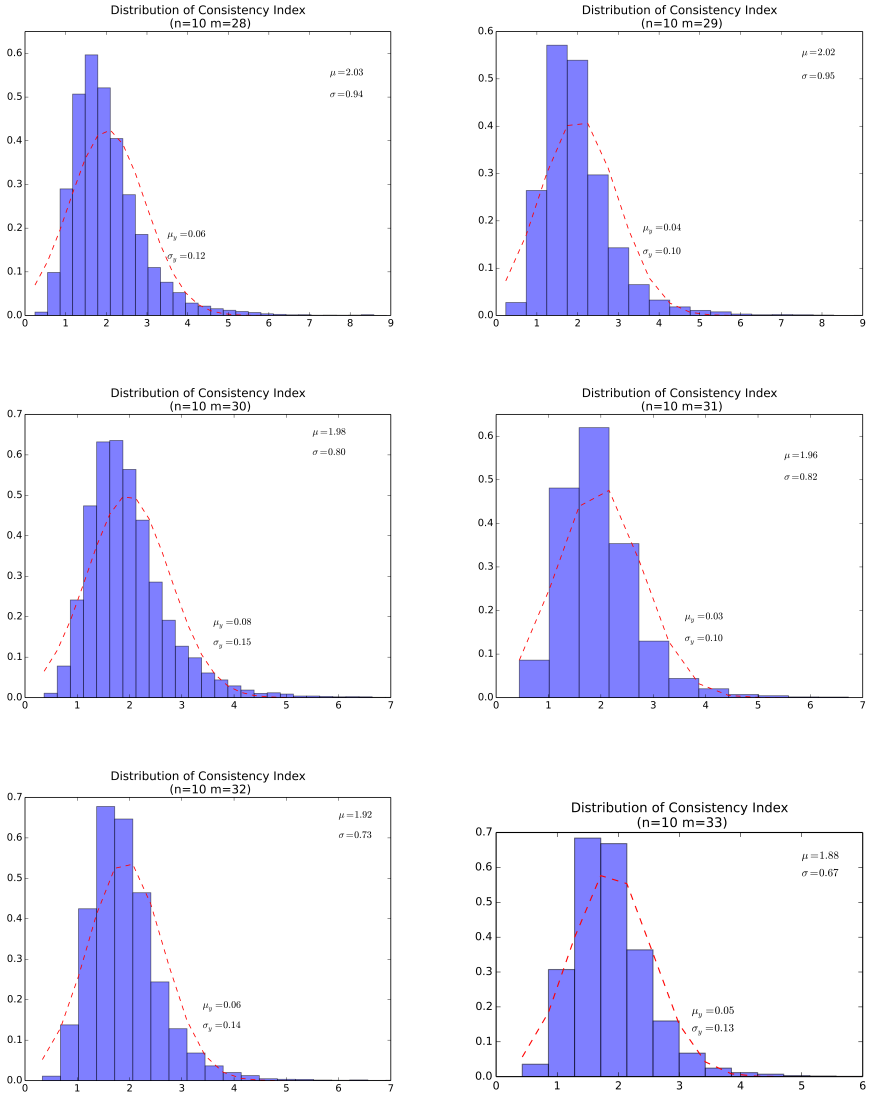


Figure B.20: Consistency index histograms for $n = 10$ (4)

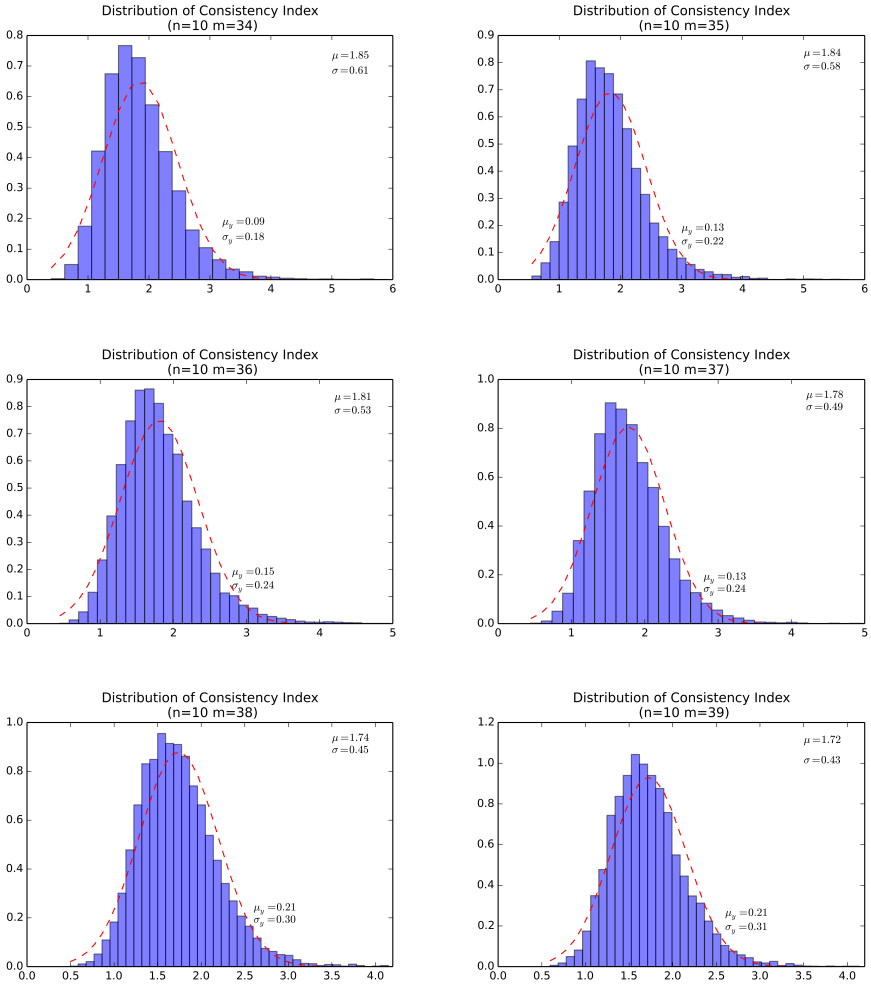


Figure B.21: Consistency index histograms for $n = 10$ (5)

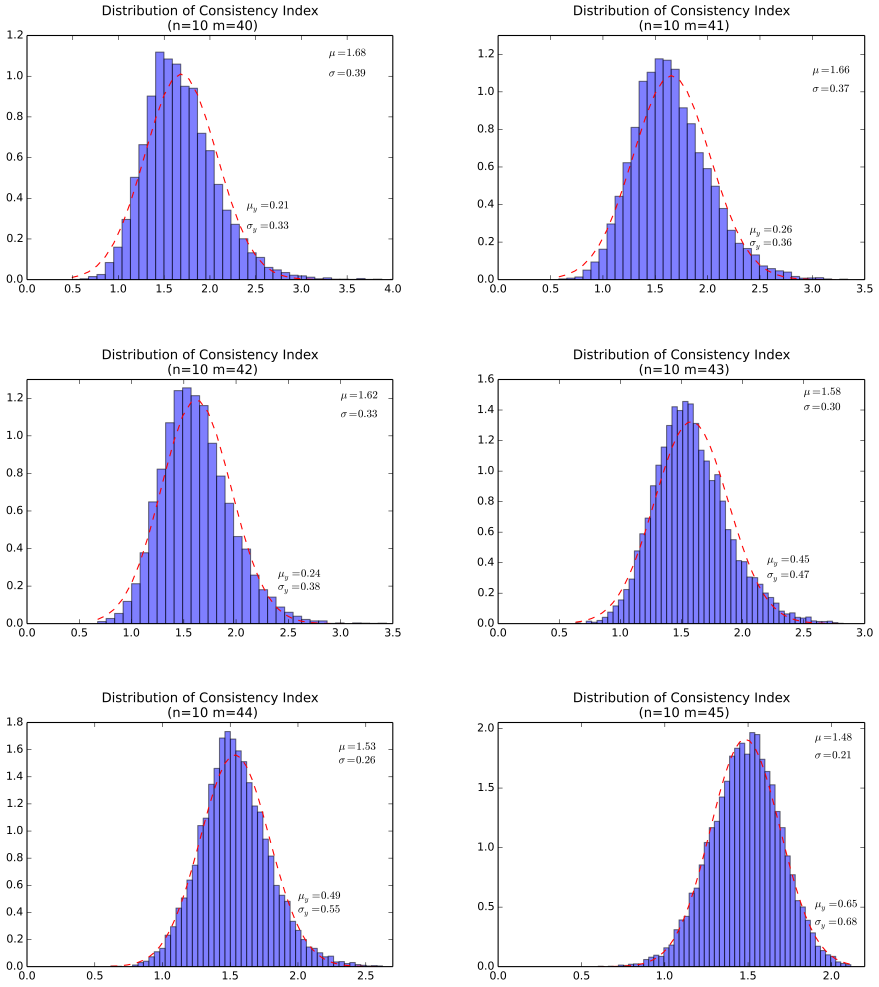


Figure B.22: Consistency index histograms for $n = 10$ (6)

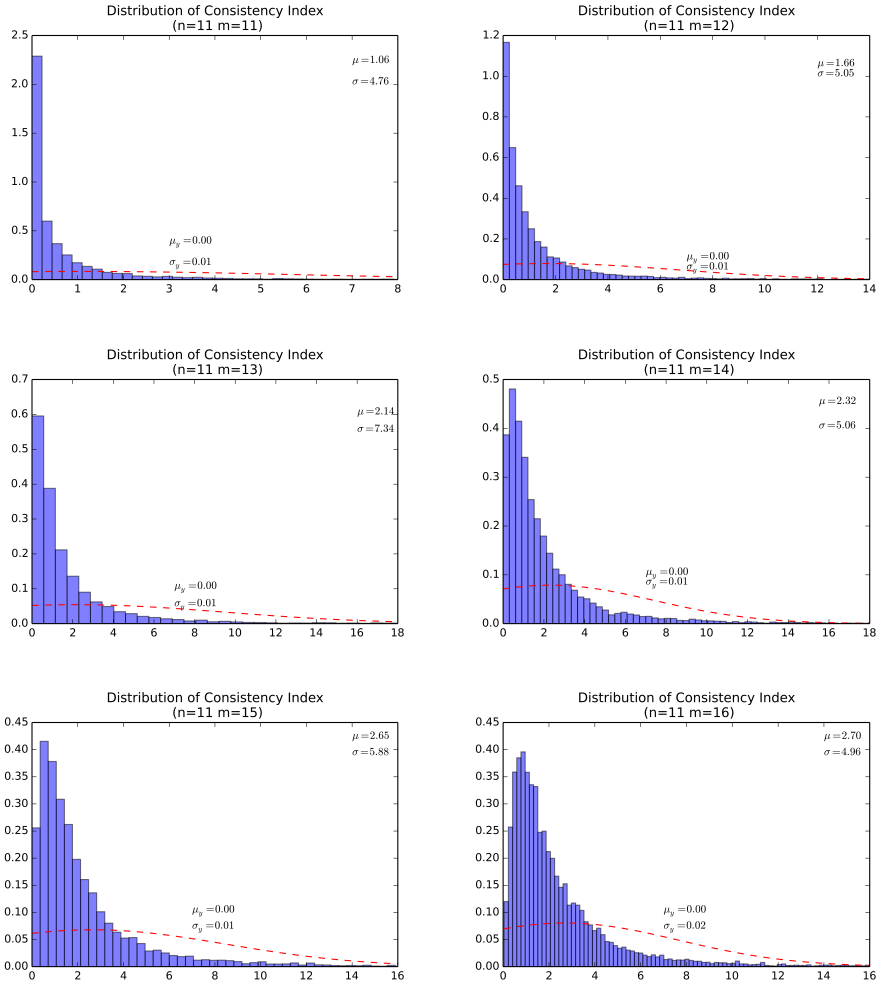


Figure B.23: Consistency index histograms for $n = 11$ (1)

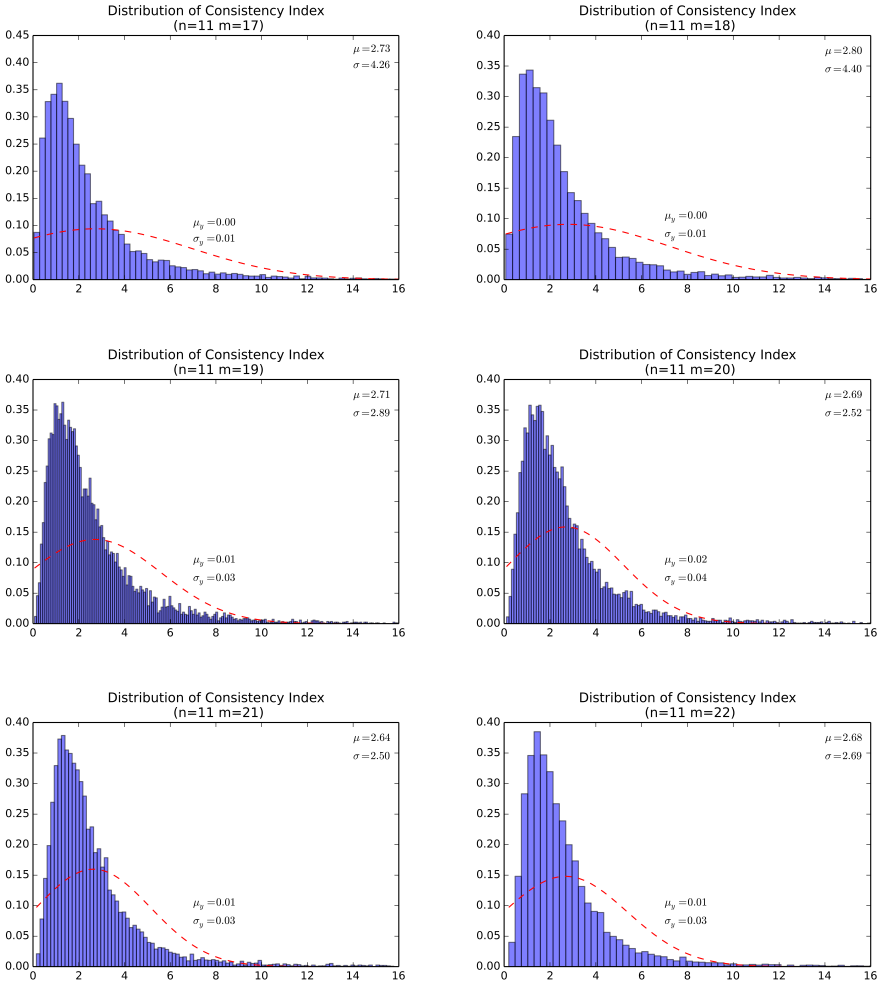


Figure B.24: Consistency index histograms for $n = 11$ (2)

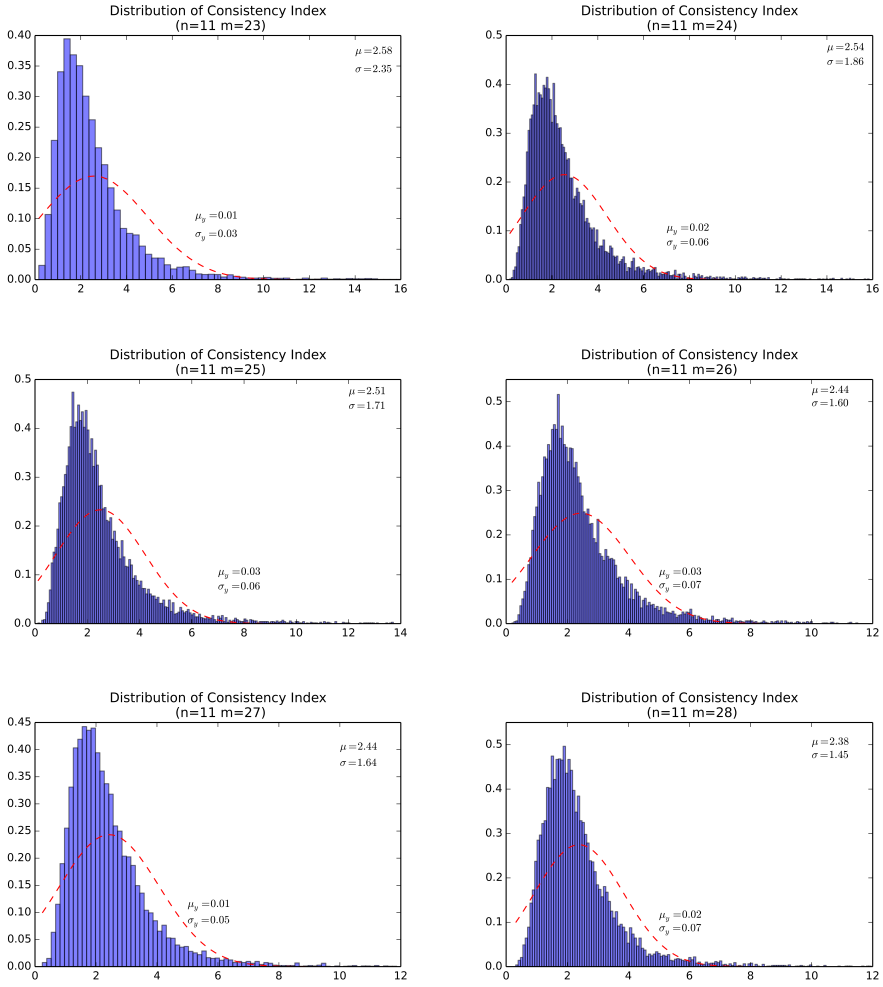


Figure B.25: Consistency index histograms for $n = 11$ (3)

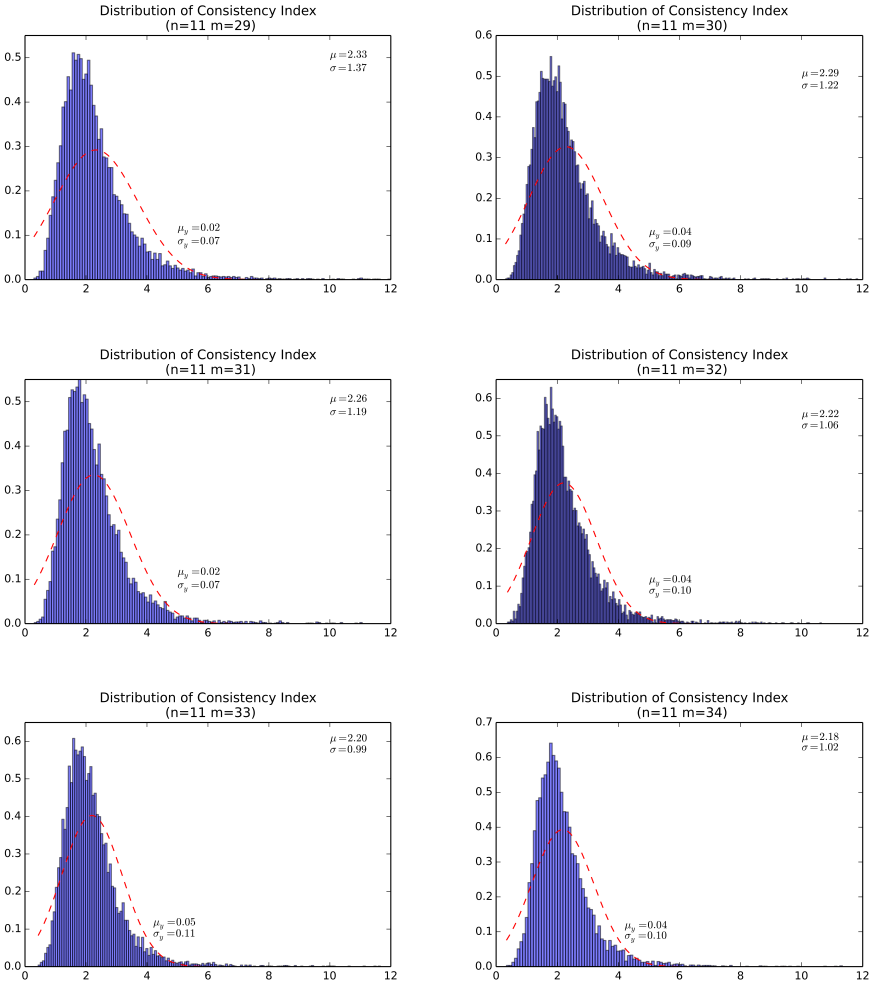


Figure B.26: Consistency index histograms for $n = 11$ (4)

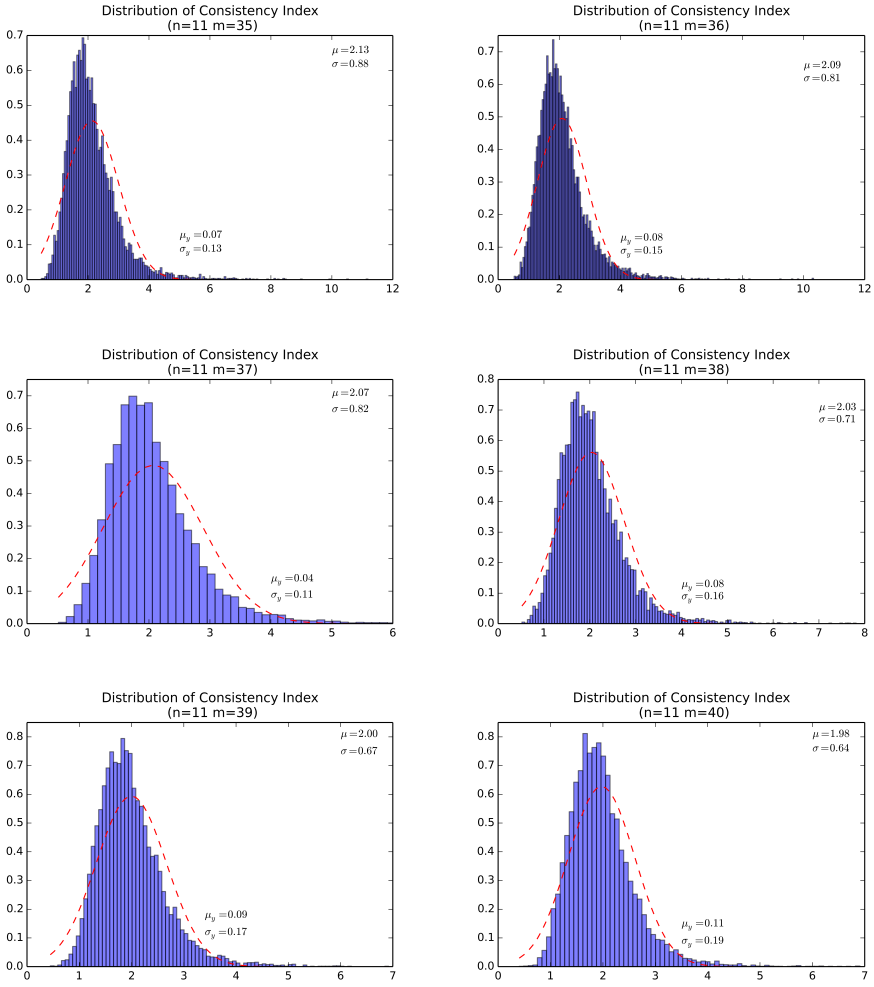


Figure B.27: Consistency index histograms for $n = 11$ (5)

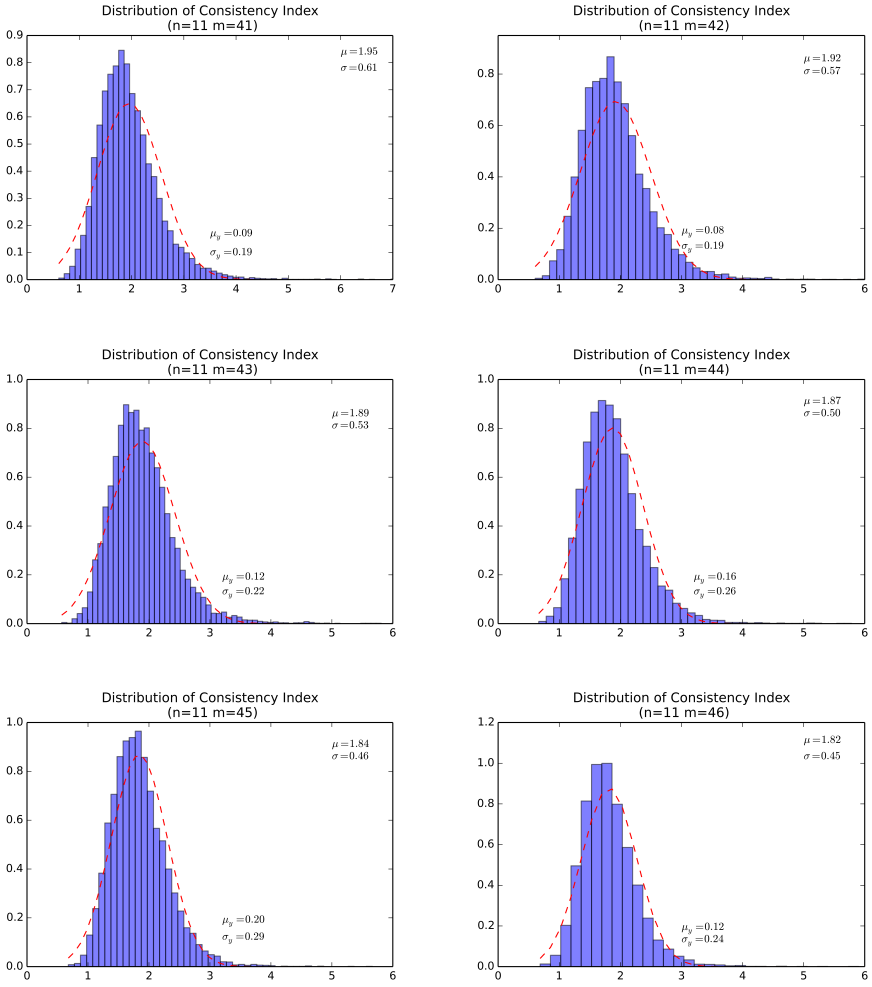


Figure B.28: Consistency index histograms for $n = 11$ (6)

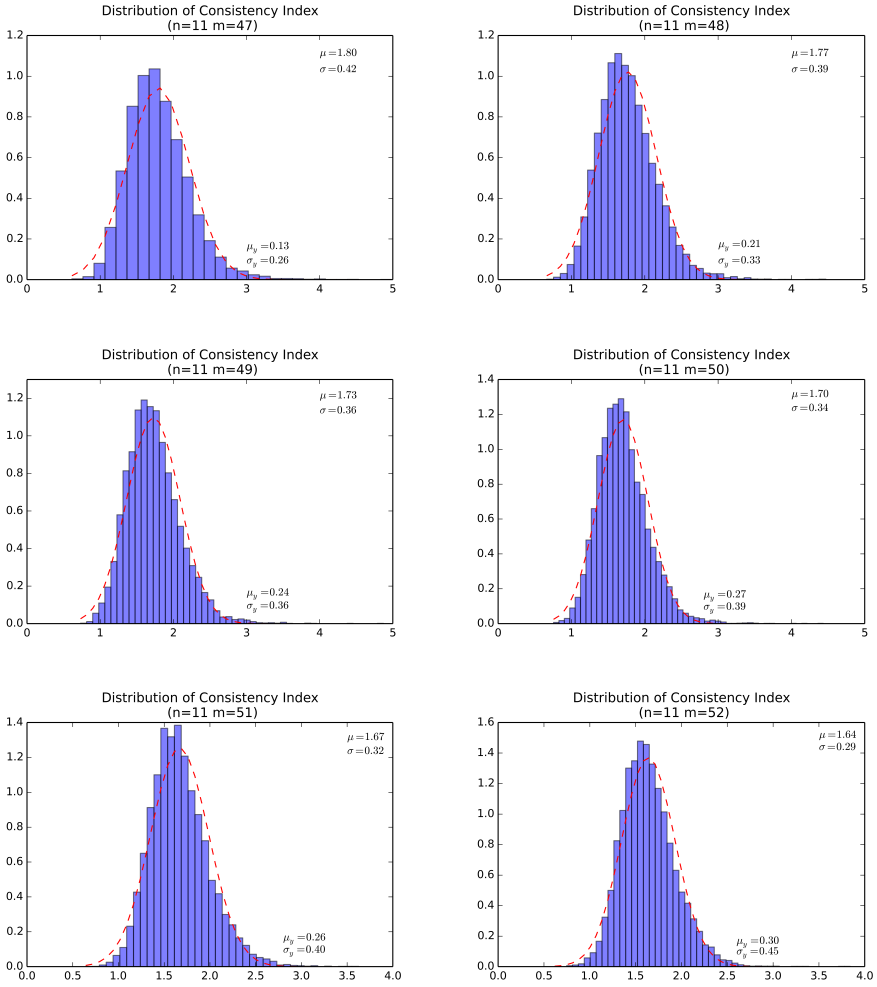


Figure B.29: Consistency index histograms for $n = 11$ (7)

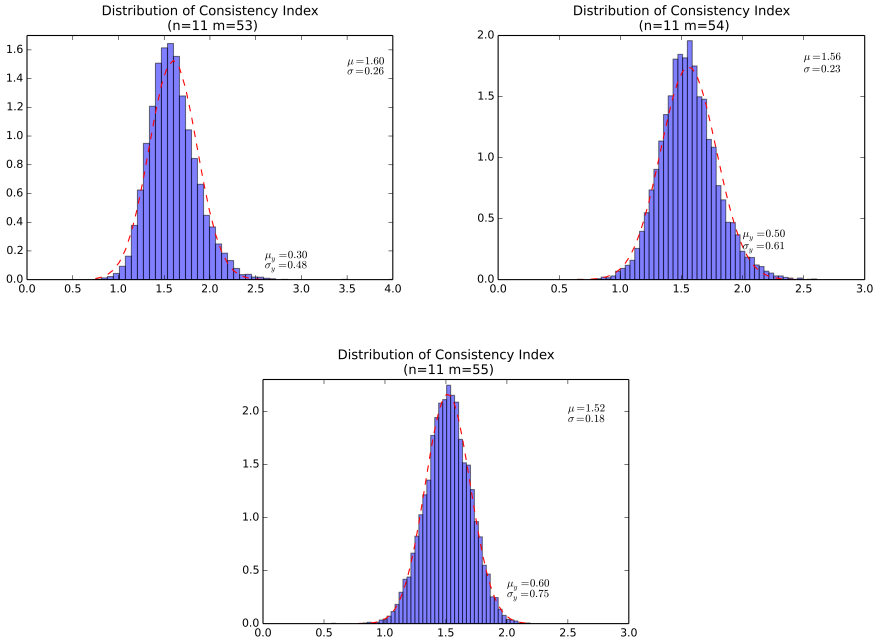


Figure B.30: Consistency index histograms for $n = 11$ (8)

APPENDIX C

Random Index Study

Table C.1: Random Index values for PCMs with the order of 4

# of PCs	Random Index	σ_{RI}
4	0,1238	0,1924
5	0,1561	0,1941
6	0,3052	0,2290

Table C.2: Random Index values for PCMs with the order of 5

# of PCs	Random Index	σ_{RI}
5	0,0675	0,1411
6	0,0930	0,1523
7	0,1021	0,1598
8	0,1410	0,1869
9	0,2196	0,2209
10	0,3615	0,2708

Table C.3: Random Index values for PCMs with the order of 6

# of PCs	Random Index	σ_{RI}
6	0,0360	0,0997
7	0,0505	0,1064
8	0,0618	0,1174
9	0,0612	0,1200
10	0,0834	0,1423
11	0,0929	0,1588
12	0,1336	0,1928
13	0,1331	0,2102
14	0,2519	0,2648
15	0,3980	0,3427

Table C.4: Random Index values for PCMs with the order of 7

# of PCs	Random Index	σ_{RI}
7	0,0335	0,0901
8	0,0405	0,0886
9	0,0442	0,0884
10	0,0294	0,0747
11	0,0400	0,0888
12	0,0596	0,1138
13	0,0860	0,1362
14	0,0931	0,1488
15	0,1144	0,1712
16	0,0970	0,1697
17	0,1441	0,2068
18	0,1871	0,2432
19	0,1859	0,2633
20	0,3144	0,3298
21	0,4495	0,4245

Table C.5: Random Index values for PCMs with the order of 8

# of PCs	Random Index	σ_{RI}
8	0,0136	0,0485
9	0,0051	0,0245
10	0,0248	0,0613
11	0,0210	0,0584
12	0,0257	0,0654
13	0,0448	0,0864
14	0,0551	0,0971
15	0,0508	0,0998
16	0,0683	0,1156
17	0,0541	0,1111
18	0,0939	0,1433
19	0,0877	0,1494
20	0,0492	0,1237
21	0,0735	0,1543
22	0,0793	0,1684
23	0,1707	0,2332
24	0,2020	0,2637
25	0,2250	0,2895
26	0,2852	0,3374
27	0,3491	0,396
28	0,4879	0,5079

Table C.6: Random Index values for PCMs with the order of 9

# of PCs	Random Index	σ_{RI}
9	0,0138	0,0459
10	0,0155	0,0418
11	0,0109	0,0354
12	0,0112	0,0366
13	0,0209	0,0507
14	0,0139	0,0436
15	0,0258	0,0595
16	0,0282	0,0655
17	0,0497	0,0868
18	0,0344	0,0780
19	0,0428	0,0878
20	0,0272	0,0775
21	0,0496	0,1040
22	0,0570	0,1170
23	0,0833	0,1393
24	0,0735	0,1406
25	0,0585	0,1319
26	0,0995	0,1726
27	0,0973	0,1786
28	0,1223	0,2052
29	0,1118	0,2101
30	0,1734	0,2593
31	0,1994	0,2815
32	0,2411	0,3151
33	0,2372	0,3381
34	0,3103	0,3943
35	0,3907	0,4610
36	0,5260	0,5860

Table C.7: Random Index values for PCMs with the order of 10

# of PCs	Random Index	σ_{RI}
10	0,0057	0,0242
11	0,0048	0,0208
12	0,0102	0,0307
13	0,0137	0,0375
14	0,0052	0,0196
15	0,0111	0,0333
16	0,0056	0,0236
17	0,0332	0,0615
18	0,0179	0,0473
19	0,0178	0,0492
20	0,0231	0,0582
21	0,0186	0,0549
22	0,0273	0,0673
23	0,0538	0,0944
24	0,0362	0,0828
25	0,0620	0,1101
26	0,0462	0,1017
27	0,0492	0,1054
28	0,0626	0,1238
29	0,0385	0,1006
30	0,0775	0,1479
31	0,0340	0,1040
32	0,0566	0,1373
33	0,0458	0,1318
34	0,0892	0,1829
35	0,1320	0,2186
36	0,1529	0,2410
37	0,1358	0,2442
38	0,2163	0,2970
39	0,2117	0,3083
40	0,2158	0,3285
41	0,2581	0,3634
42	0,2431	0,3823
43	0,4515	0,4684
44	0,4937	0,5491
45	0,6523	0,6758

Table C.8: Random Index values for PCMs with the order of 11

# of PCs	Random Index	σ_{RI}
11	0,0027	0,0128
12	0,0027	0,0124
13	0,0011	0,0066
14	0,0023	0,0115
15	0,0019	0,0099
16	0,0039	0,0151
17	0,0030	0,0146
18	0,0025	0,0130
19	0,0097	0,0307
20	0,0158	0,0410
21	0,0108	0,0349
22	0,0060	0,0256
23	0,0063	0,0279
24	0,0248	0,0585
25	0,0254	0,0616
26	0,0285	0,0671
27	0,0126	0,0463
28	0,0239	0,0658
29	0,0224	0,0660
30	0,0363	0,0862
31	0,0237	0,0726
32	0,0395	0,0963
33	0,0545	0,1141
34	0,0438	0,1030
35	0,0666	0,1324
36	0,0805	0,1494
37	0,0378	0,1089
38	0,0827	0,1626
39	0,0923	0,1748
40	0,1135	0,1943
41	0,0939	0,1865
42	0,0837	0,1854

Table C.8: Continued

# of PCs	Random Index	σ_{RI}
43	0,1165	0,2197
44	0,1600	0,2564
45	0,1958	0,287
46	0,1162	0,2428
47	0,1299	0,2645
48	0,2108	0,3286
49	0,2363	0,3579
50	0,2660	0,3869
51	0,2626	0,4049
52	0,3040	0,4497
53	0,2984	0,4830
54	0,5003	0,6056
55	0,6024	0,7473

CURRICULUM VITAE

PERSONAL INFORMATION

Name Surname	Şahika KOYUN
Date of Birth and Place	February 17, 1988 - Samsun
Foreign Languages	English, Japanese

EDUCATION

Degree	**Field/Department**	**School/University**	**Date of Graduation**
Master's	Dept. of Industrial Eng.	Yildiz Technical University	-
Undergraduate	Dept. of Industrial Eng.	Yildiz Technical University	2010
Highschool	Science	Milli Piyango Anatolian High.	2006

WORK EXPERIENCE

Year	**Corporation/Institute**	**Enrollment**
2014- ...	Yildiz Technical University, Department of Industrial Engineering	Research Assistant
2012-2014	Yildiz Technical University, Graduate School of Natural and Applied Sciences	Research Assistant
Dec 2011-Jan 2012	Lojitek Consulting and Training Servisces	Logistics Specialist
Jan 2011	Center for European Union Education and Youth Programs	Intern

WORK EXPERIENCE

May 2010-Aug 2010	Adidas Spor Malzemeleri Satis ve Pazarlama	Intern
June 2009-Aug 2009	Stratejitek Corporate Consultancy Services	Intern
Aug 2008-Sept 2008	Deloitte Turkey	Intern
June 2007-Aug 2007	Borsan Elektrik Malzemeleri San. Tic. Ltd.	Intern

PUBLISHMENTS

Conference Papers

1. Koyun, Ş. and Çetinsaya Özkır, V., (2014), 'A new algortihm for dealing with inconsistency in decision making problems', FRICO 2014 - Future Research in Combinatorial Optimization, Magdeburg, Germany

2. Koyun, Ş. and Çetinsaya Özkır, V., (2014), 'Constructing highly consistent pairwise comparison matrices in the AHP', International Symposium on the Analytic Hierarchy Process 2014, Washington DC, USA

3. Koyun, Ş. and Çetinsaya Özkır, V., (2013), 'Multi-criteria cold warehouse location selection', 26$^{\text{th}}$ European Conference on Operations Research, Rome, Italy

4. Koyun, Ş. and Çetinsaya Özkır, V., (2013), 'Performance evaluation of Yildiz Technical University Graduate School Services by using Quality Function Deployment', International IIE Conference and YA/EM 2013, Istanbul, Turkey