★ Solve the system of differential equations

$$\frac{dy}{dx} = 2y - z + 1 \quad (1)$$

$$\frac{dz}{dx} = y + e^{2x} \quad (2) \quad \text{using Elimination method.}$$

$$(D-2)\,y + z = 1$$

$$D-2 \quad -y + Dz = e^{2x}$$

$$+ \quad \overline{(1 + D^2 - 2D)\,z = 1 + 2e^{2x} - 2e^{2x}} \Rightarrow (D-1)^2 z = 1 \quad \left(z'' - 2z' + z = 1\right)$$

$$(r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \quad \Rightarrow z_h = c_1 e^x + c_2 x e^x$$

$$z_p = a \,,\; z_p' = z_p'' = 0 \quad \Big\} \quad 0 - 0 + a = 1 \Rightarrow a = 1$$

$$\boxed{z = c_1 e^x + c_2 x e^x + 1}$$

$$y = Dz - e^{2x} = c_1 e^x + c_2 e^{2x} + c_2 x e^{2x} - e^{2x} = \boxed{(c_1 + c_2 + c_2 x)e^{2x} - e^{2x}}$$

★ Using the Derivation and elimination, solve the system

$$\frac{dx}{dt} + 2y - x = 0 \qquad (D-1)\,x + 2y = 0$$

$$\frac{dy}{dt} + y - x = -\sin t \qquad (D-1) \quad -x + (D+1)\,y = -\sin t$$

$$+ \quad \overline{(D^2 - 1 + 2)\,y = -\cos t + \sin t} \quad \left[\, y'' + y = -\cos t + \sin t \,\right]$$

$$r^2 + 1 = 0 \Rightarrow r_{1,2} = \mp i \qquad y_h = c_1 \cos t + c_2 \sin t$$

$$y_p = (A\cos t + B\sin t)\,t$$

$$y_p' = (-A\sin t + B\cos t)\,t + (A\cos t + B\sin t)$$

$$y_p'' = (-A\cos t - B\sin t)\,t + 2(-A\sin t + B\cos t)$$

$$(\cancel{-A\cos t} - B\sin t)t + 2(-A\sin t + B\cos t) + (\cancel{A\cos t} + B\sin t)t = -\cos t + \sin t$$

$$A = -\frac{1}{2} \qquad B = -\frac{1}{2}$$

$$y_p = -\frac{1}{2} t(\cos t + \sin t)$$

$$\boxed{y = c_1 \cos t + c_2 \sin t - \frac{t}{2}(\cos t + \sin t)} \qquad \boxed{x = c_1(\cos t - \sin t) + c_2(\cos t + \sin t) - \frac{\cos t}{2} + \frac{\sin t}{2} - t\cos t}$$

* solve the system $\left.\begin{aligned} 2\frac{dx}{dt}+\frac{dy}{dt}+x+5y=4t \\ \frac{dx}{dt}+\frac{dy}{dt}+2x+2y=2 \end{aligned}\right\}$ using determinant method.

$$\left.\begin{aligned}(2D+1)x+(D+5)y=4t \\ (D+2)x+(D+2)y=2\end{aligned}\right\} \quad \Delta=\begin{vmatrix}2D+1 & D+5 \\ D+2 & D+2\end{vmatrix}=D^2-2D-8$$

$$x=\frac{\Delta_1}{\Delta}=\frac{\begin{vmatrix}4t & D+5 \\ 2 & D+2\end{vmatrix}}{D^2-2D-8} \Rightarrow (D^2-2D-8)x=8t-6$$

$$\left.\begin{aligned}r^2-2r-8=0 \\ -4\ \ +2\end{aligned}\right\} \begin{aligned} r_1=4 \\ r_2=-2\end{aligned} \qquad x_h=c_1e^{4t}+c_2e^{-2t}$$

$$x_p=at+b,\ x_p'=a,\ x_p''=0 \ \left\} \begin{aligned} -2a-8at-8b=8t-6 \\ \boxed{a=-1}\ \ -2a-8b=-6 \\ b=1\end{aligned} \right\} x_p=-t+1$$

$$\boxed{x=c_1e^{4t}+c_2e^{-2t}-t+1}$$

$$y=\frac{\Delta_2}{\Delta}=\frac{\begin{vmatrix}2D+1 & 4t \\ D+2 & 2\end{vmatrix}}{D^2-2D-8} \Rightarrow (D^2-2D-8)y=-8t-2$$

$$y_h=c_3e^{4t}+c_4e^{-2t}$$

$$y_p=at+b,\ y_p'=a,\ y_p''=0\ \left\} \begin{aligned} -2a-8at-8b=-8t-2 \\ a=1,\ \ b=0\end{aligned}\right\} y_p=t$$

$$y=c_3e^{4t}+c_4e^{-2t}+t$$

$$4c_1e^{4t}-2c_2e^{-2t}\cancel{-1}+4c_3e^{4t}-2c_4e^{-2t}\cancel{+1}+2c_1e^{4t}+2c_2e^{-2t}\cancel{-2t}\cancel{+2}+2c_3e^{4t}+2c_4e^{-2t}$$
$$+\cancel{2t}=\cancel{2}$$

$$\underbrace{[4c_1+4c_3+2c_1+2c_3]}_{=0}e^{4t}+\underbrace{[-2c_2-2c_4+2c_2+2c_4]}_{c_2=c_4}e^{-2t}=0$$

$c_3=-c_1$

?

* Find the solution of the differential equation $y''+16y=5\sin x$ which possesses the initial conditions $y'(0)=y(0)=0$, by using the Laplace Transform.

$$L\{y''\}+16L\{y\}=L\{5\sin x\}$$

$$s^2Y(s)-s\underbrace{y(0)}_{0}-\underbrace{y'(0)}_{0}+16Y(s)=\frac{5}{s^2+1}$$

$$(s^2+16)Y(s)=\frac{5}{s^2+1} \Rightarrow Y(s)=\frac{5}{(s^2+1)(s^2+16)}$$

$$\underbrace{L^{-1}\{Y(s)\}}_{y(t)}=5L^{-1}\underbrace{\left\{\frac{1}{(s^2+1)(s^2+16)}\right\}} \longrightarrow \left(\frac{1}{s^2+1}-\frac{1}{s^2+16}\right)\frac{1}{15}$$

$$\frac{5}{15}\left[L^{-1}\left\{\frac{1}{s^2+1}\right\}-L^{-1}\left\{\frac{1}{s^2+16}\right\}\right]$$

$$y(t)=\frac{1}{3}\left[\sin x-\frac{1}{4}\sin 4x\right]$$

* solve the initial value problem $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=te^{-t}$ $y(0)=1$, $y'(0)=2$ using Laplace and inverse laplace transform.

$$L\{y''\}+2L\{y'\}+L\{y\}=L\{te^{-t}\}$$

$$s^2Y(s)-s\underbrace{y(0)}_{1}-\underbrace{y'(0)}_{2}+2sY(s)-2\underbrace{y(0)}_{1}+Y(s)=\frac{1}{(s+1)^2}$$

$$(s^2+2s+1)Y(s)=\frac{1}{(s+1)^2}+(s+4) \Rightarrow Y(s)=\frac{1}{(s+1)^4}+\frac{s+4}{(s+1)^2}$$

$$\underbrace{L^{-1}\{Y(s)\}}_{y(t)}=\underbrace{L^{-1}\left\{\frac{1}{(s+1)^4}\right\}}_{\frac{t^3}{3!}e^{-t}}+\underbrace{L^{-1}\left\{\frac{s+1}{(s+1)^2}\right\}}_{e^{-t}}+\underbrace{L^{-1}\left\{\frac{3}{(s+1)^2}\right\}}_{3te^{-t}}$$

$$y(t)=\frac{t^3}{3!}e^{-t}+e^{-t}+3te^{-t}$$

☆ Evaluate the integral $\int_0^\infty e^{-st}(1+t\sin t)\,dt$ by using Laplace transform definition and properties of Laplace transform.

$$\int_0^\infty e^{-st}(1+t\sin t)\,dt = L\{1+t\sin t\} = L\{1\}+L\{t\sin t\}$$

$$= \frac{1}{s} + (-1)\frac{d}{ds}L\{\sin t\} = \frac{1}{s} - \frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{s} + \frac{2s}{(s^2+1)^2}$$

☆ Using Laplace transform, find the solution of the initial value problem $y''+2y'+5y=0$, $y(0)=2$, $y'(0)=-1$

$$s^2Y(s) - s\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{-1} + 2[sY(s) - \underbrace{y(0)}_{2}] + 5Y(s) = 0$$

$$(s^2+2s+5)Y(s) = 2s+3 \Rightarrow Y(s) = \frac{2s+3}{s^2+2s+5}$$

$$\underbrace{L^{-1}\{Y(s)\}} = L^{-1}\left\{\frac{2s+3}{s^2+2s+5}\right\}$$

$$y(t) = L^{-1}\left\{\frac{2s+3}{(s+1)^2+4}\right\} = 2L^{-1}\left\{\frac{s+\frac{3}{2}}{(s+1)^2+4}\right\} = 2\left\{\frac{s+1}{(s+1)^2+4}\right\} + 2\left\{\frac{\frac{1}{2}}{(s+1)^2+4}\right\}$$

$$y(t) = 2e^{-t}\cos 2t + e^{-t}\frac{\sin 2t}{2}$$