Ex.: Find the local max. and min. values and saddle points of $f(x,y)=x^4+y^4-4xy+1$

$f_x = \cancel{4}x^3-\cancel{4}y=0 \Rightarrow x^3=y$

$f_y = \cancel{4}y^3-\cancel{4}x=0 \Rightarrow x^9-x=0 \Rightarrow \underbrace{x}_{0}(\underbrace{x^8-1}_{=0})=0$

$x=0 \Rightarrow y=0$ $\qquad x^8-1=(\underbrace{x^4-1}).(\underbrace{x^4+1}_{\neq 0})=0$

$x=1 \Rightarrow y=1$ $\qquad x^4-1=(\underbrace{x^2-1}).(\underbrace{x^2+1}_{\neq 0})=0$

$x=-1 \Rightarrow y=-1$

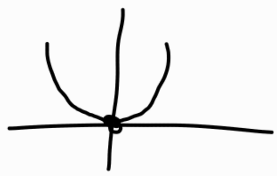
$f_{xx}=A=12x^2 \quad , \quad B=f_{xy}=-4 \quad , \quad C=f_{yy}=12y^2$

	$A=12x^2$	$B=-4$	$C=12y^2$	$D=AC-B^2$
$(0,0)$	0	-4	0	$-16<0 \Rightarrow$ saddle point
$(1,1)$	$\underline{12}$	-4	$\underline{12}$	$128>0 \Rightarrow$ local min. (-1)
$(-1,-1)$	$\underline{12}$	-4	$\underline{12}$	$128>0 \Rightarrow$ local min. (-1)

Ex.: Find the absolute max. and min. of the function $f(x,y)=x^2-2xy+2y$ on the rectangle $D=\{(x,y)\,|\,0\leq x\leq 3,\ 0\leq y\leq 2\}$

$f_x = 2x-2y=0 \Rightarrow y=1$ $\qquad (1,1) \Rightarrow$ critical point

$f_y=-2x+2=0 \Rightarrow x=1$ $\qquad f(1,1)=\underline{1}$

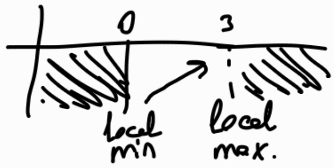


$L_1:\ f(x,0)=x^2 \quad (0\leq x\leq 3) \Rightarrow f_{\underline{x}}(x,0)=2x=0 \Rightarrow x=0$

$f''(x,0)=2 \qquad x=0 \Rightarrow$ local min $\quad y=0 \quad (0,0) \Rightarrow$ local min.

$f(3,0)=\underline{9} \Rightarrow$ local max

$f(0,0)=\underline{0} \Rightarrow$ local min



$L_2 : f(3,y) = 9-4y \Rightarrow f(3,0) = \underline{9}$
$(0 \leq y \leq 2) \qquad f(3,2) = \underline{1}$

$L_3 : (x,2) = x^2-4x+4 = (x-2)^2 \qquad (0 \leq x \leq 3)$
$f(0,2) = \underline{4} \quad , \quad f(3,2) = \underline{1}$

$L_4 : (0,y) = 2y \Rightarrow f(0,0) = \underline{0}$
$(0 \leq y \leq 2) \Rightarrow f(0,2) = \underline{4}$

$(3,0) \Rightarrow$ abs. max (with the velue 9)
$(0,0) \Rightarrow$ abs. min (with the value 0)
$(2,0)$

Ex.: Find the local max. and local min values and saddle points of the function $f(x,y) = x^2+y^4+2xy$

$f_x = 2x+2y = 0 \Rightarrow x = -y$
$f_y = 4y^3+2x = 0 \Longrightarrow 4y^3-2y = 0 \Rightarrow y(2y^2-1) = 0$
$0 \qquad 2y^2 = 1 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \mp\frac{1}{\sqrt{2}}$

$(0,0)$, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$f_{xx} = 2, \quad f_{xy} = 2, \quad f_{yy} = 12y^2 \Rightarrow D = 24y^2-4 = 4(6y^2-1)$

$(0,0) \Rightarrow D<0 \Rightarrow (0,0)$ is saddle point
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \Rightarrow D>0, A>0 \Rightarrow \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is local min. $f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \Rightarrow D>0, A>0 \Rightarrow \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ is local min. $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{4}$

Ex.: Find the local max. and local min values and saddle points of the function $f(x,y) = e^x \cos y$

$f_x = e^x \cos y = 0 \qquad e^x \neq 0 \qquad \left.\begin{array}{l}\cos y = 0 \\ \sin y = 0\end{array}\right\} y \neq 0 \Rightarrow$ no critical point.
$f_y = -e^x \sin y = 0$

Ex.: For which value(s) of k does $f(x,y)=x^2+kxy+4y^2$ have a local min. at $(0,0)$?

$$f_x = 2x+ky=0 \;/k \qquad\qquad 2kx+k^2y=0$$
$$f_y = kx+8y=0 \;/-2 \qquad\qquad + \; -2kx-16y=0$$
$$(k^2-16)\cdot y=0$$

Case 1: $y=0$ $(k^2-16\neq 0)$ $\quad 2x+k\underset{0}{\underset{\parallel}{y}}=0 \Rightarrow 2x=0 \Rightarrow x=0$

$f_{xx}=2$, $\quad f_{xy}=k$, $\quad f_{yy}=8 \qquad D=16-k^2$

If $D<0 \Rightarrow (0,0)$ is saddle point

If $D>0 \Rightarrow (0,0)$ is local min $\Leftrightarrow 16-k^2>0 \Rightarrow \boxed{16>k^2}$

Case 2: $k^2-16=0 \Rightarrow k=\mp 4 \Rightarrow f(x,y)=x^2\mp 4xy+4y^2$
$$=(x\mp 2y)^2$$

min. value of $f(x,y)=0$, $f(0,0)=0$

Ex.: Find the abs. max.-min values of f on the set $D=\{(x,y) \mid |x|\leq 1,\ |y|\leq 1\}$ where $f(x,y)=x^2+y^2+x^2y+4$

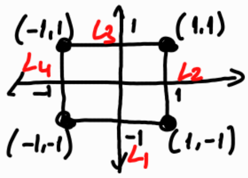
$f_x=2x+2xy=0 \Rightarrow 2x(1+y)=0 \quad (x=0) \quad 1+y=0 \Rightarrow y=-1$

$f_y=2y+x^2=0 \Rightarrow 2y=-x^2$

If $x=0 \Rightarrow 2y=0 \Rightarrow y=0$ $\quad(0,0)$ is a critical point

If $y=-1 \Rightarrow x^2=2 \Rightarrow x=\mp\sqrt{2}$ $\quad(\mp\sqrt{2},-1)$ is nothing. (outside of D)

$f(0,0)=4$ $\quad$ 4 is abs. min. value.



L_1: $f(x,-1)=5$, constant

L_2: $f(1,y)=y^2+y+5 \qquad f(1,1)=7,\ f(1,-1)=5$

$\qquad f'(1,y)=2y+1=0 \Rightarrow y=-\frac{1}{2} \qquad f(1,-\frac{1}{2})=\frac{19}{4}$

L_3: $f(x,1)=2x^2+5 \Rightarrow f(-1,1)=f(1,1)=7$, $\quad f'(x,1)=4x=0 \Rightarrow x=0$

$\qquad f(0,1)=5 \qquad$ 7 is abs. max value

L_4: $f(-1,y)=y^2+y+5 \quad f(-1,1)=7 \quad f(-1,-1)=5 \quad f(-1,-\frac{1}{2})=\frac{19}{4}$

Ex.: $$\int_0^3\left[\int_1^2 x^2y\,dy\right]dx=\int_0^3 x^2\frac{y^2}{2}\Big|_{y=1}^{y=2}dx=\int_0^3 x^2\underbrace{\left(\frac{4-1}{2}\right)}_{\frac{3}{2}}dx=\frac{3}{2}\cdot\frac{x^3}{3}\Big|_0^3=\frac{27}{2}$$

$$\int_1^2\int_0^3 x^2y\,dx\,dy=\frac{27}{2}$$

Ex.: $\iint_R (x-3y^2)\,dA$ where $R=\{(x,y)\mid 0\le x\le 2,\ 1\le y\le 2\}$

$$\int_1^2\int_0^2(x-3y^2)\,dx\,dy=\int_0^2\int_1^2(x-3y^2)\,dy\,dx=\int_0^2(xy-y^3)\Big|_1^2dx=\int_0^2\Big[\underbrace{(2x-8)-(x-1)}_{x-7}\Big]dx$$

$$\frac{x^2}{2}-7x\Big|_0^2=2-14=-12$$

H.W.: $\iint_R y\sin(xy)\,dA$ where $R=\underbrace{[1,2]}_{x}\times\underbrace{[0,\pi]}_{y}$ $\quad(=0)$

$dx\,dy \qquad dy\,dx$

Remark: $\iint_R g(x)h(y)\,dA=\int_a^b g(x)\,dx\cdot\int_c^d h(y)\,dy$ where $R=[a,b]\times[c,d]$

Ex.: If $R=[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]$

$$\iint_R \sin x\cos y\,dA=\underbrace{\int_0^{\pi/2}\sin x\,dx}_{\underbrace{-\cos x\Big|_0^{\pi/2}}_{1}}\cdot\underbrace{\int_0^{\pi/2}\cos y\,dy}_{\underbrace{\sin y\Big|_0^{\pi/2}}_{1}}=1$$