HIGH ORDER LINEAR DEs

An n-th order linear DE has the form:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = F(x)$$  \hspace{1cm} (1)

where $a_0$ is not identically zero.

- Observe that $F(x)$ and the coefficients $a_i(x)$ ($i = 0, 1, 2, \ldots, n$) depend solely on the variable $x$.

- The right-hand member $F(x)$ is called the non-homogeneous term. If $F$ is identically zero, Equation (1) reduces to

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \ldots + a_{n-1}(x)y' + a_n(x)y = 0$$  \hspace{1cm} (2)

and is called homogeneous.

- For $n=2$, (1) reduces to the second-order non-homogeneous linear DE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x)$$ and (2) reduces to the corresponding second-order homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0.$$

Example: $y'' + 3xy' + x^2y = e^x$ ... a linear ordinary DE of the second order.

$$y''' + xy'' + 3x^2y' - 5y = \sin x$$ ... a linear ordinary DE of the third order.
Theorem 1: Consider the n-th order linear DE (1), where \( a_0, a_1, \ldots, a_n \) and \( F \) are continuous real functions on a real interval \( a \leq x \leq b \) and \( q_0(x) \neq 0 \) for any \( x \) on \( a \leq x \leq b \). Let \( x_0 \) be any point of the interval \( a \leq x \leq b \) and let \( c_0, c_1, \ldots, c_{n-1} \) be \( n \) arbitrary real constants. There exists a unique solution \( f \) of (1) s.t.

\[
f(x_0) = c_0, \quad f'(x_0) = c_1, \ldots, \quad f^{(n-1)}(x_0) = c_{n-1}
\]

and this solution is defined over the entire interval \( a \leq x \leq b \).

Example: Consider the initial-value problem:

\[
y'' + 3xy' + x^3y = e^x, \quad y(1) = 2, \quad y'(1) = -5
\]

The coefficients \( 1, 3x, x^3 \), as well as the non-homogeneous term \( e^x \) are all cont. functions for all \( x \) values in \((-\infty, \infty)\). The \( x_0 \) point is 1 here.

\( \{ 1 \in (-\infty, \infty) \} \) Real numbers \( c_0 \) and \( c_1 \) are \( 2 \) and \( -5 \) respectively. Thus Theorem 1 assures us that a solution of the given problem exists, is unique, and is defined for all \( x \in (-\infty, \infty) \).

Corollary: Let \( f \) be a solution of the n-th order homogeneous linear DE (2) such that

\[
f(x_0) = 0, \quad f'(x_0) = 0, \ldots, \quad f^{(n-1)}(x_0) = 0,
\]

where \( x_0 \) is a point of the interval \( a \leq x \leq b \) in which the coefficients \( a_0, a_1, \ldots, a_n \) are all cont. and \( a_0(x) \neq 0 \).

Then \( f(x) = 0 \) for all \( x \in [a, b] \).

This corollary states that this solution is the "trivial" solution \( f \) s.t. \( f(x) = 0 \) for all \( x \) on the above-mentioned interval.
Theorem-2: Let $f_1, f_2, \ldots, f_m$ be any $m$ solutions of the homogeneous linear DE (2). Then
\[ c_1f_1 + c_2f_2 + \cdots + c_mf_m \] is also a solution of (2), where $c_1, c_2, \ldots, c_m$ are $m$ arbitrary constants. Here the expression $\sum c_if_i$ is called as a linear combination of $f_1, f_2, \ldots, f_m$.

Example: $\sin x$ and $\cos x$ are solutions of $y'' + y = 0$.

Theorem-2 states that the linear combination $c_1\sin x + c_2\cos x$ is also a solution for any constants $c_1$ and $c_2$.

Let's prove that $y = \sin x$ is a solution.

$y = \sin x$, $y' = \cos x$, $y'' = -\sin x \implies y'' + y = 0$

$y = \cos x$ is a solution, too.

$y = \cos x$, $y' = -\sin x$, $y'' = -\cos x \implies y'' + y = 0$

Let's prove that $y = c_1\sin x + c_2\cos x$ is a solution.

$y = c_1\sin x + c_2\cos x$, $y' = c_1\cos x - c_2\sin x$, $y'' = -c_1\sin x - c_2\cos x$

We now consider what constitutes the general solution of (2). We first introduce the concepts of linear dependence and independence.
Definition: The n functions $f_1, f_2, \ldots, f_n$ are called linearly dependent on $a \leq x \leq b$ if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that
\[ c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \quad \text{for all } x \]
such that $a \leq x \leq b$.

Example: Observe that $x$ and $2x$ are linearly dependent on the interval $0 \leq x \leq 1$.
\[ c_1 (x) + c_2 (2x) = 0 \quad \Rightarrow \quad c_1 = 2, \quad c_2 = -1 \]

Definition: The n functions $f_1, f_2, \ldots, f_n$ are called linearly independent on $a \leq x \leq b$ if the relation
\[ c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) = 0 \quad \text{for all } x \]
such that $a \leq x \leq b$ implies that $c_1 = c_2 = \ldots = c_n = 0$.

In other words, the only linear combination of $f_1, \ldots, f_n$ that is identically zero on $a \leq x \leq b$ is the trivial linear combination $0f_1 + 0f_2 + \ldots + 0f_n$.

Example: $x$ and $x^2$ are linearly independent on $0 \leq x \leq 1$ since $c_1 x + c_2 x^2 = 0$ for all $x$ on $0 \leq x \leq 1$ implies both $c_1 = 0$ and $c_2 = 0$.

Theorem 3: The n-th order linear homogeneous DE (2) always has n linearly independent solutions. If $f_1, f_2, \ldots, f_n$ represent these solutions, then the general solution of (2) is
\[ f(x) = c_1 f_1(x) + c_2 f_2(x) + \ldots + c_n f_n(x) \quad \text{where} \]
$c_1, c_2, \ldots, c_n$ are arbitrary constants.
Here, the set \( f_1, \ldots, f_n \) is called a fundamental set of solutions of (2).

Therefore, if we can find \( n \) linearly independent solutions of (2), we can at once write the general solution of (2) as a general linear combination of these \( n \) solutions.

**Example:** The solutions \( e^x, e^{-x} \) and \( e^{2x} \) of \( y''' - 2y'' - y' + 2y = 0 \) may be shown to be linearly independent. Thus \( e^x, e^{-x} \) and \( e^{2x} \) constitute a fundamental set of the given DE and its general solution may be expressed as the linear combination

\[
y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}
\]

where \( c_1, c_2, c_3 \) are arbitrary constants.

**Definition:** The Wronskian of a set of functions \( f_1, f_2, \ldots, f_n \) on the interval \( a \leq x \leq b \), having the property that each function possesses \( (n-1) \) derivatives on the interval, is the determinant:

\[
W(f_1, f_2, \ldots, f_n) = \begin{vmatrix}
f_1 & f_2 & \cdots & f_n \\
f_1' & f_2' & \cdots & f_n' \\
\vdots & \vdots & \ddots & \vdots \\
\xi^{(n-1)} & \xi^{(n-1)} & \cdots & \xi^{(n)}
\end{vmatrix}
\]

**Theorem:** The \( n \) solutions \( f_1, f_2, \ldots, f_n \) of (2) are linearly independent on \( a \leq x \leq b \) if and only if the Wronskian of \( f_1, \ldots, f_n \) is different from zero for some \( x \) on the interval \( a \leq x \leq b \).

\( \xi \) If the Wronskian of a set of \( n \) functions defined on the interval \( a \leq x \leq b \) is non-zero for at least one point in this interval, then the set of functions is linearly independent there.
If the Wronskian is identically zero on this interval and if each of the functions is a solution to the same linear DE, then the set of functions is linearly dependent.

**Example:** The solutions $e^x$, $e^{-x}$ and $e^{2x}$ of

$y'' - 2y'' - y' + 2y = 0$ are linearly independent on every real interval, for

$$w(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^x e^{-x} e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2x} \neq 0 \text{ for all real } x.$$