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Research Article

ON CYCLIC CODES AND CYCLIC LCD CODES FROM THE FAMILY OF GROUP RINGS $\mathbb{Z}_4C_n$

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ABSTRACT

In this work, we study the structure of cyclic zero divisor codes over a family of group rings. We determine the number of elements of these codes and introduce the dual codes. Moreover, we show that there is no non-free cyclic LCD $\mathbb{Z}_4$ codes.
Keywords: Group rings, cyclic codes, zero divisors, linear complementary dual (LCD) codes.

1. INTRODUCTION

In [9], Hurley et al. described and presented a new construction technique for codes from group rings. This technique is essentially based on zero divisors in group rings. In addition to their general algebraic structure, group rings have a rich source of zero divisors and unit elements. Further, the well-known structural linear codes such as cyclic codes, quasi-cyclic codes are within the family of group ring codes. Thus, group rings offer a rich source for structural codes that may lead to linear codes with good properties.

Linear codes with complementary-duals (LCD codes) (see [16]) have many applications in communication systems, data storage, cryptography and consumer electronics. A linear code $\mathcal{C}$ is called an LCD code if $\mathcal{C}^{\perp} \cap \mathcal{C} = \{0\}$. In [16], it was shown that LCD codes provide an optimum linear coding solution for binary adder channel, and in [17], Massey showed that asymptotically good LCD codes exist. Moreover, in [20] Sendrier has proved that LCD codes meet the Gilbert-Varshamov bound. In [22], Yang and Massey have given a necessary and sufficient condition for a cyclic code to have a complementary dual. All LCD constacyclic codes of length $2l^t p^s$ was determined in [4]. The LCD condition for a certain class of quasi cyclic codes has been studied in [7]. In [6], Dougherty et al. gave a linear programming bound on the largest size of an LCD code of given length and minimum distance. In [12], Li constructed some non MDS cyclic Hermitian LCD codes over finite fields and analyzed their parameters. In [19], a class of MDS negacyclic LCD codes of even length $n|q-1$ was given. Carlet and Guiley studied an application of LCD codes against side-channel attacks and presented particular constructions for LCD codes in [2]. In

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[1], Beelen and Jin gave an explicit construction of several classes of LCD MDS codes. MDS LCD codes over finite field $\mathbb{F}_q$ with even q were completely solved in [10]. In [14], Li et al. explored two special families of LCD cyclic codes, which are both BCH codes. The authors of [13] constructed several families of reversible cyclic codes over finite fields and analyzed their parameters. Galvez et al. ([8]), have given exact values of dimension k and length n of a binary LCD code, where $1 \le k \le n \le 12$. In [5], Chen and Liu have proposed a different approach to obtain new LCD MDS codes from generalized Reed-Solomon codes. In [21], Sok et al. proved the existence of optimal LCD codes over large finite fields and they have also given methods to generate orthogonal matrices over finite fields and then use them to construct LCD codes. In [3], Carlet et al. studied several constructions of new Euclidean and Hermitian LCD MDS codes. In [15],

Liu and Liu provided a necessary condition for an LCD linear code $\mathcal{C}$ over a finite chain ring. Under suitable conditions, they have given a sufficient condition under which a linear code $\mathcal{C}$ over a finite chain ring is LCD. Especially, they have derived a necessary and sufficient condition for free linear codes over a finite chain ring to be LCD. In [11], a condition for codes obtained from units of group rings to be LCD has been provided. It is also shown that a special decomposition of group rings meet the LCD condition and a construction of linear complementary pair (LCP) of codes has proposed.

In this study, we extend the encoding method originally given in [9], for group ring codes over fields to group ring codes on $\mathbb{Z}_4 C_n$ where n is an odd integer and further we explore their structures. We define the generators of these codes and their duals. We determine the cardinality of these codes. Further, we show that there is no non-free cyclic LCD $\mathbb{Z}_4$ codes.

2. PREMILINARIES

Let q be a prime power and $\mathbb{F}_q$ be the finite field with q elements. An $[n,k]_q$ linear code $\mathcal{C}$ of length n over $\mathbb{F}_q$ is a k -dimensional subspace of the vector space $\mathbb{F}_q^n$. The elements of $\mathcal{C}$ are of the form $(c_0, c_1, \ldots, c_{n-1})$ and called codewords. The Hamming weight of any $c \in \mathcal{C}$ is the number of nonzero coordinates of c and denoted by $w(c)$. The minimum distance of $\mathcal{C}$ is defined as $d = \min\{w(c) | 0 \neq c \in \mathcal{C}\}$. An $[n,k]_q$ linear code with minimum distance d is said to be MDS (maximum distance separable) if $n+1 = k+d$. The Euclidean dual code of $\mathcal{C}$ is defined to be $\mathcal{C}^{\perp} = \left\{ \mathbf{x} \in \mathbb{F}_q^n \middle| \sum_{i=0}^{n-1} x_i y_i = 0, \forall \mathbf{y} \in \mathcal{C} \right\}$.

3. GROUP RINGS BASICS

In this subsection we present some basics about group rings. For further information reader may refer to [18].

Let R be a ring and G a group define the group ring RG to be the set of all elements of the form $u = \sum_{g \in G} \alpha_g g$ where $\alpha_g \in R$ and only finitely many of the $\alpha_g \neq 0$. For given two elements $\alpha = \sum_{g \in G} \alpha_g g$ and $\beta = \sum_{g \in G} \beta_g g$ in RG, the binary operations addition and multiplication are defined as below:

$$\alpha+\beta=\sum_{g\in G}\alpha_g g+\sum_{g\in G}\beta_g g=\sum_{g\in G}\left(\alpha_g+\beta_g\right)g,$$

$$\alpha\beta=\left(\sum_{g\in G}\alpha_g g\right)\left(\sum_{h\in G}\beta_h h\right)=\sum_{g,h\in G}\alpha_g\beta_h gh.$$

Example 3.1. Le t R be the integer ring $\mathbb{Z}_3=\{0,1,2\}$ and G be the cyclic group $C_4=\{1,g,g^2,g^4\}$ of order four with the generator element g. Then, the group ring $\mathbb{Z}_3C_4$ is

$$\mathbb{Z}_3C_4=\left\{a_0+a_1g+a_2g^2+a_3g^3\middle|\, a_i\in\mathbb{Z}_3,\, g^j\in C_4\right\}.$$

A non-zero element z in a commutative ring R is a zero-divisor if there exists a non-zero $r\in R$ such that $zr=0$. The transpose of an element $u=\sum_{g\in G}\alpha_g g$ in RG is $u^T=\sum_{g\in G}\alpha_g g^{-1}$ or $u^T=\sum_{g\in G}\alpha_{g^{-1}}g$.

For example, the transpose of the element $w=1+g^2\in\mathbb{Z}_2C_3$, is $w^T=1+g$.

4. CODES FROM ZERO-DIVISORS

In this section we give a brief summary of zero divisor codes. For more detailed information the reader may consult the reference [9].

The following definition of zero-divisor codes is based on reference [9].

Definition 4.1. Let u,v be a zero-divisors in RG and W be a submodule of RG with basis of group elements $S\subseteq G$. A zero-divisor code is the set $\mathcal{C}=\{ux|\,x\in W\}=uW$ or $\mathcal{C}=\{xu|\,x\in W\}=Wu$. Here, the element u is a generator element of the code $\mathcal{C}=Wu$ relative to the submodule W.

Let $T\subset RG$. T is linearly independent if for all $x\in T$ and for $\alpha_x\in R$, $\sum_{x\in T}\alpha_x x=0$ only when $\alpha_x=0$. Apart from that T is linearly dependent. The maximum number of linearly independent elements of T is called rank of T and denoted by $rank(T)$. Therefore, $rank(T)=|T|$ if and only if T is linearly independent. Notice that a zero-divisor code $\mathcal{C}=Wu$ is a submodule of RG and consists of all elements of the form $\sum_{g\in S}\alpha_g gu$.

Thus, the dimension of this code is the rank (Su) [9].

Example 4.1. Let RG be the group ring given in Example 3.1 Suppose that $u=1+g^2$ and $v=2+g^2$ be two zero-divisors in $\mathbb{Z}_3C_4$ and let W be the submodule of $\mathbb{Z}_3C_4$ generated by the set $S=\{1,g\}$, i.e $W=\langle S\rangle=\{0,1,2,g,2g,1+g,1+2g,2+g,2+2g\}$. Since $(Su)=\{1,g\}(1+g^2)=\{1+g,g+g^2\}$, the rank of Su is 2. Moreover, the zero-divisor code generated by the element u is

$$\mathcal{C}=\left\{\begin{matrix}0,g^2+1,2g^2+2,g^3+g,g^3+g^2+g+1,g^3+2g^2+g+2,\\ 2g^3+2g,2g^3+g^2+2g+1,2g^3+2g^2+2g+2\end{matrix}\right\}=Wu.$$

Definition 4.2. [9] Let u be a zero-divisor and the rank of u be rank $(Su)=r$. Then we called u as a principal zero-divisor if there exists a $v \in RG$ such that $uv=0$ and $rank\left((G-S)v\right)=n-r$.

For example, the elements $u=1+g$ and $v=2+g^2$ in $\mathbb{Z}_3C_4$ are principal zero-divisors.

Theorem 4.1. [9] Let $\mathcal{C}=\{xu \mid x \in W\}$ zero-divisor code, where $W=\langle S \rangle$ and rank $(Su)=r$. Assume that $uv=0$ in the group ring RG so that $rank\left((G-S)v\right)=n-r$. Then y is a codeword if and only if $yv=0$.

The element $v \in RG$ is called the check element of the code $\mathcal{C}$. For example, the zero-divisor element $v=2+g^2$ is the check element of the code $\mathcal{C}$ given in Example 4.1.

Corollary 4.1. [9] The zero-divisor code $\mathcal{C}=\{xu \mid x \in W\}$ has a single check element if and only if u is a principal zero-divisor.

The inner product of any two elements x,y in RG is given by term-by-term multiplication of the ring elements, that is to say $\langle x,y \rangle = \sum_{g \in G} \alpha_g \beta_g$ where $x=\sum_{g \in G} \alpha_g g$ and $y=\sum_{g \in G} \beta_g g$. Hence, the dual code of a zero-divisor code is defined as $\mathcal{C}^{\perp}=\{y \in RG \mid \langle ux,y \rangle = 0, \forall x \in W\}$.

Theorem 4.2. [9] Let $u,v \in RG$ be two principal zero-divisors such that for $S \subset G$, $rank(Su)=r$ and $rank\left((G-S)v\right)=n-r$. Let $W=\langle S \rangle$ be a submodule of dimension r and $W^{\perp}=\langle G \setminus S \rangle$ be the submodule of rank $n-r$. Then the dual code of the zero-divisor code $\mathcal{C}=\{xu \mid x \in W\}$ is $\mathcal{C}^{\perp}=\{xv^T \mid x \in W^{\perp}\}=\{y \in RG \mid yu^T=0\}$.

Example 4.2. The dual code of the zero-divisor code $\mathcal{C}$ given in the Example 4.1} is

$$\begin{aligned}\mathcal{C}^{\perp} &= \{xv^T \mid x \in W^{\perp}\} \\ &= \left\{ \begin{matrix} 0, g^2+2, 2g^2+1, g^3+2g, g^3+g^2+2g+2, \\ g^3+2g^2+2g+1, 2g^3+g, 2g^3+g^2+g+2, 2g^3+2g^2+g+1 \end{matrix} \right\},\end{aligned}$$

where

$$\begin{aligned}W^{\perp} &= \langle G \setminus S \rangle = \langle \{g^2, g^3\} \rangle \\ &= \{0, g, 2g, g^2, 2g^2, g^2+g, g^2+2g, 2g^2+g, 2g^2+2g\},\end{aligned}$$

and $v^T=1+g+g^2$. Note that, $rank\left((G-S)v\right)=2$.

5. CYCLIC CODES FROM ZERO-DIVISORS IN GROUP RINGS $\mathbb{Z}_4C_n$

In this section, we determine necessary and sufficient conditions to zero divisor generators of $\mathbb{Z}_4$-cyclic linear group ring codes. Further, we describe the structure of $\mathbb{Z}_4$-cyclic linear group ring codes with odd length and their duals. This section presents an extension to the results given by Hurley et al. in [9].

The definition in the following is a revised version of the notion rank of an element in a group ring given in [9].

Definition 5.1. Let n be an odd integer and C_n be the cyclic group of order n. Then the rank of $u \in \mathbb{Z}_4 C_n$ is defined to be $rank(u) = n - \dim_F \left(\left(\mathbb{Z}_4 C_n \right) u \right)$ where $\dim_F \left(\left(\mathbb{Z}_4 C_n \right) u \right)$ is the free dimension of the module $\left(\mathbb{Z}_4 C_n \right) u$.

Example 5.1. Let $u = g^2 + 3g \in \mathbb{Z}_4 C_3$. Then

$$\left(\mathbb{Z}_4 C_3 \right) u = \begin{Bmatrix} 0, 3+g, 2+2g, 1+3g, 3+g^2, 2+g+g^2, 1+2g+g^2, \\ 3g+g^2, 2+2g^2, 1+g+2g^2, 2g+2g^2, 3+3g+2g^2, \\ 1+3g^2, g+3g^2, 3+2g+3g^2, 2+3g+3g^2 \end{Bmatrix}.$$

It can be easily seen that $\left| \left(\mathbb{Z}_4 C_3 \right) u \right| = 4^2$ and $\dim_F \left(\left(\mathbb{Z}_4 C_3 \right) u \right) = 2$ and so we have $rank(u) = 3 - dim_F \left(\left(\mathbb{Z}_4 C_3 \right) u \right) = 3 - 2 = 1$.

In order to define $\mathbb{Z}_4$-cyclic linear group ring codes and their duals, we need to following restrictions and definitions.

Let n be an odd integer and $\{g_1, g_2, \ldots, g_n\}$ be a fixed list of the elements of cyclic group C_n. Assume that $W = \langle S \rangle$ and $W^{\perp} = \langle G - S \rangle$ are submodules of $\mathbb{Z}_4 C_n$, where $S \subset C_n$ and $S(uw)$, $S(2uv)$ are linearly independent. Further, assume $uvw = 0$, such that $rank(u) + rank(v) + rank(w) = n$ and $u, v, w \in \mathbb{Z}_4 C_n$.

Definition 5.2. Let u, v, w be zero-divisors in $\mathbb{Z}_4 C_n$, such that $uvw = 0$, and $rank(u) + rank(v) + rank(w) = n$. Let W be a submodule of $\mathbb{Z}_4 C_n$ with basis of group elements $S \subseteq C_n$. The code $\mathcal{C} = \{ xuw + y2uv \,|\, x, y \in W \} = W(uw) + W(2uv)$ is called a zero divisor group ring code. This code can be viewed as a $\mathbb{Z}_4$-cyclic linear code.

In Theorem 5.1, we give the number of the elements of zero-divisor code $\mathcal{C} = W(uw) + W(2uv)$.

Theorem 5.1. Let u, v, w be zero-divisors in $\mathbb{Z}_4 C_n$, such that $uvw = 0$, and $rank(u) + rank(v) + rank(w) = n$, where n is an odd integer. Let W be a submodule of $\mathbb{Z}_4 C_n$ generated by $S \subset C_n$ such that $S(uw)$ and $S(2uv)$ are linearly independent. The number of elements of the code $\mathcal{C} = \{ xuw + y2uv \,|\, x, y \in W \} = W(uw) + W(2uv)$ is $4^{rank(v)} 2^{rank(w)}$.

Proof. Notice that the zero-divisor code $\mathcal{C} = Wuw + W2uv$ is a submodule of RG and consists of all elements of the form $\sum_{g \in S} \alpha_g g(uw) + \sum_{g \in S} \alpha_g g(2uv)$. Also we know that $W(uw) \cap W(2uv) = W(2uvw) = \{0\}$, because $S(uw)$ and $S(2uv)$ are linearly independent. Thus, the dimension of this submodule is the rank $\left(S(uw) + S(2uv) \right)$. Therefore, we have that $|\mathcal{C}| = |W(uw)||W(2uv)| = |S(uw)||S(2uv)| = 4^{rank(v)} 2^{rank(w)}$. This completes the proof.

Theorem 5.2. The dual code of the $\mathbb{Z}_4$-zero divisor code $\mathcal{C}=\{xuw+y2uv \mid x,y\in W\}$ $=W(uw)+W(2uv)$ given in Theorem 5.1 is $\mathcal{C}^{\perp}=\{x(v^Tw^T)+y(2u^Tv^T) \mid x,y\in W^{\perp}\}$ $=W^{\perp}(v^Tw^T)+W^{\perp}(2u^Tv^T)$, where $W^{\perp}$ is the $\mathbb{Z}_4$ submodule of $\mathbb{Z}_4C_n$ generated by C_n-S.

Proof. By the definition of a zero divisor code from Theorem 4.2, it can be written that $(Wu)^{\perp}=W^{\perp}(v^Tw^T)$. Since $W(uw)$ and $W(2uv)\subset W(u)$ we have $\mathcal{C}=W(uw)+W(2uv)\subset W(u)$. This proves that $(W(u))^{\perp}\subset\mathcal{C}^{\perp}$. So, $W^{\perp}(v^Tw^T)\subseteq\mathcal{C}^{\perp}$. Similarly, we have $W^{\perp}(2u^Tv^T)\subseteq W^{\perp}(v^T)=(W(uw))^{\perp}$ and $W^{\perp}(2u^Tv^T)\subseteq(W(2uv))^{\perp}$. Hence, $W^{\perp}(2u^Tv^T)\subseteq(W(uw))^{\perp}\cap(W(2uv))^{\perp}=\mathcal{C}^{\perp}$. Consequently, we get $W^{\perp}(v^Tw^T)+W^{\perp}(2u^Tv^T)\subseteq\mathcal{C}^{\perp}$.

In Corollary 5.1, we give the number of elements of dual of a $\mathbb{Z}_4$-cyclic zero divisor code.

Corollary 5.1. The number of elements of the code $\mathcal{C}^{\perp}=\{x(v^Tw^T)+y(2u^Tv^T) \mid x,y\in W^{\perp}\}$ $=W^{\perp}(v^Tw^T)+W^{\perp}(2u^Tv^T)$ given in Theorem 5.2 is $4^{rank(u)}2^{rank(w)}$.

Proof. From Theorem 5.1, we know that $|\mathcal{C}|=4^{rank(v)}2^{rank(w)}$. Further, we have $|W^{\perp}(v^Tw^T)+W^{\perp}(2u^Tv^T)|=4^{rank(u)}2^{rank(w)}=|\mathcal{C}^{\perp}|$. Observe that $|\mathcal{C}^{\perp}|.|\mathcal{C}|=4^n$. This is the desired result.

Example 5.2. Let $u,v,w\in\mathbb{Z}_4C_7$ be zero divisors in the following with property $uvw=0$, and $rank(u)+rank(v)+rank(w)=1+3+3=7$.

$$u=g^2+3g \qquad u^T=3g^6+g^5$$
$$v=g^5+2g^4+g^3+3g^2 \qquad v^T=3g^5+g^4+2g^3+g^2$$
$$w=g^6+3g^5+2g^4+3g^3 \qquad w^T=3g^4+2g^3+3g^2+g$$

Then we have the submodules given in below:

$$uw=2+g+g^4+g^5+3g^6$$
$$2uv=2+2g^3+2g^5+2g^6$$
$$v^Tw^T=1+g+g^2+g^3+g^4+g^5+g^6$$
$$2u^Tv^T=2+2g+2g^2+2g^4$$

$$uw+2uv=g+2g^3+g^4+3g^5+g^6$$
$$v^Tw^T+2u^Tv^T=3+3g+3g^2+g^3+3g^4+g^5+g^6$$

Sub module	Dimension	Sub module	Dimension
$W(uw)$	4^3	$W(uw)+W(2uv)$	4^32^3
$W(2uv)$	2^3		
$W^{\perp}(v^Tw^T)$	4^1	$W^{\perp}(v^Tw^T)+W^{\perp}(2u^Tv^T)$	4^12^3
$W^{\perp}(2u^Tv^T)$	2^3		

W and $W^{\perp}$ are $\mathbb{Z}_4$-submodules of $\mathbb{Z}_4C_7$ spanned by $S=\{1,g,g^2,g^3\}$ and $S^{\perp}=\{g^4,g^5,g^6\}\subset C_7$ respectively. The submodule $\mathcal{C}=W(uw)+W(2uv)$ is a $\mathbb{Z}_4$-cyclic linear code generated by the matrix

$$G=\begin{pmatrix}1&0&0&1&1&1&0\\0&1&0&0&1&1&1\\0&0&1&1&1&0&1\\0&0&0&2&0&0&2\\0&0&0&0&2&0&2\\0&0&0&0&0&2&2\end{pmatrix}.$$

$\mathcal{C}$ has length 7, and cardinality $4^3 2^3$. Moreover, $\mathcal{C}^{\perp}=W^{\perp}(v^Tw^T)+W^{\perp}(2u^Tv^T)$ is a $\mathbb{Z}_4$-cyclic linear code generated by the matrix

$$H=\begin{pmatrix}1&1&1&1&1&1&1\\0&2&0&0&2&2&2\\0&0&2&0&2&2&0\\0&0&0&2&0&2&2\end{pmatrix}.$$

Clearly $|\mathcal{C}^{\perp}|=4^{rank(u)}2^{rank(w)}=4^1 2^3$. It can be easily seen that $\forall c\in W(uw)+W(2uv)$ and $\forall c'\in W^{\perp}(v^Tw^T)+W^{\perp}(2u^Tv^T)$ we have $\langle c,c'\rangle=0$. We conclude that the dual code of $\mathcal{C}$ is $\mathcal{C}^{\perp}$.

6. LCD CODES FROM $\mathbb{Z}_4C_n$

In [15], Liu and Liu provided a necessary condition for an LCD linear code $\mathcal{C}$ over a finite chain ring. Under suitable conditions, they have given a sufficient condition under which a linear code $\mathcal{C}$ over a finite chain ring is LCD. Especially, they have derived a necessary and sufficient condition for free linear codes over a finite chain ring to be LCD. In this section, we have showed that there doesn't exist non-free cyclic LCD $\mathbb{Z}_4$ codes.

Theorem 6.1. Let $\mathcal{C}=\{xuw+y2uv\,|\,x,y\in W\}=W(uw)+W(2uv)$ and $\mathcal{C}^{\perp}=\{x(v^Tw^T)+y(2u^Tv^T)\,|\,x,y\in W^{\perp}\}=W^{\perp}(v^Tw^T)+W^{\perp}(2u^Tv^T)$, be zero divisor codes given in Theorem 5.1 and Theorem 5.2. If $\mathcal{C}\cap\mathcal{C}^{\perp}=\{0\}$, then $w=1$ and $u=u^T$ or $u=v^T$.

Proof. Observe that

$$\mathcal{C}\cap\mathcal{C}^{\perp}=\left\langle\begin{matrix}lcm(uw,v^Tw^T),2lcm(uv,u^Tv^T),\\2lcm(uw,u^Tv^T),2lcm(uv,v^Tw^T)\end{matrix}\right\rangle$$
$$=\left\langle\begin{matrix}lcm(uw,v^Tw^T),\\2\gcd(lcm(uv,u^Tv^T),lcm(uw,u^Tv^T),lcm(uv,v^Tw^T))\end{matrix}\right\rangle.$$

$\mathcal{C} \cap \mathcal{C}^{\perp} = \{0\}$ if

$$uvw \mid lcm\left(uw, v^T w^T\right) \tag{6.1}$$

and

$$uvw \mid 2\gcd\left(lcm\left(uv, u^T v^T\right), lcm\left(uw, u^T v^T\right), lcm\left(uv, v^T w^T\right)\right) \tag{6.2}$$

or

$u^T v^T w^T \mid lcm\left(uw, v^T w^T\right)$ and $u^T v^T w^T \mid 2\gcd\left(lcm\left(uv, u^T v^T\right), lcm\left(uw, u^T v^T\right), lcm\left(uv, v^T w^T\right)\right)$.

We need to consider two cases separately.

Case 1: If $u = u^T$, then from Equation (6.1), we have $v \mid v^T w^T$, $w \mid u^T v^T = uv$, $v \mid u^T v^T$ and $u^T \mid uv$. When we consider all these results together, we conclude that $v = v^T$ and $w = 1$. This means that if $\mathcal{C} \cap \mathcal{C}^{\perp} = \{0\}$, then $\mathcal{C}$ is a free $\mathbb{Z}_4$-cyclic zero divisor code.

Case 2: If $u \neq u^T$, then from Equation (6.1), we have $v \mid v^T w^T$, $w \mid u^T v^T = uv$, $v \mid u^T v^T$ and $u^T \mid uv$. Here, we conclude that $v = u^T$ and $u = v^T$. This requires that $w = 1$. We conclude that if $\mathcal{C} \cap \mathcal{C}^{\perp} = \{0\}$, then $\mathcal{C}$ is a free $\mathbb{Z}_4$-cyclic zero divisor code.

The following result is immediate from Theorem 6.1.

Corollary 6.1. There is no non-free cyclic LCD $\mathbb{Z}_4$ codes.

7. CONCLUSION AND FUTURE REMARKS

In this paper, we present the structure of cyclic zero divisor $\mathbb{Z}_4$-codes over the class of group rings $\mathbb{Z}_4 C_n$. We determine the number of elements of these codes and we introduce the dual codes. Further, we show that there is no non-free cyclic LCD $\mathbb{Z}_4$ codes. To investigate the structure of zero divisor group ring codes in a non-commutative group ring can have interesting results. Also, the generalization of these codes to the group rings $\mathbb{Z}_{p^k} C_n$ awaits researchers.

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