ENGINEERING SYSTEM MODELLING AND SIMULATION

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Laplace Transforms

The Laplace transform is a powerful tool for solving *linear differential equations with constant coefficients* and to handle several of them simultaneously.

This last property develops from the fact that in solving differential equations the Laplace transform method first converts them into algebraic equations, and the resulting equations are then manipulated algebraically before obtaining the final result; algebraic manipulations are much easier, and more often possible, than dealing directly with differential equations.

The Laplace transform method is similar to the procedure with logarithms in that it first “transforms” the differential equation(s) into a “different domain” (the Laplace domain); it then works algebraically with the resulting equation(s); and finally it “transforms back” or “inverse Laplace” the final equation(s) to the original domain.

Laplace Transforms

Definition of the Laplace Transform

The Laplace transform of a function f(x), where x is the independent variable, is defined by the following formula:

$$F(s) = L[f(x)] = \int_0^\infty f(x) e^{-sx}\, dx$$

where F (s) = the Laplace transform of f(x) and s is the Laplace transform variable.

For the unit of the power of the exponential to be dimensionless, the unit of s is the inverse of the unit of the independent variable.

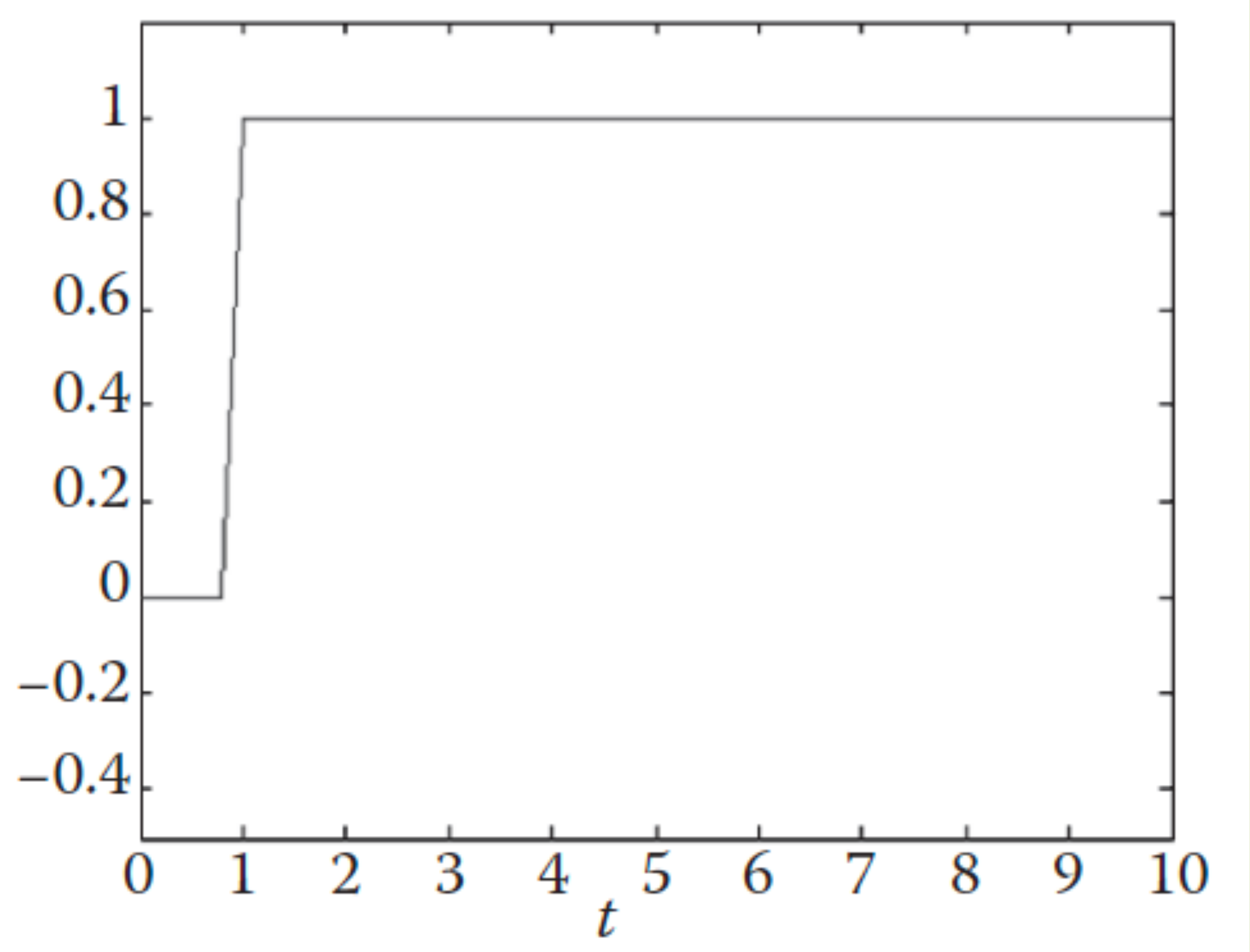
The limits of integration show that the Laplace transform contains information on the function f(x) for only positive values of the independent variable x.
In the analysis of system dynamics, the independent variable is time t. The Laplace transform of a function of time, f(t), is then

$$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

and consequently in this case the unit of s is time inverse (time^{-1}).

Example

Figure below shows four functions commonly applied as inputs to systems to study their responses. We now use the definition of the Laplace transform to derive their transforms.



Common input signals: (a) unit step function

(a) Unit Step Function

This is a sudden change of unit magnitude as sketched in figure. The mathematical representation of the unit step change is

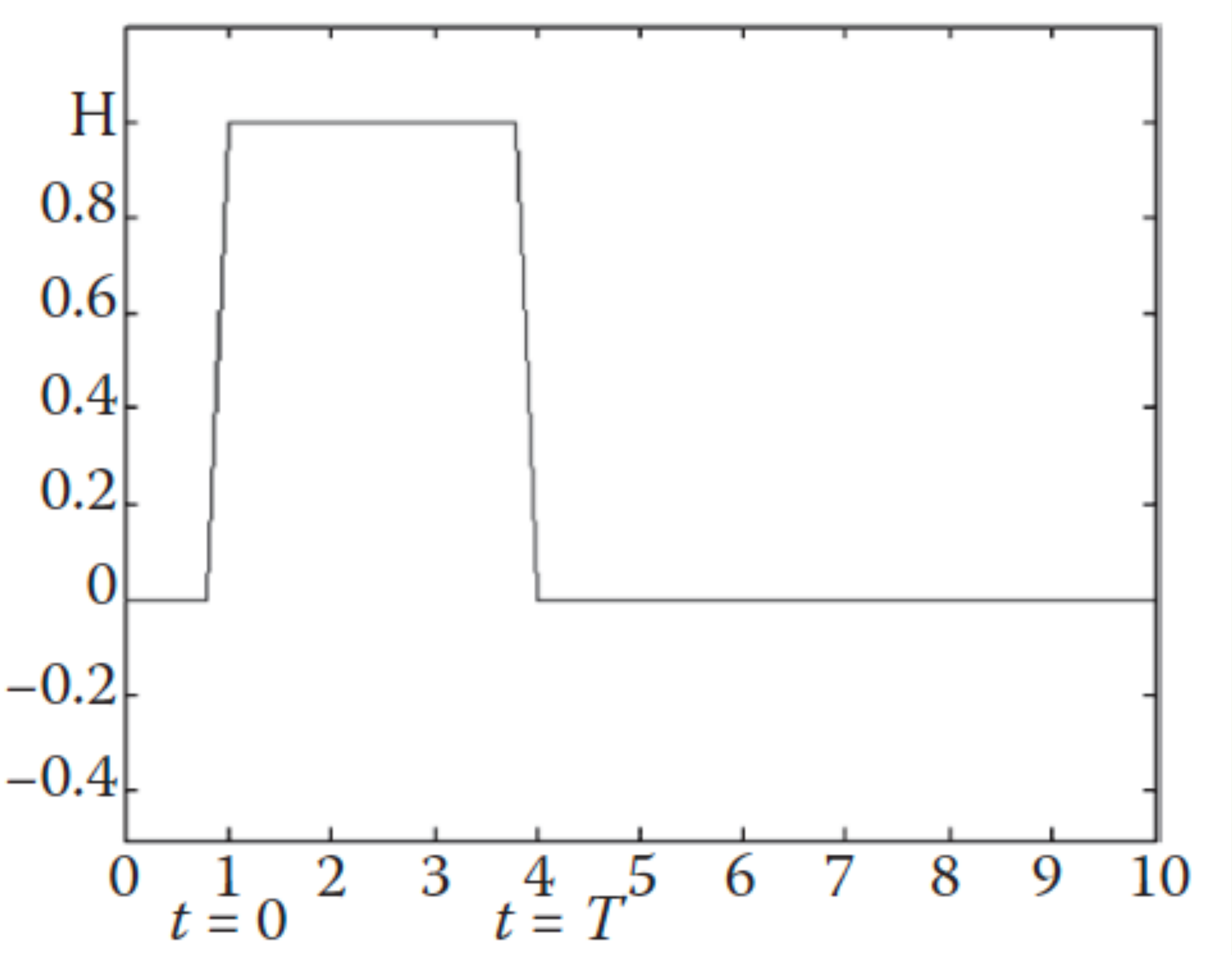
$$u(t-a)=\begin{cases}0 & t<a\\ 1 & t\geq a\end{cases}$$

In figure a = 1. Assuming a = 0 and substituting into

$$F(s)=L[f(t)]=\int_0^\infty f(t)e^{-st}\,dt$$

$$L[u(t)]=\int_0^\infty u(t)e^{-st}\,dt=-\frac{1}{s}e^{-st}\Big|_0^\infty=-\frac{1}{s}(0-1)=\frac{1}{s}$$

$$L[u(t)]=\frac{1}{s}$$



Common input signals: (b) pulse

(b) A Pulse of Magnitude H and Duration T

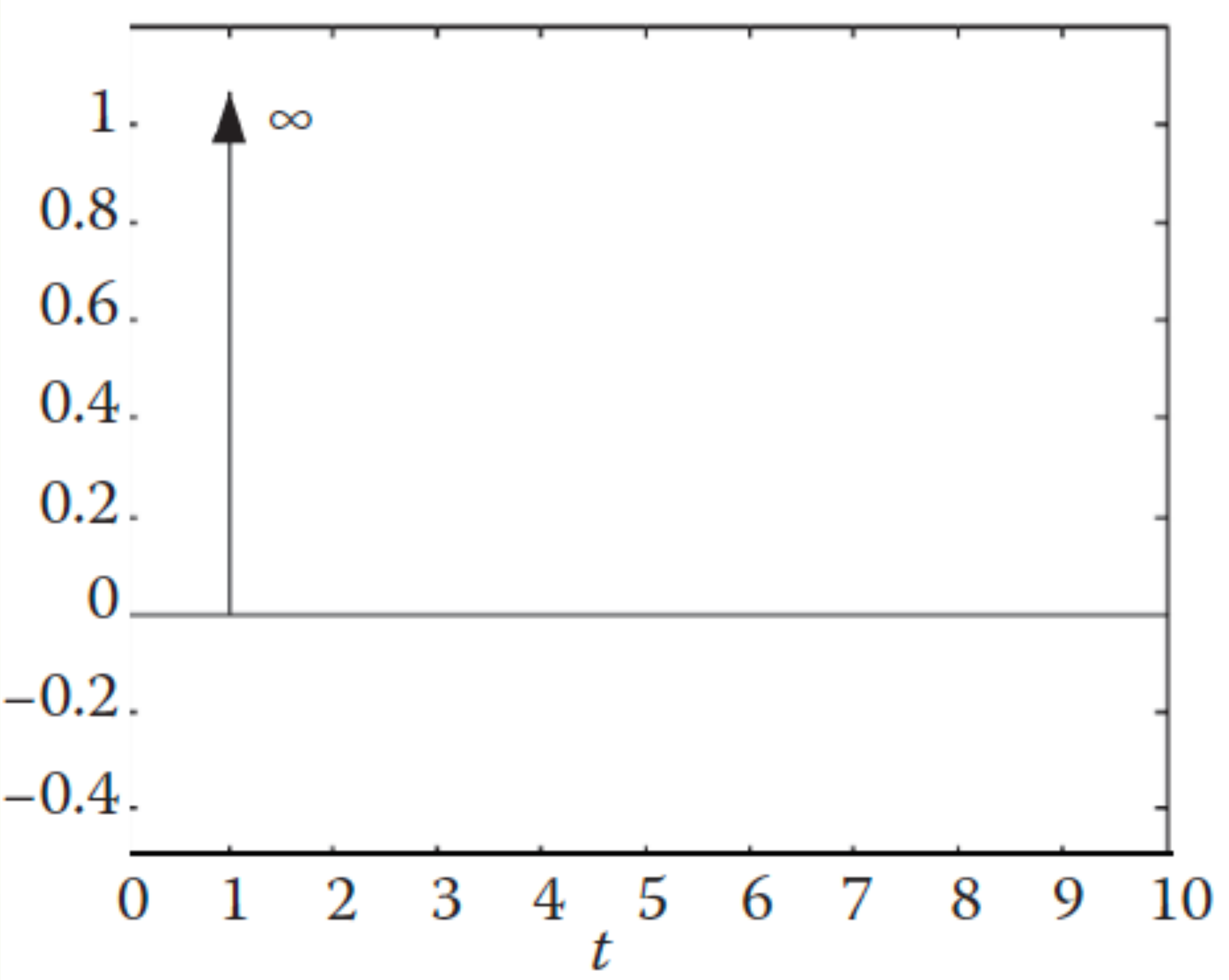
The pulse sketched in figure is represented by

$$f(t)=\begin{cases}0 & t<0, t\geq T\\ H & 0\leq t\leq T\end{cases}$$

Substituting into $F(s)=L[f(t)]=\int_0^{\infty} f(t)e^{-st}\,dt$

$$L\left[f(t)\right]=\int_0^{\infty} f(t)e^{-st}\,dt=\int_0^{T} He^{-st}\,dt=-\frac{H}{s}e^{-st}\Big|_0^T=-\frac{H}{s}\left(e^{-sT}-1\right)=H\left[\frac{1-e^{-sT}}{s}\right]$$

$$L\left[f(t)\right]=H\left[\frac{1-e^{-sT}}{s}\right]$$



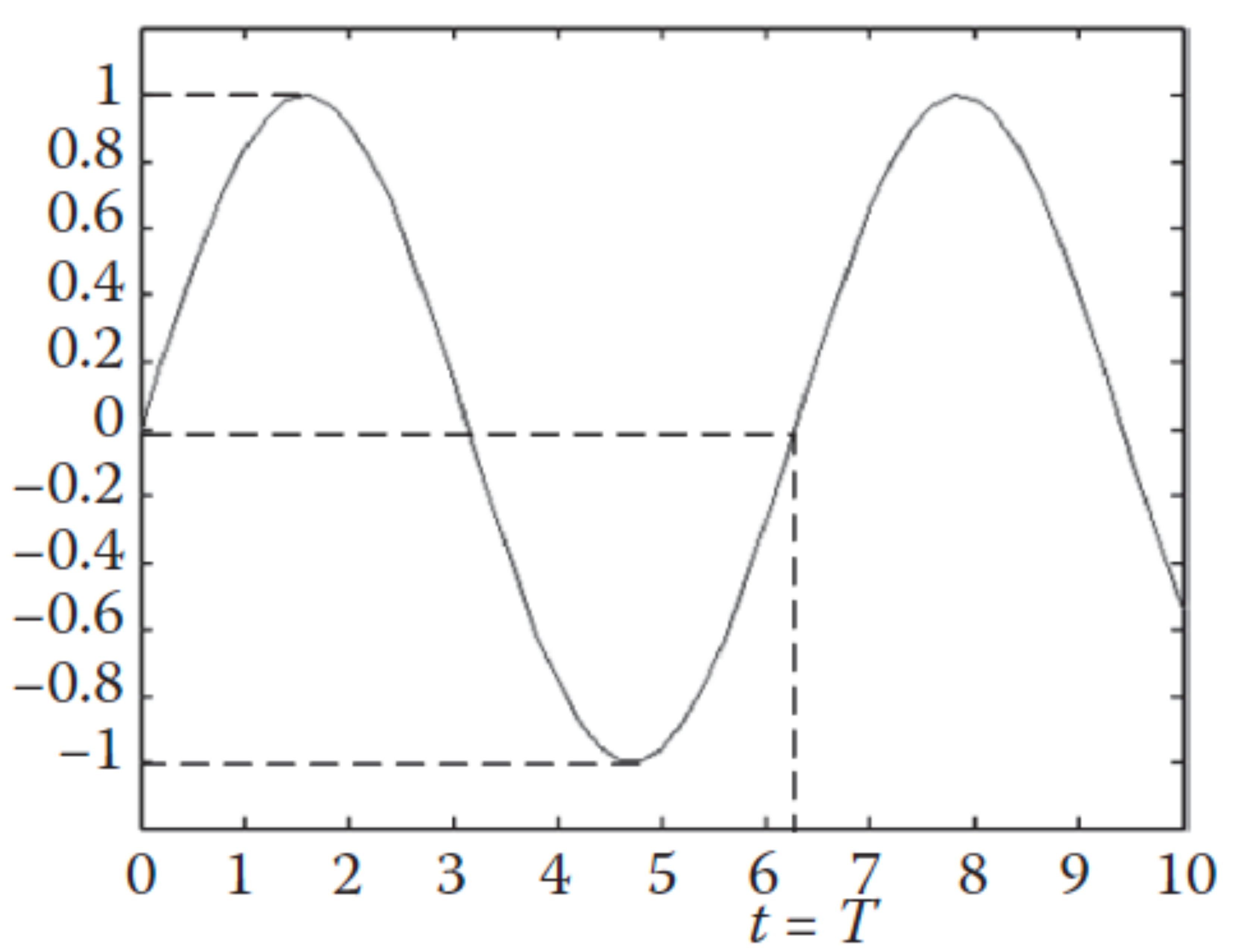
Common input signals: (c) unit impulse function

(c) A Unit Impulse Function

This unit impulse function, also known as the *Dirac delta function* and represented by $\delta(t)$, is sketched in figure. It is an ideal pulse with zero duration and unit area. All of its area is concentrated at time zero. Because the function is zero at all times except at zero, and the term e^{-st} in Equation $F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$ is equal to unity at t = 0, the Laplace transform is

$$L[\delta(t)] = \int_0^\infty \delta(t)e^{-st}\,dt = 1$$

The result of the integration, 1, is the area of the impulse.



Common input signals: (d) sine wave

(d) A Sine Wave of Unity Amplitude and Frequency ω

The sine wave is sketched in figure and is represented in exponential form by

$$\sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$$

where i= $\sqrt{-1}$ is the symbol for imaginary numbers. Substituting into equation

$$F(s) = L[f(t)] = \int_0^\infty f(t) e^{-st}\, dt$$

$$L[\sin\omega t] = \int_0^\infty \frac{e^{i\omega t} - e^{-i\omega t}}{2i} e^{-st}\, dt$$

$$= \frac{1}{2i}\int_0^\infty \left[e^{-(s-i\omega)t} - e^{-(s+i\omega)t}\right] dt$$

$$= \frac{1}{2i}\left[-\frac{e^{-(s-i\omega)t}}{s-i\omega} + \frac{e^{-(s+i\omega)t}}{s+i\omega}\right]_0^\infty$$

$$= \frac{1}{2i}\left[-\frac{0-1}{s-i\omega} + \frac{0-1}{s+i\omega}\right]$$

$$= \frac{1}{2i}\frac{2i\omega}{s^2+\omega^2}$$

$$= \frac{\omega}{s^2+\omega^2}$$

Laplace Transforms

Properties and Theorems of the Laplace Transform

Laplace Transforms of Common Functions are seen in the table.

$f(t)$	$F(s) = [f(t)]$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Linearity and the real differentiation and integration theorems are necessary for transforming differential equations into algebraic equations.

The final value theorem is useful for predicting the final steady-state value of a time function from its Laplace transform.

The real translation theorem is useful for dealing with functions delayed in time.

Linearity Property

Laplace transform is a linear operation. This means that if *a* is a constant,

$$L[af(t)] = aL[f(t)] = aF(s)$$

The distributive property of addition also follows from the linearity property:

$$L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)] = aF(s) + bG(s)$$

where a and b are constants.

Real Differentiation Theorem

This theorem establishes a relationship between the Laplace transform of a function and that of its derivatives, and it is most important in transforming differential equations into algebraic equations. It states that

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

From the definition of the Laplace transform, equation

$$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

$$L\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt} e^{-st}\, dt$$

Integration by parts yields

$$u(t) = e^{-st} \qquad dv = \frac{df(t)}{dt} dt$$

$$du = -se^{-st} dt \qquad v = f(t)$$

$$L\left[\frac{df(t)}{dt}\right] = \left[f(t)e^{-st}\right]_0^\infty - \int_0^\infty f(t)\left(-se^{-st} dt\right)$$

$$= \left[0 - f(0)\right] + s\int_0^\infty f(t)e^{-st}\, dt$$

$$= sF(s) - f(0)$$

The extension to higher derivatives is

$$L\left[\frac{d^2 f(t)}{dt^2}\right] = L\left[\frac{d}{dt}\left(\frac{df(t)}{dt}\right)\right]$$

$$= sL\left[\frac{df(t)}{dt}\right] - \left.\frac{df}{dt}\right|_{t-0}$$

$$= s\left[sF(s) - f(0)\right] - \left.\frac{df}{dt}\right|_{t=0}$$

$$L\left[\frac{d^2 f(t)}{dt^2}\right] = s^2F(s) - sf(0) - \left.\frac{df}{dt}\right|_{t=0}$$

In general,

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} \left.\frac{df}{dt}\right|_{t=0} - \cdots - \left.\frac{d^{n-1} f}{dt^{n-1}}\right|_{t=0}$$

As mentioned before, often at the initial state (initial condition), the variable is at steady state, meaning that all time derivatives are zero and that the variable itself is at some value that it is defined as zero. For this common case the preceding expression reduces to

$$L\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s)$$

Real Integration Theorem

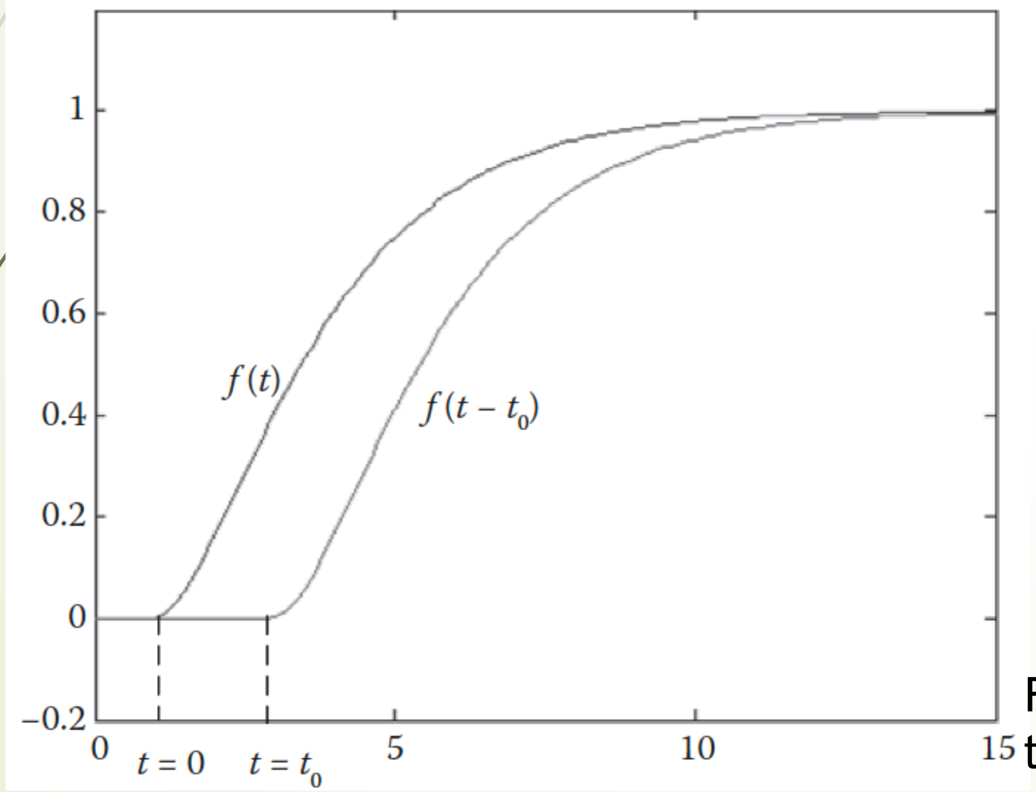
This theorem establishes the relationship between the Laplace transform of a function and that of its integral. It states that

$$L\left[\int_0^t f(t)dt\right] = \frac{1}{s}F(s)$$

The Laplace transform of the *n*th integral of a function is the transform of the function divided by s^n.

Real Translation Theorem

This theorem deals with the translation of a function in the time axis as shown in figure. The translated function is the original function delayed in time.



Function delayed in time is zero for all times less than the time delay t_0.

The theorem states that

$$L\left[f\left(t-t_0\right)\right]=e^{-st_0}F(s)$$

Because the Laplace transform does not contain information about the original function for negative time, the delayed function must be zero for all times less than the time delay.

This condition is satisfied if the initial condition(s) of the dependent variable(s) is (are) zero, or if not, the variable(s) is (are) expressed as deviation(s) from the initial steady-state condition(s).

From the definition of the Laplace transform, equation

$$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

$$L\left[f\left(t-t_0\right)\right] = \int_0^\infty f\left(t-t_0\right)e^{-st}\,dt$$

Let τ = t – t_0 (or t = t_0 – τ) and substitute.

$$L\left[f\left(t-t_0\right)\right] = \int_{\tau=-t_0}^{\infty} f(\tau)e^{-s\left(t_0+\tau\right)}d\left(t_0+\tau\right)$$

$$= \int_{\tau=0}^{\infty} f(\tau)e^{-st_0}e^{-s\tau}\,d\tau$$

$$= e^{-st_0}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau$$

$$= e^{-st_0}F(s)$$

f(τ)=0 for τ< 0 (t<t_0)

Final Value Theorem

This theorem allows us to figure out the final value of a function from its transform. If the limit of f(t) as t approaches ∞ exists, the final value can be found from its Laplace transform as follows:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}\left[s\,F(s) \right]$$

Complex Differentiation Theorem

This theorem is useful for evaluating the transforms of functions involving powers of the independent variable t. It states that

$$L\left[tf(t)\right]=-\frac{d}{ds}F(s)$$

Complex Translation Theorem

This theorem is useful for evaluating transforms of functions involving exponential functions of time.
It states that

$$L[e^{at} f(t)] = F(s - a)$$

Initial Value Theorem

This theorem allows the calculation of the initial value of a function from its transform. It would provide another check of the validity of derived transforms were it not for the fact that often the initial conditions of the variables are zero.

The theorem states that

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$$