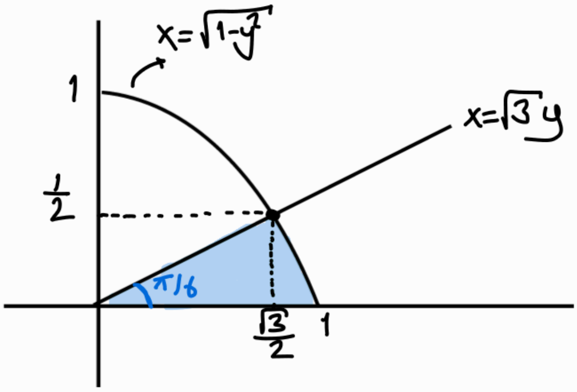
Ex.: $\displaystyle\int_0^{1/2}\int_{\sqrt{3}y}^{\sqrt{1-y^2}} x y^2 \, dx\, dy$



$\frac{y}{x} = \frac{1}{\sqrt{3}}$

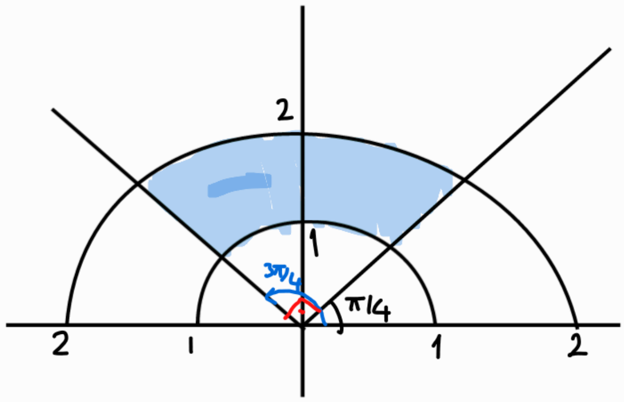
$$\int_0^{\pi/6}\int_0^{1} r^3\cos\theta\sin^2\theta \, r\,dr\,d\theta$$

$$\int_0^{\pi/6}\cos\theta\sin^2\theta\, d\theta \cdot \int_0^{1} r^4\, dr = \frac{\sin^3\theta}{3}\Big|_0^{\pi/6} \cdot \frac{r^5}{5}\Big|_0^{1} = \frac{\sin^3\frac{\pi}{6}}{3}\cdot\frac{1}{5} = \frac{1}{120}$$

$\sin\theta = u$

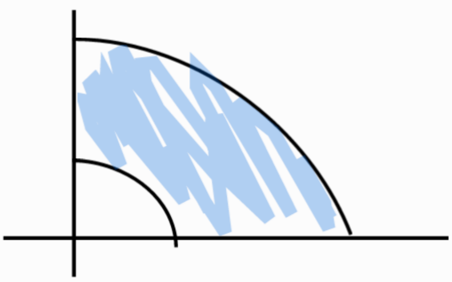
$\cos\theta\, d\theta = du$

Ex.: $\displaystyle\int_{\pi/4}^{3\pi/4}\int_1^2 r\,dr\,d\theta$



$$\int_{\pi/4}^{3\pi/4} d\theta \cdot \int_1^2 r\,dr = \theta\Big|_{\pi/4}^{3\pi/4} \cdot \frac{r^2}{2}\Big|_1^2 = \frac{\pi}{2}\cdot\left(\frac{4-1}{2}\right) = \frac{3\pi}{4}$$

Ex.: Evaluate the integral $\iint_R \sin(x^2+y^2)\,dA$ where R is the region in the first quadrant between the circles with center the origin and radii 1 and 3

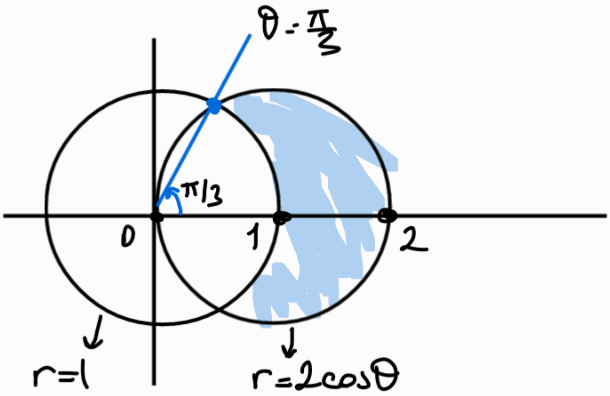


$$\int_0^{\pi/2}\int_1^3 \sin(r^2)\cdot r\,dr\,d\theta = -\frac{\cos(r^2)}{2}\Big|_1^3 \cdot \frac{\pi}{2}$$

$\frac{\pi}{2}$

$$= \frac{\pi}{4}(\cos 1 - \cos 9)$$

Ex.: Use a double integral to find the area of the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.



$$A = 2\int_0^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

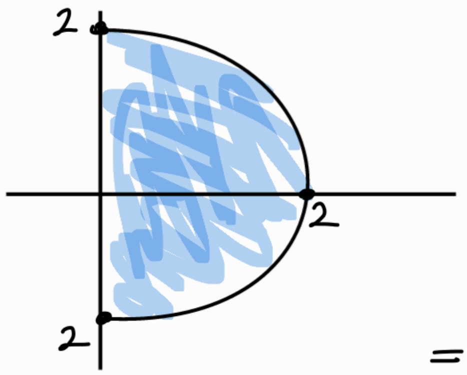
$$= \cancel{2}\int_0^{\pi/3} \frac{r^2}{\cancel{2}}\Big|_1^{2\cos\theta} d\theta = \int_0^{\pi/3}(4\cos^2\theta - 1)\,d\theta$$

$2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$$= \int_0^{\pi/3}\left(\cancel{2}^{\,1} + 2\cos(2\theta) - \cancel{1}\right)d\theta = \theta + \sin(2\theta)\Big|_0^{\pi/3}$$

$$A = \frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

Ex.: Evaluate the integral $\iint_R e^{-x^2-y^2}\,dA$ where R is the region bounded by the semicircle $x=\sqrt{4-y^2}$ and the y-axis.

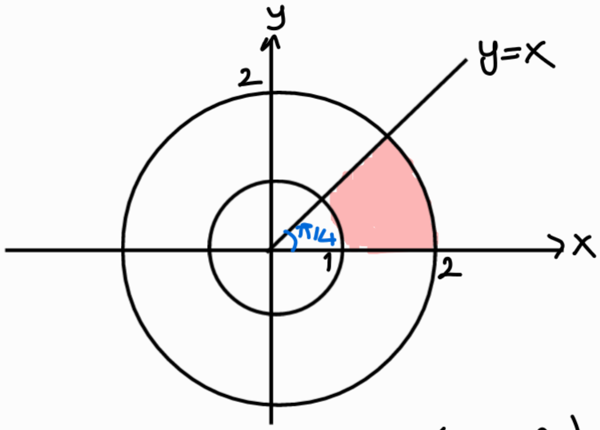


$$I = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta \cdot \int_0^2 e^{-r^2}\cdot r\,dr$$

$$= 2\cdot\int_0^{\pi/2} d\theta \cdot \int_0^2 e^{-r^2}\cdot r\,dr = \cancel{2}\cdot\frac{\pi}{\cancel{2}}\cdot\left(-\frac{e^{-r^2}}{2}\right)\Big|_0^2$$

$$\frac{\pi}{2}\cdot\left(-e^{-r^2}\right)\Big|_0^2 = \frac{\pi}{2}\left(-e^{-4}+e^0\right) = \frac{\pi}{2}\left(1-e^{-4}\right) = \frac{\pi}{2}\left(1-\frac{1}{e^4}\right)$$

Ex.: $\iint_R \arctan\left(\frac{y}{x}\right) dA$, where $R=\{(x,y) \mid 1\le x^2+y^2\le 4,\ 0\le y\le x\}$



$$\arctan\left(\frac{y}{x}\right)=\arctan\left(\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}\right)$$

$$=\arctan(\tan\theta)=\theta$$

$$I=\int_0^{\pi/4}\int_1^2 \theta\, r\,dr\,d\theta$$

$$=\int_0^{\pi/4}\theta\, d\theta\cdot\int_1^2 r\,dr$$

$$=\frac{\theta^2}{2}\Big|_0^{\pi/4}\cdot\frac{r^2}{2}\Big|_1^2=\frac{1}{4}\cdot\frac{\pi^2}{16}\cdot(4-1)$$

$$=\frac{3\pi^2}{64}$$

Ex.: Find the volume of the given solid below the plane $2x+y+z=4$ and above the disk $x^2+y^2\le 1$.

$$2x+y+z=4 \Rightarrow z=4-2x-y$$

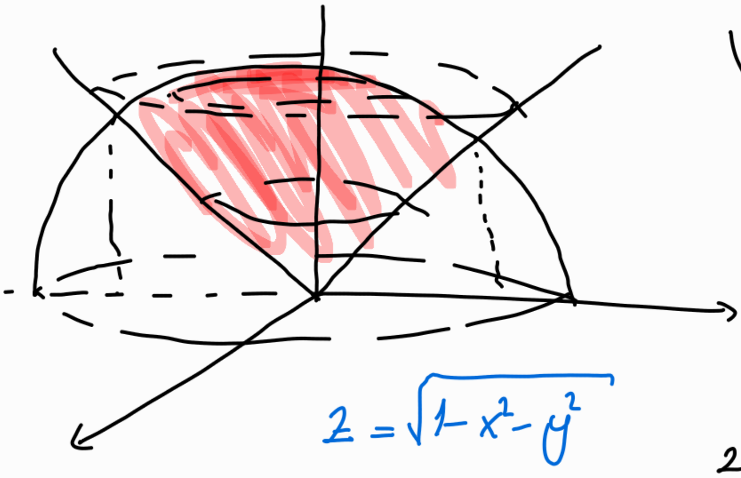
$$V=\iint_R (4-2x-y)\,dA=\int_0^{2\pi}\int_0^1 \underbrace{(4-2r\cos\theta-r\sin\theta)r}_{4r-2r^2\cos\theta-r^2\sin\theta}\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[2r^2-\frac{2}{3}r^3\cos\theta-\frac{r^3}{3}\sin\theta\right]_{r=0}^{r=1}d\theta=\int_0^{2\pi}\left[2-\frac{2}{3}\cos\theta-\frac{1}{3}\sin\theta\right]d\theta$$

$$=2\theta-\frac{2}{3}\sin\theta+\frac{1}{3}\cos\theta\Big|_0^{2\pi}=\left[4\pi-\underbrace{\frac{2}{3}\sin(2\pi)}_{0}+\frac{1}{3}\cos(2\pi)\right]-\left[\frac{1}{3}\cos 0\right]$$

$$=\underline{\underline{4\pi}}$$

Ex.: Use polar coordinates to find the volume of the given solid above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$.



$$V=\iint_R \left(\sqrt{1-x^2-y^2}-\sqrt{x^2+y^2}\right)dA$$

$$\sqrt{1-x^2-y^2}=\sqrt{x^2+y^2}$$

$$\sqrt{1-r^2}=\sqrt{r^2}$$

$$1-r^2=r^2$$

$$2r^2=1 \Rightarrow r=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

$$V=\int_0^{2\pi}\int_0^{\sqrt{2}/2}\underbrace{\left(\sqrt{1-r^2}-r\right)r}_{r\sqrt{1-r^2}-r^2}\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[-\frac{(1-r^2)^{3/2}}{3}-\frac{r^3}{3}\right]_0^{\sqrt{2}/2}d\theta$$

$$=\int_0^{2\pi}d\theta\cdot\frac{1}{3}\cdot\left[-(1-r^2)^{3/2}-r^3\right]_0^{\sqrt{2}/2}$$

$$=\frac{2\pi}{3}\left[-\underbrace{\left(1-\frac{1}{2}\right)}_{\frac{1}{2}}{}^{3/2}-\frac{2\sqrt{2}}{8}+1^{3/2}\right]$$

$$=\frac{2\pi}{3}\left[\underbrace{\frac{-1}{2\sqrt{2}}}_{\frac{\sqrt{2}}{4}}-\frac{\sqrt{2}}{4}+1\right]=\frac{2\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right)=\frac{\pi}{3}\left(2-\sqrt{2}\right)$$