Example: Solve the following system via Smith-Normal form.

$$\frac{dx_1}{dt} = x_1 + 2x_2 + x_3$$

$$\frac{dx_2}{dt} = x_1 + 3x_2 + 2x_3$$

$$\frac{dx_3}{dt} = -2x_2$$

$$(D-1)x_1 - 2x_2 - x_3 = 0$$

$$-x_1 + (D-3)x_2 - 2x_3 = 0$$

$$2x_2 + Dx_3 = 0$$

$$\underbrace{\begin{bmatrix} D-1 & -2 & -1 \\ -1 & D-3 & -2 \\ 0 & 2 & D \end{bmatrix}}_{B} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{H}$$

$$\boxed{B \cdot X = 0}$$

$$P \cdot B \cdot Q = N$$

$$\boxed{X = Q \cdot Y}$$

$$B \cdot X = 0$$

$$B \cdot Q \cdot Y = B \cdot X = 0$$

$$\underbrace{P \cdot B \cdot Q}_{N} \cdot Y = 0 \quad \Rightarrow \quad \boxed{N \cdot Y = 0}$$

$$B = \begin{bmatrix} D-1 & -2 & -1 \\ -1 & D-3 & -2 \\ 0 & 2 & D \end{bmatrix} \sim \begin{bmatrix} 1 & -D+3 & 2 \\ D-1 & -2 & -1 \\ 0 & 2 & D \end{bmatrix}$$

R_{12} , $R_1(-1)$ $\qquad$ $C_{21}(D-3)$, $C_{31}(-2)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ D-1 & D^2-4D+1 & -2D+1 \\ 0 & 2 & D \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & D^2-4D+1 & -2D+1 \\ 0 & 2 & D \end{bmatrix}$$

$R_{21}(-D+1)$ $\qquad$ $R_{23}(2)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & D^2-4D+5 & 1 \\ 0 & 2 & D \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & D^2-4D+5 \\ 0 & D & 2 \end{bmatrix}$$

C_{23} $\qquad$ $C_{32}(-D^2+4D-5)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & D & -D^3+4D^2-5D+2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -D^3+4D^2-5D+2 \end{bmatrix}$$

$R_{32}(-D)$ $\qquad$ $R_3(-1)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D^3-4D^2+5D-2 \end{bmatrix} = N$$

$$Q = \begin{bmatrix} 1 & -2 & 2D^2-7D+7 \\ 0 & 0 & 1 \\ 0 & 1 & -D^2+4D-5 \end{bmatrix} \leftarrow \underline{\text{find it}}$$

$$N.Y = 0$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D^3-4D^2+5D-2 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y_1 = 0$$

$$y_2 = 0$$

$$(D^3-4D^2+5D-2)\cdot y_3 = 0$$

$$y_3''' - 4y_3'' + 5y_3' - 2y_3 = 0$$

$$r^3 - 4r^2 + 5r - 2 = 0$$

$r_1 = 1$

	1	−4	5	−2
		1	−3	2
1	1	−3	2	0
	1			

$$r^2 - 3r + 2 = 0$$

$$r_2 = 1 \quad r_3 = 2$$

$$y_3 = c_1 e^t + c_2 \cdot t \cdot e^t + c_3 e^{2t}$$

$$X = Q \cdot Y$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2D^2-7D+7 \\ 0 & 0 & 1 \\ 0 & 1 & -D^2+4D-5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ c_1 e^t + c_2 \cdot t \cdot e^t + c_3 e^{2t} \end{bmatrix}$$

$$x_1 = (2D^2-7D+7)\cdot(c_1 e^t + c_2 \cdot t \cdot e^t + c_3 e^{2t})$$

$$x_2 = c_1 e^t + c_2 \cdot t \cdot e^t + c_3 e^{2t}$$

$$x_3 = (-D^2+4D-5)(c_1 e^t + c_2 t e^t + c_3 e^{2t})$$

Find the final result. !!