LAPLACE TRANSFORMS

The transformation

$$T\ \{ f(t) \} = \int_{-\infty}^{+\infty} f(t)\ .\ K(s,t)\ dt = F(s)$$

is called "Integral Transform". Here $K(s,t)$ is the Kernel function.

$$T\ \{ f(t) \} = F(s)$$

For Fourier transform, the kernel function is

$$K(s,t) = \begin{cases} 0 & , \quad t \prec 0 \\ e^{-2\pi i s t} \quad (i = \sqrt{-1}) & , \quad t \geq 0 \end{cases}$$

For Laplace transform, the kernel function is

$$K(s,t) = \begin{cases} 0 & , \quad t \prec 0 \\ e^{-st} & , \quad t \geq 0 \end{cases}$$

$$L\ \{ f(t) \} = \int_{-\infty}^{+\infty} f(t)\ .\ K(s,t)\ dt$$

$$L\ \{ f(t) \} = \int_{-\infty}^{0} f(t)\ .\ 0\ dt + \int_{0}^{+\infty} f(t)\ .\ e^{-st}\ dt$$

$$L\ \{ f(t) \} = \int_{0}^{+\infty} f(t)\ .\ e^{-st}\ dt = F(s)$$

$$L\ \{ f(t) \} = F(s)$$

LAPLACE TRANSFORM OF FUNDAMENTAL FUNCTIONS

$f(t)$	$L\{f(t)\} = F(s)$
$f(t) = C$	$L\{C\} = \dfrac{C}{s} \qquad (C \in R)$
$f(t) = e^{at}$	$L\{e^{at}\} = \dfrac{1}{s-a}$
$f(t) = t^n$	$L\{t^n\} = \int_0^\infty t^n e^{-st}\,dt = \dfrac{n!}{s^{n+1}} \quad , \qquad (n \in N^+)$
$f(t) = \sin kt$	$L\{Sin(kt)\} = \dfrac{k}{s^2+k^2} \quad , \qquad (k \in R)$
$f(t) = \cos kt$	$L\{Cos(kt)\} = \dfrac{s}{s^2+k^2} \quad , \qquad (k \in R)$
$f(t) = \sinh kt$	$L\{Sinh(kt)\} = \dfrac{k}{s^2-k^2} \quad , \qquad (k \in R)$
$f(t) = \cosh kt$	$L\{Cosh(kt)\} = \dfrac{s}{s^2-k^2} \quad , \qquad (k \in R)$

PROPERTIES OF LAPLACE TRANSFORMS

1. Linearity

$$L\{C_1 . f_1(t) + C_2 . f_2(t)\} = L\{C_1 . f_1(t)\} + L\{C_2 . f_2(t)\} = C_1 .\ L\{f_1(t)\} + C_2 . L\{f_2(t)\}$$

$$= C_1 .\ F_1(s) + C_2 .\ F_2(s)$$

Ex. $L\{\ 3t^2 - 4\ Sin(3t) - 7e^{-2t} + 2\ Cosh(t) + 5\ \}$

$$= \frac{6}{s^3} - \frac{12}{s^2+9} - \frac{7}{s+2} + \frac{2s}{s^2-1} + \frac{5}{s}$$

Ex. $f(t) = Sin(kt) .\ Cos(kt)$

$$L\left\{\frac{1}{2}\left[Sin(2kt)\right]\right\} = \frac{1}{2} L\{\ Sin(2kt)\} = \frac{1}{2}\ \frac{2k}{s^2+(2k)^2}$$

2. Shift

$L\{f(t)\} = F(s)$, $L\{e^{at} .\ f(t)\} = F(s-a)$

Ex. $L\left\{ t \, . \, e^{t} \right\}$

$$F(s) = L\{f(t)\} = L\{t\} = \frac{1}{s^2}$$

$$L\left\{ t \, . \, e^{t} \right\} = F(s-1) = \frac{1}{(s-1)^2}$$

Ex. $L\left\{ e^{3t} \, . \, Sin(4t) \right\}$

$$L\{Sin(4t)\} = \frac{4}{s^2 + 16}$$

$$L\left\{ e^{3t} \, . \, Sin(4t) \right\} = \frac{4}{(s-3)^2 + 16} = \frac{4}{s^2 - 6s + 25}$$

Ex. $L\left\{ e^{-t} \, . \, Cos(2t) \right\}$

$$F(s) = L\{Cos(2t)\} = \frac{s}{s^2 + 4}$$

$$L\left\{ e^{-t} \, Cos(2t) \right\} = F(s+1) = \frac{s+1}{(s+1)^2 + 4}$$

Ex. $h(t) = e^{2t} \, . \, Sinh(3t)$

$$L\{Sinh(3t)\} = \frac{3}{s^2 - 3^2} = \frac{3}{s^2 - 9}$$

$$L\{h(t)\} = L\left\{ e^{2t} \, . \, Sin(3t) \right\} = G(s-2) = \frac{3}{(s-2)^2 - 9}$$

3. Multiplication with t^n

$$L\{F(t)\} = f(s)$$

$$L\left\{ t^n \, . \, f(t) \right\} = (-1)^n \, . \, \frac{d^n}{ds^n}\left[F(s)\right] = (-1)^n \, . \, F^{(n)}(s)$$

Ex. $L\left\{ t^2 \, . \, e^{2t} \right\}$

$$L\left\{ e^{2t} \right\} = \frac{1}{s-2}$$

$$L\left\{ t e^{2t} \right\} = -\frac{d}{ds}\left(\frac{1}{s-2}\right) = \frac{1}{(s-2)^2}$$

$$L\left\{ t^2 e^{2t} \right\} = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{2}{(s-2)^3}$$

4. Laplace Transforms of Derivatives

$$L\{ f(t) \} = F(s)$$

$$L\left\{ f^{(n)}(t) \right\} = s^n . F(s) - s^{n-1} . f(0) - s^{n-2} . f'(0) - ..... - s . f^{(n-2)}(0) - f^{(n-1)}(0)$$

Ex. $f(t) = Cos\,3t$, $L\{ f'(t) \}$

$L\{ F'(t) \} = = s\, f(s) - F(0)$, $L\{ Cos\ 3t \} = \frac{s}{s^2+9}$ $F(0) = Cos\,0 = 1$

$$L\{ F'(t) \} = s . \frac{s}{s^2+9} - 1 = \frac{s^2 - s^2 - 9}{s^2+9} = -\frac{9}{s^2+9}$$

Ex. $L\left\{ (e^{2t} \sin 2t)'' \right\} = ?$

$$L\{ F''(t) \} = s^2 f(s) - s F(0) - F'(0)$$

$L\left\{ (e^{2t} \sin 2t) \right\} = \frac{2}{(s-2)^2+4} = f(s)$ $F(0) = 0$

$F'(t) = 2e^{2t} Sin\,2t + 2e^{2t} Cos\,2t$ $F'(0) = 0$

$$L\left\{ (e^{2t} \sin 2t)'' \right\} = s^2 \frac{2}{(s-2)^2+4} - 2 = \frac{2s^2 - 2s^2 + 8s - 16}{s^2 - 4s + 8}$$

$$= \frac{8s - 16}{s^2 - 4s + 8}$$

INVERSE LAPLACE TRANSFORMS

$$L^{-1}\{F(s)\}$$

$$L\{f(t)\} = F(s). \quad \Leftrightarrow \quad L^{-1}\{F(s)\} = f(t)$$

INVERSE LAPLACE TRANSFORM TABLE

$F(s) = \frac{1}{s}$ ise $L^{-1}\{F(s)\} = f(t) = 1$

$F(s) = \frac{1}{s^2}$ ise $L^{-1}\{F(s)\} = f(t) = t$

$F(s) = \frac{1}{s^3}$ ise $L^{-1}\{F(s)\} = f(t) = \frac{t^2}{2!}$

$F(s) = \frac{1}{s^{n+1}} (n \in N^+)$ ise $L^{-1}\{F(s)\} = f(t) = \frac{t^n}{n!}$

$F(s) = \frac{1}{s-a}$ ise $L^{-1}\{F(s)\} = f(t) = e^{at}$

$F(s) = \frac{1}{(s-a)^2}$ ise $L^{-1}\{F(s)\} = f(t) = te^{at}$

$F(s) = \frac{1}{(s-a)^3}$ ise $L^{-1}\{F(s)\} = f(t) = \frac{1}{2!} t^2 e^a$

....

....

$F(s) = \frac{1}{(s-a)^{n+1}}$ ise $L^{-1}\{F(s)\} = f(t) = \frac{1}{n!} t^n e^{at}$

$F(s) = \frac{1}{s^2+k^2}$ ise $L^{-1}\{F(s)\} = f(t) = \frac{Sin(kt)}{k}$

$F(s) = \frac{s}{s^2+k^2}$ ise $L^{-1}\{F(s)\} = f(t) = Cos(kt)$

$F(s) = \frac{1}{s^2-k^2}$ ise $L^{-1}\{F(s)\} = f(t) = \frac{Sinh(kt)}{k}$

$F(s) = \frac{s}{s^2-k^2}$ ise $L^{-1}\{F(s)\} = f(t) = Cosh(kt)$

Ex. $$L^{-1}\left\{\frac{1}{s^2+4}\right\}=L^{-1}\left\{\frac{1}{s^2+2^2}\right\}=\frac{Sin\,(2t)}{2}$$

Ex. $$L^{-1}\left\{\frac{s}{s^2-9}\right\}=L^{-1}\left\{\frac{s}{s^2-3^2}\right\}=Cosh\,(3t)$$,

Ex. $$L^{-1}\left\{\frac{1}{s+3}\right\}=e^{-3t}$$

PROPERTIES OF INVERSE LAPLACE TRANSFORMS

1. Linearity

$$L^{-1}\{c_1 f_1(s)+c_2 f_2(s)\}=c_1 L^{-1}\{f_1(s)\}+c_2 L^{-1}\{f_2(s)\}$$

$$=c_1 F_1(t)+c_2 F_2(t)$$

Ex. $$L^{-1}\left\{\frac{4}{s-2}-\frac{3s}{s^2+16}+\frac{5}{s^2+4}\right\}=?\quad 4L^{-1}\left\{\frac{1}{s-2}\right\}-3L^{-1}\left\{\frac{s}{s^2+4^2}\right\}+5L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=4e^{2t}-3\,Cos\,4t+\frac{5}{2}Sin\,2t$$

Ex. $$L\{(Cos\,3t-Sin\,3t)^2\}=?$$

$$L\{Cos^2 3t-2\,Cos\,3t.\,Sin\,3t+Sin^2 3t\}=L\left\{\underbrace{Cos^2 3t+Sin^2 3t}_{1}-\underbrace{2\,Cos\,3t.\,Sin\,3t}_{Sin\,6t}\right\}$$

$$=\frac{1}{s}-\frac{6}{s^2+36}=\frac{s^2+36-6s}{s(s^2+36)}$$

2. Shift

$L^{-1}\{f(s)\}=F(t)$ ise

$$L^{-1}\{f(s-a)\}=e^{at}F(t)$$

Ex. $$L^{-1}\left\{\frac{1}{s^2-2s+5}\right\}=?$$

$$L^{-1}\left\{\frac{1}{s^2-2s+5}\right\}=L^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}=\frac{e^t\,Sin\,2t}{2}$$

3. Inverse Laplace Transform of Derived functions

$L^{-1}\{F(s)\} = f(t)$ ise

$$L^{-1}\{F^{(n)}(s)\} = L^{-1}\left\{\frac{d^n}{ds^n}[F(s)]\right\} = (-1)^n t^n f(t)$$

Ex. $F(s) = \dfrac{-2s}{(s^2+1)^2}$

$$\frac{-2s}{(s^2+1)^2} = \frac{d}{ds}\left(\frac{1}{s^2+1}\right) \qquad L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

Ex. $L\{t^3 e^{2t}\} = ?$

$$L\{e^{2t}\} = F(s) = \frac{1}{s-2}$$

$$L\{t\,e^{2t}\} = -F'(s) = \frac{1}{(s-2)^2}$$

$$L\{t^2 e^{2t}\} = -f''(s) = \frac{2}{(s-2)^3}$$

$$L\{t^3 e^{2t}\} = -f'''(s) = \frac{6}{(s-2)^4}$$

Ex. $F(s) = \dfrac{2s^2-1}{(s^2-s-2)(s-3)}$ ise $L^{-1}\{F(s)\} = f(t) = ?$

$$\frac{2s^2-1}{(s^2-s-2)(s-3)} = \frac{A}{(s+1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$2s^2-1 = As^2-5As+6A+Bs^2-2Bs-3B+Cs^2-Cs-2C$$

$$\left.\begin{array}{l} A+B+C=2 \\ -5A-2B-C=0 \\ 6A-3B-2C=1 \end{array}\right\} \quad A=\frac{1}{12} \quad B=-\frac{7}{3} \quad C=\frac{17}{4}$$

$$L^{-1}\left\{\frac{\frac{1}{12}}{s+1}+\frac{\frac{-7}{3}}{s-2}+\frac{\frac{17}{4}}{s-3}\right\}$$

$$=\frac{1}{12}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{7}{3}L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{17}{4}L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$=\frac{1}{12}e^{-t}-\frac{7}{3}e^{2t}+\frac{17}{4}e^{3t}$$

Ex. $f(s)=\dfrac{s}{s^2-s+1}$ ise $L^{-1}\{f(s)\}=F(t)=?$

$$=L^{-1}\left\{\frac{s-\frac{1}{2}+\frac{1}{2}}{(s-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\}=L^{-1}\left\{\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{(s-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\}$$

$$=e^{\frac{1}{2}t}\,Cos\,\frac{\sqrt{3}}{2}t+\frac{1}{2}e^{\frac{1}{2}t}\,\frac{Sin\,\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}$$

$$=e^{\frac{1}{2}t}\,Cos\,\frac{\sqrt{3}}{2}t+\frac{1}{\sqrt{3}}e^{\frac{1}{2}t}\,Sin\,\frac{\sqrt{3}}{2}t$$

$$=e^{\frac{1}{2}t}\left(Cos\,\frac{\sqrt{3}}{2}t+\frac{\sqrt{3}}{3}\,Sin\,\frac{\sqrt{3}}{2}t\right)$$