SeMA Journal
https://doi.org/10.1007/s40324-022-00290-5

On exact solutions of some important nonlinear conformable time-fractional differential equations

Erdoğan Mehmet Özkan[1] · **Ayten Özkan**[1]

Received: 20 June 2021 / Accepted: 27 February 2022
© The Author(s), under exclusive licence to Sociedad Española de Matemática Aplicada 2022

Abstract
The nonlinear fractional Boussinesq equations are known as the fractional differential equation class that has an important place in mathematical physics. In this study, a method called $\left(\frac{G'}{G^2}\right)$-extension method which works well and reveals exact solutions is used to examine nonlinear Boussinesq equations with conformable time-fractional derivative. This method is a very useful approach and extremely utility compared to other analytical methods. With the proposed method, there are three unique types of solutions such as hyperbolic, trigonometric and rational solutions. This approach can similarly be applied to other nonlinear fractional models.

Keywords Conformable fractional derivative · $\left(\frac{G'}{G^2}\right)$-expansion method · Exact solutions

Mathematics Subject Classification 26A33 · 35R11 · 83C15

1 Introduction

Nonlinear partial differential equations (NPDEs) are very beneficial equations that are important in different fields including engineering sciences, mathematics, fluid dynamics, and physics. Many real world problems have been modeled thanks to NPDEs. Numerous different and robust mathematical methods have been applied to obtain exact solutions of NPDEs [4, 5, 16–18, 23, 26, 29, 35–38, 40–43]. Fractional calculus is also a new field that has grown in popularity over the past few decades. Various physical phenomena are made easily solvable in fractional differential equations (FDEs), such as viscoelasticity, plasma, solid mechanics, optical fibers, signal processing, electromagnetic waves, fluid dynamics, biomedical sciences and diffusion processes. Researchers have made these equations attractive by using various methods to obtain exact solutions of FDEs. Some of these methods and the important ones in the literature can be listed as $\left(\frac{G'}{G}\right)$-expansion method [10], the improved extended tanh-coth

✉ Erdoğan Mehmet Özkan
mozkan@yildiz.edu.tr

[1] Department of Mathematics, Yildiz Technical University, Istanbul, Turkey

method [20], Lie group analysis method [15], first integral method [27], exp-function method [44], sub-equation method [9], functional variable method [30].

Boussinesq type equations can be considered as the first model for nonlinear, dispersive wave propagation and describe the surface water waves whose horizontal scale is much larger than the depth of the water [28]. They can be noted as a critical class of fractional differential equations in mathematical physics. Recently, different techniques have been used to find analytical and numerical solutions of Boussinesq equations. These can be mentioned as invariant subspace method [32], a newly developed method called the expansion method [22], exp-function method [39].

Conformable time fractional Boussinesq equations, which can be employed to describe the propagation of the wave in the magnetic field, describes surface water waves. In addition, it is an important nonlinear model that occurs in physics, hydromechanics and optics. It is also known that it can be used to describe the physical direction of wave propagation in plasma and nonlinear wave [1, 2, 11, 12, 34]. The coupled Boussinesq equations occur in shallow water waves for bilayer fluid flow. This happens when there is an accidental oil spill from a ship, which causes a slick of oil to float above the water slide [24].

The study includes five sections. In the first section, fractional calculations are mentioned along with brief information about nonlinear partial and fractional differential equations. In addition, Boussinesq equations and coupled Bousinesq equations that form the basis of the publication are explained. In the second section, definition and theories of comformable fractional derivative are given. In the third section $\left(\frac{G'}{G^2}\right)$-expansion method which is an important method for the solution is explained. In section four, the application of finding exact solutions of comformable time fractional Boussinesq equation and coupled conformable time fractional Boussinesq equations with this method has been made. Finally, an explanation of the results is given in the last section.

2 Conformable fractional derivative and its properties

Not only from a theoretical viewpoint, but also in terms of applications, the fractional order type is crucial. Physical, chemical, biological, and ecological phenomena all produce them naturally. They're used to describe how diverse materials and processes remember and pass on their qualities. Because of the nature of the material microstructure, viscoelasticity, for example, possesses both elastic solid and viscous liquid response qualities [6–8]. Various definitions have been introduced for the fractional derivative such as Riemann-Liouville, Grünwald-Letnikov and Caputo [13, 31, 33]. All fractional derivatives provide linearity, but these definitions have some flaws. Khalil et al. [25] described a new, simple fractional derivative named the comformable fractional derivative that overcomes these shortcomings. The conformable derivative has the following definition and properties as given in [3, 19, 25].

Definition: Suppose $f : [0, \infty) \to \mathbf{R}$ be a function. The αth-order conformable fractional derivative of f is defined by

$$ {}_tD^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0, \quad \alpha \in (0, 1). $$

Theorem 1 *Suppose $\alpha \in (0, 1]$ and f, g be α-differentiable at $t > 0$. Then,*

(1) ${}_tD^{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g), \quad \forall a, b \in \mathbf{R}.$
(2) ${}_tD^{\alpha}(t^p) = pt^{p-\alpha}, \quad \forall p \in \mathbf{R}.$

(3) ${}_t D^{\alpha}(\lambda) = 0$ (λ *is any constant).*
(4) ${}_t D^{\alpha}(fg) = f({}_t D^{\alpha}(g)) + g({}_t D^{\alpha}(f))$.
(5) ${}_t D^{\alpha}(\frac{f}{g}) = \frac{g({}_t D^{\alpha}(f)) - f({}_t D^{\alpha}(g))}{g^2}$.
(6) *If* f *is differentiable, then* ${}_t D^{\alpha}(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

Theorem 2 *Let* $f, g : (0, \infty) \to \mathbf{R}$ *be* α*-differentiable functions, when* $0 < \alpha \leq 1$ *. Then,*

$$ {}_t D^{\alpha}(f \circ g)(t) = t^{1-\alpha} g'(t) f'(g(t)). $$

3 Algorithm of $\left(\frac{G'}{G^2}\right)$-expansion method

In this section, we give an presentation of $\left(\frac{G'}{G^2}\right)$-expansion method for solving a general space time fractional differential equation as

$$ H\left(u, \frac{\partial^{\alpha} u}{\partial t^{\alpha}}, \frac{\partial u}{\partial x}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2} \ldots\right) = 0, \tag{3.1} $$

where u is an unknown function and H is a polynomial of u and its partial fractional derivatives [14]. It can be introduced as follows.

Step 1: By applying the transformation

$$ \begin{aligned} u(x,t) &= U(\varepsilon), \\ \varepsilon &= x + k\frac{t^{\alpha}}{\alpha}, \end{aligned} \tag{3.2} $$

which k is nonzero constant coefficient, Eq. (3.1) can be reduced into the ordinary differential equation

$$ Q\left(U, \frac{dU}{d\varepsilon}, \frac{d^2U}{d\varepsilon^2}, \frac{d^3U}{d\varepsilon^3}, \ldots\right) = 0. \tag{3.3} $$

Step 2: Assume that Eq. (3.3) has the solution

$$ U(\varepsilon) = a_0 + \sum_{i=1}^{M} \left[a_i \left(\frac{G'}{G^2}\right)^{i} + b_i \left(\frac{G'}{G^2}\right)^{-i} \right], \tag{3.4} $$

and

$$ \left(\frac{G'}{G^2}\right)' = \mu + \lambda \left(\frac{G'}{G^2}\right)^2 \tag{3.5} $$

where $\lambda \neq 0$ $\mu \neq 1$ are integers and $a_0, a_i, b_i (i = 1, 2, \ldots, M)$ are real constants to be determined. The balancing number M is a positive integer which can be determined by balancing the highest derivative terms with the highest power nonlinear terms in Eq. (3.3).

Step 3: Substituting (3.4) into (3.3) using (3.5) and setting the coefficients of all powers of $\frac{G'}{G^2}$ to zero, the system of algebraic equations will be obtained. This algebraic system is solved by using programs such as Maple to obtain the values of the unknown constants a_0, a_i and $b_i (i = 1, 2, \ldots, M)$. More clearly, the value of M can be found from Eq. (3.3) as follows

where $D(U(\varepsilon)) = M$ is degree of $U(\varepsilon)$.

$$D\left[\frac{d^qU}{d\varepsilon^q}\right] = M + q,$$
$$D\left[U^r\left(\frac{d^qU}{d\varepsilon^q}\right)^s\right] = Mr + s(q + M).$$

Step 4: On the basis of the general solution of Eq. (3.5), the required exact solutions will be obtained in the following three cases:
Case 1. If $\lambda\mu > 0$, then

$$\frac{G'}{G^2} = \sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right), \tag{3.6}$$

Case 2. If $\lambda\mu < 0$, then

$$\frac{G'}{G^2} = -\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right), \tag{3.7}$$

Case 3. If $\lambda \neq 0$ and $\mu = 0$, then

$$\frac{G'}{G^2} = -\frac{C}{\lambda(C\varepsilon + D)}, \tag{3.8}$$

where C, D are nonzero constants.

4 Applications

In this section, two important nonlinear fractional differential equations will be solved by $\left(\frac{G'}{G^2}\right)$-expansion method.

4.1 Conformable time fractional Boussinesq equation

Let us consider the following nonlinear equation [21]:

$$\frac{\partial^{2\alpha}u}{\partial t^{2\alpha}} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0, \quad 0 < \alpha \leq 1. \tag{4.1}$$

Applying the transformation Eq. (3.2) to this equation, the following ordinary differential equation is obtained.

$$(k^2 - 1)\frac{d^2U}{d\varepsilon^2} - \frac{d^2(U^2)}{d\varepsilon^2} + \frac{d^4U}{d\varepsilon^4} = 0. \tag{4.2}$$

Integrating Eq. (4.2) twice and taking constants of integration as zero, the following equation is obtained.

$$\frac{d^2U}{d\varepsilon^2} + (k^2 - 1)U - U^2 = 0. \tag{4.3}$$

Balancing the terms of U^2 and $\frac{d^2U}{d\varepsilon^2}$ in Eq. (4.3), we get $M = 2$. For $M = 2$, solution will get the form

$$U(\varepsilon) = a_0 + a_1\left(\frac{G'}{G^2}\right) + a_2\left(\frac{G'}{G^2}\right)^2 + b_1\left(\frac{G'}{G^2}\right)^{-1} + b_2\left(\frac{G'}{G^2}\right)^{-2}, \tag{4.4}$$

where a_0, a_1, a_2, b_1 and b_2 are unknown constants.
Substituting (4.4) using (3.5) into (4.3), setting the coefficients of all powers of $\left(\frac{G'}{G^2}\right)$ to be zero, nonlinear algebraic equations are achieved. The occuring algebraic system is solved by aid of Maple to find the values of unknown constants.

$$\begin{aligned}
\left(\frac{G'}{G^2}\right)^{-4} &: \quad 6\mu^2 b_2 - b_2^2 = 0,\\
\left(\frac{G'}{G^2}\right)^{-3} &: \quad 2\mu^2 b_1 - 2b_1 b_2 = 0,\\
\left(\frac{G'}{G^2}\right)^{-2} &: \quad 8\mu\lambda b_2 + (k^2-1)b_2 - b_1^2 - 2b_2 a_0 = 0,\\
\left(\frac{G'}{G^2}\right)^{-1} &: \quad 2\mu\lambda b_1 + (k^2-1)b_1 - 2b_2 a_1 - 2b_1 a_0 = 0,\\
\left(\frac{G'}{G^2}\right)^{0} &: \quad 2\lambda^2 b_2 + 2\mu^2 a_2 + (k^2-1)a_0 - a_0^2 - 2b_2 a_2 - 2b_1 a_1 = 0,\\
\left(\frac{G'}{G^2}\right)^{1} &: \quad 2\mu\lambda a_1 + (k^2-1)a_1 - 2b_1 a_2 + 2a_1 a_0 = 0,\\
\left(\frac{G'}{G^2}\right)^{2} &: \quad 8\mu\lambda a_2 + (k^2-1)a_2 - a_1^2 - 2a_2 a_0 = 0,\\
\left(\frac{G'}{G^2}\right)^{3} &: \quad 2\lambda^2 a_1 - 2a_1 a_2 = 0,\\
\left(\frac{G'}{G^2}\right)^{-4} &: \quad 6\lambda^2 a_2 - a_2^2 = 0.
\end{aligned}$$

Solving this system of equations through Maple, we get the following results;
Set 1: $k = \mp\sqrt{16\lambda\mu + 1}$, $a_0 = 12\lambda\mu$, $a_1 = b_1 = 0$, $a_2 = 6\lambda^2$, $b_2 = 6\mu^2$,
Set 2: $k = \mp\sqrt{4\lambda\mu + 1}$, $a_0 = 6\lambda\mu$, $a_1 = b_1 = b_2 = 0$, $a_2 = 6\lambda^2$,
Set 3: $k = \mp\sqrt{4\lambda\mu + 1}$, $a_0 = 6\lambda\mu$, $a_1 = b_1 = a_2 = 0$, $b_2 = 6\mu^2$,
Set 4: $k = \mp\sqrt{1 - 4\lambda\mu}$, $a_0 = 2\lambda\mu$, $a_1 = b_1 = b_2 = 0$, $a_2 = 6\lambda^2$,
Set 5: $k = \mp\sqrt{1 - 4\lambda\mu}$, $a_0 = 2\lambda\mu$, $a_1 = b_1 = a_2 = 0$, $b_2 = 6\mu^2$,
Set 6: $k = \mp\sqrt{1 - 16\lambda\mu}$, $a_0 = -4\lambda\mu$, $a_1 = b_1 = 0$, $a_2 = 6\lambda^2$, $b_2 = 6\mu^2$.
Substituting these results in (4.4) and considering (3.6), (3.7) and (3.8), we get the following exact solutions for conformable time fractional Boussinesq equation

Solution 1:
(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_1(x,t) = U_{11}(\varepsilon) = 12\lambda\mu + 6\lambda^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2 + 6\mu^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$\begin{aligned}
&u_2(x,t)\\
&= U_{12}(\varepsilon) = 12\lambda\mu + 6\lambda^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2\\
&+ 6\mu^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.
\end{aligned}$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is achieved

$$u_3(x,t) = U_{13}(\varepsilon) = 6\left(\frac{C}{C\varepsilon + D}\right)^2$$

In (i)–(iii), $\varepsilon = x \mp \sqrt{16\lambda\mu + 1}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 2:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_4(x,t) = U_{21}(\varepsilon) = 6\lambda\mu + 6\lambda^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2.$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$\begin{aligned} u_5(x,t) &= U_{22}(\varepsilon) \\ &= 6\lambda\mu + 6\lambda^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2. \end{aligned}$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is acquired

$$u_6(x,t) = U_{33}(\varepsilon) = 6\left(\frac{C}{C\varepsilon + D}\right)^2$$

In (i)–(iii), $\varepsilon = x \mp \sqrt{4\lambda\mu + 1}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 3:

(i) If $\lambda\mu > 0$, the trigonometric solution is obtained

$$\begin{aligned} u_7(x,t) &= U_{31}(\varepsilon) \\ &= 6\lambda\mu + 6\mu^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}. \end{aligned}$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is found

$$\begin{aligned} u_8(x,t) &= U_{32}(\varepsilon) \\ &= 6\lambda\mu + 6\mu^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}. \end{aligned}$$

In (i)–(ii), $\varepsilon = x \mp \sqrt{4\lambda\mu + 1}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 4:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_9(x,t) = U_{41}(\varepsilon) = 2\lambda\mu + 6\lambda^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$\begin{aligned} u_{10}(x,t) &= U_{42}(\varepsilon) \\ &= 2\lambda\mu + 6\lambda^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2 \end{aligned}$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is achieved

$$u_{11}(x,t) = U_{43}(\varepsilon) = 6\left(\frac{C}{C\varepsilon + D}\right)^2$$

In (i)–(iii), $\varepsilon = x \mp \sqrt{1-4\lambda\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 5:

(i) If $\lambda\mu > 0$, the trigonometric solution is acquired

$$u_{12}(x,t) = U_{51}(\varepsilon) = 2\lambda\mu + 6\mu^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$\begin{aligned} u_{13}(x,t) &= U_{52}(\varepsilon) \\ &= 2\lambda\mu + 6\mu^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}. \end{aligned}$$

In (i)–(ii), $\varepsilon = x \mp \sqrt{1-4\lambda\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 6:

(i) If $\lambda\mu > 0$, the trigonometric solution is achieved

$$\begin{aligned} u_{14}(x,t) = U_{61}(\varepsilon) = -4\lambda\mu + 6\lambda^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2 \\ + 6\mu^2\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}. \end{aligned}$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$\begin{aligned} u_{15}(x,t) &= U_{62}(\varepsilon) \\ &= -4\lambda\mu + 6\lambda^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2 \\ &\quad + 6\mu^2\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(-\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}. \end{aligned}$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is found

$$u_{16}(x,t) = U_{63}(\varepsilon) = 6\left(\frac{C}{C\varepsilon + D}\right)^2$$

In (i)–(iii), $\varepsilon = x \mp \sqrt{1-16\lambda\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

In Fig. 1, the physical properties of Eq. (4.1) whose solutions are utilized in **Solution 2** have been shown for $\alpha = 0.5$ with some special values.

4.2 Coupled conformable time fractional Boussinesq equations

Let us consider the following nonlinear equation system [21]:

$$\begin{aligned} &\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial v}{\partial x} = 0, \\ &\frac{\partial^\alpha v}{\partial t^\alpha} + \beta\frac{\partial}{\partial x}(u^2) - \gamma\frac{\partial^3 u}{\partial x^3} = 0, \end{aligned} \tag{4.5}$$

where the time fractional derivative is αth order and $0 < \alpha \leq 1$.

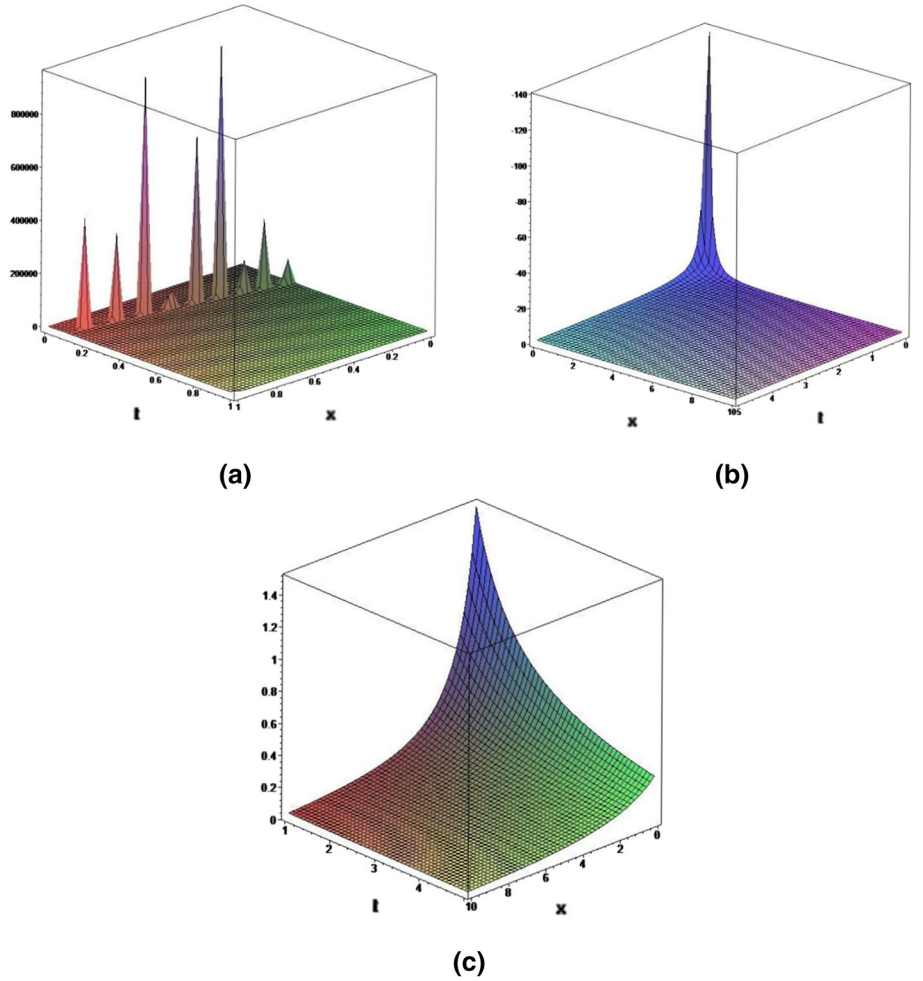


Fig. 1 **a** The trigonometric solution of $u(x,t)$ when $\lambda = \mu = C = D = 1$. **b** The hyperbolic solution of $u(x,t)$ when $\lambda = 0.5$, $\mu = -0.3$, $C = D = 1$. **c** The rational solution of $u(x,t)$ when $\lambda = 1$, $\mu = 0$, $C = D = 1$

Applying the transformation Eq. (3.2) to this system, it is found the following ordinary differential equation system

$$\begin{aligned} k\frac{dU}{d\varepsilon} + \frac{dV}{d\varepsilon} &= 0, \\ k\frac{dV}{d\varepsilon} + \beta\frac{d}{d\varepsilon}(U^2) - \gamma\frac{d^3U}{d\varepsilon^3} &= 0. \end{aligned} \tag{4.6}$$

Integrating Eq. (4.6) once and taking constant of integration as zero, the following equations are obtained:

$$kU + V = 0, \tag{4.7a}$$

$$kV + \beta U^2 - \gamma\frac{d^2U}{d\varepsilon^2} = 0. \tag{4.7b}$$

From Eq. (4.7a), we get

$$V = -kU. \tag{4.8}$$

Substituting Eq. (4.8) into Eq. (4.7b), it is obtained the following differential equation

$$-k^2U + \beta U^2 - \gamma \frac{d^2U}{d\varepsilon^2} = 0. \tag{4.9}$$

Balancing the terms of U^2 and $\frac{d^2U}{d\varepsilon^2}$ in Eq. (4.9), we get $M = 2$. For $M = 2$, solution will obtain the form

$$U(\varepsilon) = a_0 + a_1\Big(\frac{G'}{G^2}\Big) + a_2\Big(\frac{G'}{G^2}\Big)^2 + b_1\Big(\frac{G'}{G^2}\Big)^{-1} + b_2\Big(\frac{G'}{G^2}\Big)^{-2}, \tag{4.10}$$

where a_0, a_1, a_2, b_1 and b_2 are unknown constants.
Substituting (4.10) using (3.5) into (4.9), setting the coefficients of all powers of $\Big(\frac{G'}{G^2}\Big)$ to zero, nonlinear algebraic equations are obtained. The occurring algebraic system is solved by help of Maple to find the values of unknown constants.

$$\begin{aligned}
\Big(\frac{G'}{G^2}\Big)^{-4} &: \beta b_2^2 - 6\gamma\mu^2 b_2 = 0,\\
\Big(\frac{G'}{G^2}\Big)^{-3} &: 2\beta b_1 b_2 - 2\gamma\mu^2 b_1 = 0,\\
\Big(\frac{G'}{G^2}\Big)^{-2} &: -k^2 b_2 + \beta b_1^2 + 2\beta b_2 a_0 - 8\gamma\mu\lambda b_2 = 0,\\
\Big(\frac{G'}{G^2}\Big)^{-1} &: -k^2 b_1 + 2\beta b_2 a_1 + 2\beta b_1 a_0 - 2\gamma\mu\lambda b_1 = 0,\\
\Big(\frac{G'}{G^2}\Big)^{0} &: -k^2 a_0 + \beta a_0^2 + 2\beta b_2 a_2 + 2\beta b_1 a_1 - 2\gamma\lambda^2 b_2 - 2\gamma\mu^2 a_2 = 0,\\
\Big(\frac{G'}{G^2}\Big)^{1} &: -k^2 a_1 + 2\beta b_1 a_2 + 2\beta a_1 a_0 - 2\gamma\mu\lambda a_1 = 0,\\
\Big(\frac{G'}{G^2}\Big)^{2} &: -k^2 a_2 + \beta a_1^2 + 2\beta a_2 a_0 - 8\gamma\mu\lambda a_2 = 0,\\
\Big(\frac{G'}{G^2}\Big)^{3} &: 2\beta a_1 a_2 - 2\gamma\lambda^2 a_1 = 0,\\
\Big(\frac{G'}{G^2}\Big)^{4} &: \beta a_2^2 - 6\gamma\lambda^2 a_2 = 0.
\end{aligned}$$

Solving this system of equations through Maple, we get the following results;

Set 1: $k = \mp 2\sqrt{\lambda\gamma\mu}$, $a_0 = \frac{6\lambda\gamma\mu}{\beta}$, $a_1 = 0$, $a_2 = \frac{6\lambda^2\gamma}{\beta}$, $b_1 = 0$, $b_2 = 0$,

Set 2: $k = \mp 2\sqrt{\lambda\gamma\mu}$, $a_0 = \frac{6\lambda\gamma\mu}{\beta}$, $a_1 = 0$, $a_2 = 0$, $b_1 = 0$, $b_2 = \frac{6\mu^2\gamma}{\beta}$,

Set 3: $k = \mp 2\sqrt{-\lambda\gamma\mu}$, $a_0 = \frac{2\lambda\gamma\mu}{\beta}$, $a_1 = 0$, $a_2 = \frac{6\lambda^2\gamma}{\beta}$, $b_1 = 0$, $b_2 = 0$,

Set 4: $k = \mp 2\sqrt{-\lambda\gamma\mu}$, $a_0 = \frac{2\lambda\gamma\mu}{\beta}$, $a_1 = 0$, $a_2 = 0$, $b_1 = 0$, $b_2 = \frac{6\mu^2\gamma}{\beta}$,

Set 5: $k = \mp 4\sqrt{\lambda\gamma\mu}$, $a_0 = \frac{12\lambda\gamma\mu}{\beta}$, $a_1 = 0$, $a_2 = \frac{6\lambda^2\gamma}{\beta}$, $b_1 = 0$, $b_2 = \frac{6\mu^2\gamma}{\beta}$,

Set 6: $k = \mp 4\sqrt{-\lambda\gamma\mu}$, $a_0 = \frac{-4\lambda\gamma\mu}{\beta}$, $a_1 = 0$, $a_2 = \frac{6\lambda^2\gamma}{\beta}$, $b_1 = 0$, $b_2 = \frac{6\mu^2\gamma}{\beta}$.

We achieve the following exact solutions for the conformable time-fractional Boussinesq equation by substituting these results in (4.10) and considering (3.6), (3.7) and (3.8).

Solution 1:
(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_1(x,t) = U_{11}(\varepsilon) = \frac{6\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2.$$

From Eq. (4.8),

$$v_1(x,t) = -k\left[\frac{6\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_2(x,t) = U_{12}(\varepsilon)$$
$$= \frac{6\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2.$$

From Eq. (4.8),

$$v_2(x,t)$$
$$= -k\left[\frac{6\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2\right].$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is acquired

$$u_3(x,t) = U_{13}(\varepsilon) = \frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2.$$

From Eq. (4.8),

$$v_3(x,t) = -k\left[\frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2\right],$$

where $\varepsilon = x \mp 2\sqrt{\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 2:
(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_4(x,t) = U_{21}(\varepsilon) = \frac{6\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

From Eq. (4.8),

$$v_4(x,t) = -k\left[\frac{6\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_5(x,t) = U_{22}(\varepsilon)$$
$$= \frac{6\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

From Eq. (4.8),

$$v_5(x,t) = -k\left[\frac{6\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}\right],$$

where $\varepsilon = x \mp 2\sqrt{\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 3:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_6(x,t) = U_{31}(\varepsilon) = \frac{2\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2.$$

From Eq. (4.8),

$$v_6(x,t) = -k\left[\frac{2\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_7(x,t) = U_{32}(\varepsilon) = \frac{2\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2.$$

From Eq. (4.8),

$$v_7(x,t) = -k\left[\frac{2\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2\right].$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is achieved

$$u_8(x,t) = U_{33}(\varepsilon) = \frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2.$$

From Eq. (4.8),

$$v_8(x,t) = -k\left[\frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2\right],$$

where $\varepsilon = x \mp 2\sqrt{-\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 4:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_9(x,t) = U_{41}(\varepsilon) = \frac{2\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

From Eq. (4.8),

$$v_9(x,t) = -k\left[\frac{2\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_{10}(x,t) = U_{42}(\varepsilon) = \frac{2\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

From Eq. (4.8),

$$v_{10}(x,t) = -k\left[\frac{2\lambda\gamma\mu}{\beta} + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}\right],$$

where $\varepsilon = x \mp 2\sqrt{-\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 5:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_{11}(x,t) = U_{51}(\varepsilon) = \frac{12\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2 + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

From Eq. (4.8),

$$v_{11}(x,t) = -k\left[\frac{12\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2 + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_{12}(x,t) = U_{52}(\varepsilon) = \frac{12\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2 + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(-\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

From Eq. (4.8),

$$v_{12}(x,t) = -k\left[\frac{12\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2 + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(-\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}\right].$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is acquired

$$u_{13}(x,t) = U_{53}(\varepsilon) = \frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2.$$

From Eq. (4.8),

$$v_{13}(x,t) = -k\left[\frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2\right],$$

where $\varepsilon = x \mp 4\sqrt{\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

Solution 6:

(i) If $\lambda\mu > 0$, the trigonometric solution is found

$$u_{14}(x,t) = U_{61}(\varepsilon) = \frac{-4\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2 + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}.$$

From Eq. (4.8),

$$v_{14}(x,t) = -k\left[\frac{-4\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^2 + \frac{6\mu^2\gamma}{\beta}\left(\sqrt{\frac{\mu}{\lambda}}\left(\frac{C\cos(\sqrt{\mu\lambda}\varepsilon) + D\sin(\sqrt{\mu\lambda}\varepsilon)}{D\cos(\sqrt{\mu\lambda}\varepsilon) - C\sin(\sqrt{\mu\lambda}\varepsilon)}\right)\right)^{-2}\right].$$

(ii) If $\lambda\mu < 0$, the hyperbolic solution is obtained

$$u_{15}(x,t) = U_{62}(\varepsilon) = \frac{-4\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2 + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(-\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}.$$

From Eq. (4.8),

$$v_{15}(x,t) = -k\left[\frac{-4\lambda\gamma\mu}{\beta} + \frac{6\lambda^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^2 + \frac{6\mu^2\gamma}{\beta}\left(-\frac{\sqrt{|\mu\lambda|}}{\lambda}\left(-\frac{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) + D}{C\sinh(2\sqrt{|\mu\lambda|}\varepsilon) + C\cosh(2\sqrt{|\mu\lambda|}\varepsilon) - D}\right)\right)^{-2}\right].$$

(iii) If $\lambda \neq 0, \quad \mu = 0$, the rational solution is achieved

$$u_{16}(x,t) = U_{63}(\varepsilon) = \frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2.$$

From Eq. (4.8),

$$v_{16}(x,t) = -k\left[\frac{6\gamma}{\beta}\left(\frac{C}{C\varepsilon + D}\right)^2\right],$$

where $\varepsilon = x \mp 4\sqrt{-\lambda\gamma\mu}\left(\frac{t^\alpha}{\alpha}\right)$.

In Fig. 2, the physical characteristics of Eq. (4.5) whose solutions are employed in **Solution 3** have been shown for $\alpha = 0.5$ with some special values.

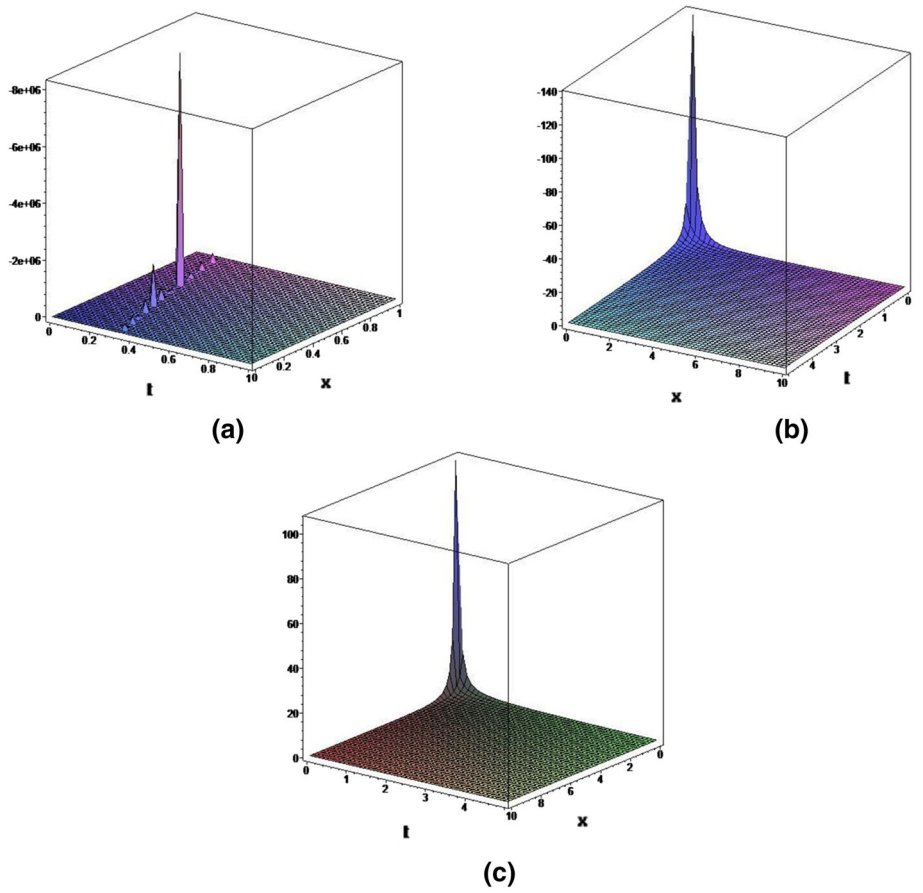


Fig. 2 **a** The trigonometric solution of $u(x,t)$ when $\lambda = \mu = \beta = C = D = 1, \gamma = -1$. **b** The hyperbolic solution of $u(x,t)$ when $\lambda = 0.5, \mu = -0.3, C = D = \beta = \gamma = 1$. **c** The rational solution of $u(x,t)$ when $\lambda = 1, \mu = 0, C = D = \beta = 1, \gamma = -1$

5 Conlusion

In this paper, analytical solutions of conformable time-fractional nonlinear Boussinesq equations which known as an important class of fractional differential equations in mathematical physics were obtained by $\left(\frac{G'}{G^2}\right)$-expansion method. Different types of solutions were officially found, including trigonometric, hyperbolic and rational function solutions. The obtained solutions were not the same and they were much recent than previously results. It was ensured that the solutions were correct by replacing the original equations. Since $\left(\frac{G'}{G^2}\right)$-expansion method is efficient and not complicated, it can also be used to get solutions for other fractional differential equations encountered in mathematical physics.

References

1. Abazari, R., Jamshidzadeh, S., Biswas, A.: Solitary wave solutions of coupled Boussinesq equation. Complexity **21**(52), 151–155 (2016)
2. Abazari, R., Jamshidzadeh, S., Biswas, A.: Multi soliton solutions based on interactions of basic traveling waves with an application to the nonlocal Boussinesq equation. Acta Physica Polonica B **47**, 1101–1112 (2016)
3. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)
4. Abdou, M., Soliman, A.: New applications of variational iteration method. Physica D **211**(1), 1–8 (2005)
5. Akbulut, A., Kaplan, M., Tascan, F.: The investigation of exact solutions of nonlinear partial differential equations by using exp(-$\Phi(\varepsilon)$) method, Optik - Int. J. Light Electr. Opt. **132**, 382–387 (2017)
6. Akil, M., Wehbe, A.: Stabilization of multidimensional wave equation with locally boundary fractional dissipation law under geometric conditions. Math. Control Relat. Fields **9**(1), 97–116 (2018)
7. Akil, M., Chitour, Y., Ghader, M., Wehbe, A.: Stability and exact controllability of a Timoshenko system with only one fractional damping on the boundary. Asymptot. Anal. **119**(3–4), 221–280 (2020)
8. Akil, M., Ghader, M., Wehbe, A.: The influence of the coefficients of a system of wave equations coupled by velocities on its stabilization. SeMA J. **78**, 287–333 (2021)
9. Aksoy, E., Çevikel, A.C., Bekir, A.: Soliton solutions of (2+1)-dimensional time-fractional Zoomeron equation. Optik - Int. J. Light Electr. Opt. **127**, 6933–6942 (2016)
10. Bekir, A., Güner, Ö.: Exact solutions of nonlinear fractional differential equations by (G'/ G)-expansion method. Chin. Phys. B **22**(11), 110202-1-110202–6 (2013)
11. Biswas, A., Kara, A.H., Moraru, L., Triki, H., Moshokoa, S.P.: Shallow water waves modeled by the Boussinesq equation having logarithmic non linearity. Proc. Romanian Acad. Ser. A **18**(2), 144–149 (2017)
12. Biswas, A., Ekici, M., Sonmezoglu, A.: Gaussian solitary waves to Boussinesq equation with dual dispersion and logarithmic non linearity. Nonlinear Anal.: Model. Control **23**(6), 942–950 (2018)
13. Caputo, M.: Vibrations of an infinite plate with a frequency independant. Q. J. Acoustic Soc. Am. **60**, 634–639 (1976)
14. Chen, J., Chen, H.: The (G'/G^2) method and its application to coupled nonlinear Klein-Gordon equation. Journal of South China normal University (Natural Science Edition) **2**, 013 (2012)
15. Chen, C., Jiang, Y.L.: Lie group analysis method for two classes of fractional partial differential equations. Commun. Nonlinear Sci. Numer. Simul. **26**(1), 24–35 (2015)
16. Cheng, X.Y., Wang, L.Z.: Invariant analysis, exact solutions and conservation laws of (2+1)-dimensional time fractional Navier-Stokes equations. Proc. R. Soc. A **477**, 20210220 (2021)
17. Cheng, X.Y., Hou, J., Wang, L.Z.: Lie symmetry analysis, invariant subspace method and q-homotopy analysis method for solving fractional system of single-walled carbon nanotube. Comput. Appl. Math. **40**, 1–17 (2021)
18. Elwakil, S., El-Labany, S., Zahran, M., Sabry, R.: Modified extended tanh-function method for solving nonlinear partial differential equations. Phys. Lett. A **299**(2), 179–188 (2002)
19. Eslami, M., Rezazadeh, H.: The first integral method for Wu-Zhang system with conformable time-fractional derivative. Calcolo **53**, 475–485 (2016)
20. Gómez, S., Cesar, A.: A nonlinear fractional Sharma-Tasso-Olver equation. App. Math. Comput. **266**, 385–389 (2015)
21. Hosseini, K., Ansari, R.: New exact solutions of nonlinear conformable time-fractional Boussinesq equations using the modified Kudryashov method Waves in random and complex media **27**(4), 628–636 (2017)
22. Hosseini, K., Bekir, A., Ansari, R.: Exact solutions of nonlinear conformable time-fractional Boussinesq equations using the exp$(-\phi(\varepsilon))$-expansion method. Opt. Quant. Electron. **49**, 131 (2017)
23. Inan, I.E., Ugurlu, Y., Bulut, H.: Auto-Bäcklund transformation for some nonlinear partial differential equation. Optik - Int. J. Light Electr. Opt. **127**(22), 10780–10785 (2016)
24. Jawad, A.J.M., Petković, M.D., Laketa, P., Biswas, A.: Dynamics of shallow water waves with Boussinesq equation. Scientia Iranica B **20**(1), 179–184 (2013)
25. Khalil, R., Al Horani, M., Yousef, A., Sababheh, M.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
26. Kudryashov, N.A.: One method for finding exact solutions of nonlinear differential equations Commun. Nonlinear Sci. Numer. Simulat. **17**(6), 2048–2053 (2012)
27. Lu, B.: The first integral method for some time fractional differential equations. J. Math. Anal. Appl. **395**(2), 684–693 (2012)
28. Madsen, P.A., Murray, R., Sorensen, O.R.: A new form of the Boussinesq equations with improved linear dispersion characteristics. Coast Engl. J. **15**(4), 371–388 (1991)

29. Manafian, J., Lakestani, M.: Application of tan(//2)-expansion method for solvingthe Biswas-Milovic equation for Kerr law nonlinearity, Optik - Int. J. Light Electr. Opt. **127**(4), 2040–2054 (2012)
30. Matinfar, M., Eslami, M., Kordy, M.: The functional variable method for solving thefractional Korteweg de Vries equations and the coupled Korteweg de Vries equations. Pramana J Phys. **85**, 583–592 (2015)
31. Oldham, K., Spanier, J.: The Fractional Calculus, Hardcover: 234 pages. Academic Press, Cambridge (1974)
32. Sahadevan, R., Prakash, P.: Exact solutions and maximal dimension of invariant subspaces of time fractional coupled nonlinear partial differential equations. Commun. Nonlinear Sci. Numer. Simulat. **42**, 158–177 (2017)
33. Samko, S., Kilbas, A., Marichev, O.: Fractional integrals and derivatives, *Gordon and Breach, Amsterdam [Engl. Trans. from the Russian]*, (1993)
34. Triki, H., Kara, A.H., Biswas, A.: Domain walls to Boussinesq-type equations in (2 + 1)-dimensions. Indian J. Phys. **88**(7), 751–755 (2014)
35. Wang, L.Z., Wang, D.J., Shen, S.F., Huang, Q.: Lie point symmetry analysis of the Harry-Dym type equation with Riemann-Liouville fractional derivative. Acta. Math. Appl. Sinica. **34**, 469–477 (2018)
36. Wang, M.M., Shen, S.F., Wang, L.Z.: Lie symmetry analysis, optimal system and conservation laws of a new (2+1)-dimensional KdV system. Commun. Theor. Phys. **73**(8), 085004 (2021)
37. Wazwaz, A.-M.: Multiple-soliton solutions for the KP equation by Hirota's bilinear method and by the tanh-coth method. Appl. Math. Comput. **190**(1), 633–640 (2007)
38. Yang, Y., Wang, L.Z.: Lie symmetry analysis, conservation laws and separation variable type solutions of the time fractional Porous Medium equation. Wave. Random. Complex. **49**, 1–20 (2020)
39. Yaslan, H.Ç., Girgin, A.: Exp-function method for the conformable space-time fractional STO, ZKBBM and coupled Boussinesq equations. Arab J. Basic Appl. Sci. **26**, 163–170 (2019)
40. Yildirim, O., Caglak, S.: Lie point symmetries of difference equation for nonlinear sine-Gordon equation. Physica Scripta **94**, 085219(8) (2019)
41. Yildirim, O., Uzun, M.: On the numerical solutions of high order stable difference schemes for the hyperbolic multipoint nonlocal boundary value problems. Appl. Math. Comput **254**, 210–218 (2015)
42. Yildirim, O., Uzun, M.: Weak solvability of the unconditionally stable difference scheme for the coupled sine-Gordon system. Nonlinear Anal.: Model. Control **25**(6), 997–1014 (2020)
43. Zayed, E.M.E., Alurrfi, K.A.E.: A new Jacobi elliptic function expansion method for solving a nonlinear PDE describing the nonlinear low-pass electrical lines. Chaos Solitons Fractals **78**, 148–155 (2015)
44. Zheng, B.: Exp-function method for solving fractional partial differential equations. Sci. World J. **2013**, 465723 (2013)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.