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THE INVERSE SCATTERING PROBLEM FOR A SYSTEM OF n DIFFERENTIAL EQUATIONS ON THE HALF-AXIS

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Abstract

Let us consider the system of differential equations given on a half-axis

$$-i\frac{dy_k(x)}{dx}+\sum_{j=1}^{n}c_{kj}(x)y_j(x)=\lambda\xi_k y_k(x),\quad k=1,2,\ldots,n,\quad i=\sqrt{-1},$$

where $c_{kj}(x)$'s are complex-valued functions satisfying $c_{kk}(x)=0$ and

$|c_{kj}(x)|\le c\exp(-\varepsilon x)$, $\varepsilon>0$ (ε and c are constants) $k, j=1,2,\ldots,n-1$.

For the above system of equations, given $(n-1)$ boundary conditions and solution together with asymptotic expression considered matrix function $S(\lambda)$ on the half-axis, subsquare matrices obtained from this function and scattering data by means of their inverses are defined.

Relationships between scattering matrix and scattering data are determined when the problem is given on the whole axis and coefficients are zero in case of $x<0$. By using analytic representation of the solution it is shown that the coefficient of the system of equations is defined uniquely under the solvability condition of Riemann problem.

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Introduction

In this study, scattering problem for n ordinary differential equations of the form

$$-i\frac{dy_k(x)}{dx}+\sum_{j=1}^{n}c_{kj}(x)y_j(x)=\lambda\xi_k y_k(x),\quad k=1,2,\ldots,n,\quad i=\sqrt{-1} \tag{1}$$

is studied on a half-axis ($x \geq 0$). Here, functions $c_{kj}(x)(k, j = 1, 2, \ldots, n)$ are of complex valued and they satisfy the following conditions:

$$\begin{aligned} &c_{kk}(x)=0,\quad k=1,2,\ldots,n \\ &|c_{kj}(x)|\leq c\exp(-\varepsilon x),\quad \varepsilon>0 \quad (\varepsilon \text{ and } c \text{ are constants}) \\ &\xi_1>\xi_2>\cdots>\xi_{n-1}>0>\xi_n. \end{aligned} \tag{2}$$

For the system of equations (1), the following $(n-1)$ conditions are considered.

In the first condition, for the boundary condition $y_n^1(0)=y_1^1(0)$, and for $x\to\infty$, the asymptotic expressions

$$y_k^1(x,\lambda)=A_k\exp(i\xi_k\lambda x)+o(1),\quad (k=1,2,\ldots,n-1)\quad x\to+\infty$$

are given. In the second condition, for the boundary condition $y_n^2(0)=y_2^2(0)$, and for $x\to+\infty$, the same asymptotic expressions

$$y_k^2(x)=A_k\exp(i\xi_k\lambda x)+o(1),\quad (k=1,2,\ldots,n-1)\quad x\to+\infty$$

are given. In a similar way if we write $(n-1)$th condition, boundary condition $y_n^{n-1}(0)=y_{n-1}^{n-1}(0)$, and for $x\to+\infty$, asymptotic expressions

$$y_k^{n-1}(x,\lambda)=A_k\exp(i\xi_k\lambda x),\quad (k=1,2,\ldots,n-1)\quad x\to+\infty$$

are given.

We say that problem is the *scattering problem* if it is of looked for a solution for the system of equations (1) on the half-axis with respect to

the given boundary conditions and the asymptotic expressions of the solution.

One can easily show that conditions $(n-1)$ are equivalent to the following integral equation system:

$$y_k^p(x) = A_k \exp(i\xi_k\lambda x) + i\int_x^\infty \sum_{j=1}^{n} c_k^j(x')y_j^p(x')\exp(i\xi_k\lambda(x-x'))dx',$$

$$k = 1, 2, \ldots, n-1$$

$$y_n^p(x) = B_p \exp(i\xi_k\lambda x) + i\int_x^\infty \sum_{j=1}^{n} c_{nj}(x')y_j^p(x')\exp(i\xi_n\lambda(x-x'))dx', \quad (3)$$

where

$$B_p = A_p + i\int_0^\infty \sum [c_{pj}(x')\exp(-i\xi_p\lambda x') - c_{nj}(x')\exp(-i\xi_n\lambda x')]y_j^p(x')dx',$$

$$p = 1, 2, \ldots, n-1. \quad (4)$$

Theorem 1. *If the coefficients of the equation system* (1) *satisfy the condition* (2), *then for the arbitrarily given constants* $A_1, A_2, \ldots, A_{n-1}$ *for real* λ, *there is a unique solution for the* $(n-1)$ *problems on the half-axis and*

$$y_n^p(x) = B_p \exp(i\xi_n\lambda x) + o(1), \quad x \to +\infty, \quad p = 1, 2, \ldots, n-1. \quad (5)$$

Proof. The scattering problem on the half-axis for the equation system (1) is equivalent to the integral equation system (3). It is possible to indicate the existence and uniqueness of the solution of integral equation system (3) by means of the method of successive approach, using the condition (2). By considering the condition (2), expression (5) is obtained.

As is seen from the theorem (1) and the formula (5), there is a matrix function $S(\lambda)$ which converts the vector $A = (B_1, B_2, \ldots, B_{n-1})^T$. This matrix function is called the *scattering matrix* on the half-axis. It is in the form of

$$S(\lambda) = \begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda) & \cdots & s_{1,n-1}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) & \cdots & s_{2,n-1}(\lambda) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ s_{n-1,1}(\lambda) & s_{n-1,2}(\lambda) & \cdots & s_{n-1,n-1}(\lambda) \end{pmatrix}$$

and when the coefficients of the system (1) are zero, $S(\lambda)$ becomes unit matrix since $B_k = A_k$ $(k = 1, 2, \ldots, n-1)$.

Inverse scattering problem on the half-axis is the determination of coefficients of equation system (1), according to the given matrix function $S(\lambda)$ and some data.

Inverse scattering problem on the half-axis is solved as follows:

1. For the system of equations (1), inverse scattering problem is studied on the whole-axis. Its coefficients are zero on the negative axis.

When $x \in (-\infty, \infty)$, inverse scattering problem for a system of three first-order has been studied by Zakharov and Shabat [5, 6, 7], Kaup [3], Beals and Coifman [1], and others.

Inverse scattering problem for two equations (in particular Dirac system) has been investigated by Gasymov and Levitan [2], Nizhnik and Vu [4] and others.

The scattering matrix on the whole-axis is described as the matrix which converts the solution in $+\infty$ to the solution in $-\infty$. Let us represent this matrix by $S^{/1}(\lambda)$.

When the coefficients are zero for $x < 0$, then some new properties appear being different from the general case for $S^{/1}(\lambda)$. These properties make a relation possible matrix $S(\lambda)$ which we have described on the half-axis, and matrix $S^{/1}(\lambda)$.

2. The new matrix function $S^{f_2}(\lambda)$ which is represented uniquely by $S^{f_1}(\lambda)$ is described. This matrix function has the following property:

$$S^{f_2}(\lambda) = \exp(-i\lambda Jx)(I + W_+(x, \lambda))^{-1}(I + W_-(x, \lambda))\exp(i\lambda Jx) \tag{6}$$

different from $S^{f_1}(\lambda)$. Here,

$$J = diag(\xi_1, \xi_2, ..., \xi_n),$$

$$W_-(x, \lambda) = \int_{-\infty}^{0} \widetilde{W}_-(x, \tau, \lambda)\exp(i\lambda\tau)d\tau,$$

$$W_+(x, \lambda) = \int_{0}^{\infty} \widetilde{W}_+(x, \tau, \lambda)\exp(i\lambda\tau)d\tau,$$

$$\widetilde{W}_-(x, \tau, \lambda) = (\widetilde{W}_{ij-}(x, \tau, \lambda))_{i,j=1}^{n},$$

$$\widetilde{W}_+(x, \tau, \lambda) = (\widetilde{W}_{ij+}(x, \tau, \lambda))_{i,j=1}^{n},$$

3. There are the following relations between the coefficients of the equation and $W_-(x, \lambda)$ and $W_+(x, \lambda)$ functions:

$$i[J, \widetilde{W}_-(x, 0, \lambda)] = -c(x),$$

$$i[J, \widetilde{W}_+(x, 0, \lambda)] = c(x). \tag{7}$$

Here, $[A, B] = AB - BA$ denotes the commutant of the two square matrices A and B.

4. The properties of matrix function $S(\lambda)$ and its inverse are investigated. These properties are given in the following theorem.

Theorem 2. *Assume that the coefficients of the equation system* (1) *satisfy the condition* (2). *Then there is inverse of subsquare matrix obtained from functions* $S(\lambda)$ *and matrix function* $S(\lambda)$ *except for a finite number of* λ*'s.*

$$\Delta_1 = s_{11}(\lambda),$$

$$\Delta_2 = \begin{bmatrix} s_{11}(\lambda) & s_{12}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) \end{bmatrix},$$

$$\Delta_3 = \begin{bmatrix} s_{11}(\lambda) & s_{12}(\lambda) & s_{13}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) & s_{23}(\lambda) \\ s_{31}(\lambda) & s_{32}(\lambda) & s_{33}(\lambda) \end{bmatrix},$$

$$\vdots$$

$$\Delta_{n-2} = \begin{bmatrix} s_{11}(\lambda) & s_{12}(\lambda) & \cdots & s_{1,n-2}(\lambda) \\ s_{21}(\lambda) & s_{22}(\lambda) & \cdots & s_{2,n-2}(\lambda) \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ s_{n-2,1}(\lambda) & s_{n-2,2}(\lambda) & \cdots & s_{n-2,n-2}(\lambda) \end{bmatrix},$$

$$\Delta_{n-1} = S(\lambda).$$

$$(\Delta_1^{-1})_{11},\ (\Delta_2^{-1})_{22},\ (\Delta_3^{-1})_{33},\ \ldots,\ (\Delta_{n-1}^{-1})_{n-2,n-2},\ (\Delta_{n-1}^{-1})_{n-1,n-1},$$

(here $(\Delta_{n-1}^{-1})_{n-1,n-1}$ indicates $(n-1)$-row $(n-1)$-column element of inverse of the matrix Δ_{n-1})

$$\gamma_{11}(\lambda) + \gamma_{12}(\lambda) + \cdots + \gamma_{1,n-1}(\lambda).$$

These functions have the following factorisations:

$$(\Delta_1^{-1})_{11} = s_{11}^{-1}(\lambda) = (1 + R_{1+}(\tilde{\lambda}))^{-1}(1 + R_{1-}(\lambda)) \tag{8}$$

$$(\Delta_2^{-1})_{22} = (s_{22}(\lambda) - s_{11}^{-1}(\lambda)s_{12}(\lambda)s_{21}(\lambda))^{-1} = (1 + R_{2+}(\lambda))^{-1}(1 + R_{2-}(\lambda)) \tag{9}$$

$$(\Delta_3^{-1})_{33} = (1 + R_{3+}(\lambda))^{-1}(1 + R_{3-}(\lambda)) \tag{10}$$

$$(\Delta_{n-2}^{-1})_{n-2,n-2} = (1 + R_{(n-2)+}(\lambda))^{-1}(1 + R_{(n-2)-}(\lambda)) \tag{11}$$

$$(\Delta_{n-1}^{-1})_{n-1,n-1} = (1 + R_{(n-1)+}(\lambda))^{-1}(1 + R_{(n-1)-}(\lambda)) \tag{12}$$

$$\gamma_{11}(\lambda) + \gamma_{12}(\lambda) + \cdots + \gamma_{1,n-1}(\lambda) = (1 + N_{1-}(\lambda))^{-1}(1 + N_{1+}(\lambda)). \tag{13}$$

If functions are $\gamma_{11}(\lambda)$, $\gamma_{12}(\lambda)$, ..., $\gamma_{1,n-1}(\lambda)$, then we get

$$\gamma_{11}(\lambda) = (1 + N_{1-}(\lambda))^{-1}(1 + R_{1-}(\lambda))(1 + m_{1+}(\lambda)) + \tilde{r}_{1-}(\lambda) \tag{14}$$

$$\gamma_{12}(\lambda) = (1 + N_{1-}(\lambda))^{-1}(1 + R_{2-}(\lambda))(m_{2+}(\lambda)) - \tilde{r}_{2-}(\lambda) \tag{15}$$

$$\gamma_{1,n-1}(\lambda) = (1 + N_{1-}(\lambda))^{-1}(1 + R_{(n-1)-}(\lambda))(m_{(n-1)+}(\lambda)) - r_{(n-1)-}(\lambda). \tag{16}$$

Here,

$$S^{-1}(\lambda) = \|\gamma_{ij}(\lambda)\|_{i,j=1}^{n-1},$$

where the indices "–" show analytic functions in the lower plane, and the indices "+" show the analytic functions in the upper plane.

The following theorem reduces the inverse scattering problem on the half-axis to the problem whose coefficients are zero for $x < 0$ in the whole axis.

Theorem 3. *Let the scattering matrix of equation system* (1) *be* $S(\lambda) = \|s_{ij}(\lambda)\|_{i,j=1}^{n-1}$ *whose coefficients satisfy the condition* (2). *Then, the matrix function* $S^{f_1}(\lambda)$, *with the help of the matrix function* $S(\lambda)$, *subsquare matrix of* $S(\lambda)$ *and its inverse elements are expressed in the*

following way, with the analytic function $N_{1-}(\lambda)$, $N_{1+}(\lambda)$, $R_{1-}(\lambda)$, $R_{2-}(\lambda$ ..., $R_{(n-1)}(\lambda)$ *obtained from the formulae* (8), (9), (10), (11), (12), (13) *whic separate the multipliers (by the solution of Riemann problem).*

$$A = \begin{pmatrix} 1+m_{1+}(\lambda) & m_{2+}(\lambda) & \cdots & m_{(n-1)+}(\lambda) & 1+N_{1+}(\lambda) \\ m_{1+}(\lambda) & 1+m_{2+}(\lambda) & \cdots & m_{(n-1)+}(\lambda) & 1+N_{1+}(\lambda) \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ m_{1+}(\lambda) & m_{2+}(\lambda) & \cdots & 1+m_{(n-1)+}(\lambda) & 1+N_{1+}(\lambda) \\ m_{1+}(\lambda) & m_{2+}(\lambda) & \cdots & m_{(n-1)+}(\lambda) & 1+N_{1+}(\lambda) \end{pmatrix}$$

$$B = \begin{pmatrix} 1+R_{1-}(\lambda) & 0 & \cdots & 0 & 0 \\ 0 & 1+R_{2-}(\lambda) & \cdots & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & 1+R_{(n-1)-}(\lambda) & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} s_{11}(\lambda) & s_{12}(\lambda) & \cdots & s_{1,n-1}(\lambda) & -1 \\ s_{21}(\lambda) & s_{22}(\lambda) & \cdots & s_{2,n-1}(\lambda) & -1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ s_{n-1,1}(\lambda) & s_{n-1,2}(\lambda) & \cdots & s_{n-1,n-1}(\lambda) & -1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$S^{f_1}(\lambda) = ABC. \qquad (1$$

Here, the functions $m_{1+}(\lambda)$, $m_{2+}(\lambda)$, ..., $m_{(n-1)+}(\lambda)$ are obtained from th equations

$$m_{1+}(\lambda) - n_{1-}(\lambda) = (1 + N_{1-}(\lambda))(1 + R_{1-}(\lambda))^{-1}\gamma_{11}(\lambda) \tag{18}$$

$$m_{2+}(\lambda) - n_{2-}(\lambda) = (1 + N_{1-}(\lambda))(1 + R_{2-}(\lambda))^{-1}\gamma_{12}(\lambda) \tag{19}$$

$$\vdots$$

$$m_{(n-1)+}(\lambda) - n_{(n-1)-}(\lambda) = (1 + N_{1-}(\lambda))(1 + R_{(n-1)-}(\lambda))^{-1}\gamma_{1,n-1}(\lambda). \tag{20}$$

The determinant of the matrix function $S^{\Lambda}(\lambda)$ is

$$\det S^{\Lambda}(\lambda)$$
$$= \det S(\lambda)(1 + R_{1-}(\lambda))(1 + R_{2-}(\lambda))\cdots(1 + R_{(n-1)-}(\lambda))(1 + N_{1+}(\lambda)). \tag{21}$$

Here, as is seen, the zeros and pole points of the matrix function $S^{\Lambda}(\lambda)$ are formed by the matrix function $S(\lambda)$ and the zeros and pole points of functions $1 + R_{1-}(\lambda)$, $1 + R_{2-}(\lambda)$, ..., $1 + R_{(n-1)-}(\lambda)$ and $1 + N_{1+}(\lambda)$.

Zeros, pole points and scattering matrix of these functions are called *scattering data*.

Thus, the problem completed on the half-axis is extended to whole axis.

Theorem 4. *If the coefficients of the equation system* (1) *satisfy the condition* (2), *then the coefficients of the equation is described uniquely by the* $S^{\Lambda}(\lambda)$ *matrix given on the whole-axis and the zeros and pole points of this matrix.*

If we compare this theorem with the expressions (17) and (21), we obtain the following fundamental theorem.

Theorem 5. *Assume that the coefficients of the equation system* (1) *satisfy the condition* (2). *Then, the coefficients of the equation can be indicated uniquely according to the matrix* $S(\lambda)$, *the zeros and polar points of this matrix and the functions* $1 + R_{1-}(\lambda)$, $1 + R_{2-}(\lambda)$, ..., $1 + R_{(n-1)-}(\lambda)$ *and* $1 + N_{1+}(\lambda)$.

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