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Primary hyperideals of multiplicative hyperrings

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Abstract. In this article, we study prime, primary, C-hyperideals of multiplicative hyperrings in the sense of Rota [14]. Prime hyperideal avodiance lemma was proved in [4]. We mainly study union of primary hyperideals in hyperring. Among many results in this study we give the primary hyperideal avodiance lemma. Also we determine all prime, primary, C-ideals of quotient hyperring.
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1. Introduction

The study of hyperstructures dates back to [8]. In 1934, during the 8^{th} congress of Scandinavian Mathematicians, F. Marty introduced hypergroups as a generalization of groups. So far, it has been a developing area and many mathematicians has studied on hyperstructures. See [2,3,8,11]. One of the main reason for studying hyperstructures, it has a wide variety of applications to other areas such as geometry, lattices, automata, cryptography, coding theory, artificial intelligence and probabilities [3].

In [6], Krasner introduced the hyperrings in a study of approximations of complete valued fields. He defined $(R,+,\circ)$ as a hyperring with the following properties: (i) $(R,+)$ is a canonical hypergroup, (ii) $(R,\circ)$ is a semigroup and (iii) binary operation " $\circ$ " is distributive over addition. The theory of hyperrings has been developed by many researches. See [5-6,9,12-14]. There are various types of hyperrings and one of the important classes of hyperrings, called multiplicative hyperring, was introduced by Rota in [14]. Rota called $(R,+,\circ)$ a multiplicative hyperring if (i) $(R,+)$ is an abelian group, (ii) $(R,\circ)$ is a hypersemigroup, (iii) $a\circ(b+c)\subseteq a\circ b+a\circ c$ and (iv) $a\circ(-b)=(-a)\circ b=-(a\circ b)$ for all $a,b,c\in R$. Here, we mean a hypersemigroup by a nonempty set R with an associative hyperoperation $\circ$, i.e,

$$a\circ(b\circ c)=\bigcup_{t\in(b\circ c)}a\circ t=\bigcup_{s\in(a\circ b)}s\circ c=(a\circ b)\circ c$$

for all $a,b,c\in R$. Further, if R is a multiplicative hyperring with $a\circ b=b\circ a$ for all $a,b\in R$, then R is called a commutative multiplicative hyperring.

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Throughout, all hyperrings are assumed to be commutative hyperrings in the sense of Rota [14]. In particular, R always denotes such a hyperring. For any two nonempty subsets K, L of R, we define

$$K \circ L = \bigcup_{k \in K, l \in L} k \circ l.$$

Also by induction we can mention that product of finitely many nonempty subsets of R. As well we denote the sum of the set $a \circ b$ at k times by $k(a \circ b) = \{x_1 + x_2 + ... + x_k : x_i \in a \circ b \text{ for } 1 \leq i \leq k\}$. Suppose that I is an additive subgroup of R. If $r \circ a \subseteq I$ for all $r \in R$ and $a \in I$, then I is called a hyperideal of R. Let I be a hyperideal of R. Then I is said to be a C-ideal if $a_1 \circ a_2 \circ ... \circ a_n \cap I \neq \emptyset$, for $n \in \mathbb{N}, a_i \in R, 1 \leq i \leq n$, imply that $a_1 \circ a_2 \circ ... \circ a_n \subseteq I$ [4].

Prime ideal has a distinguished place in commutative algebra. It is useful tool for understanding ideal structure of rings. There are several studies on this issue. See [1,7]. Also, Primeness of hyperideals was first studied by Processi and Rota in [13]. Recall that a proper hyperideal P of R is said to be a prime hyperideal if the condition $a \circ b \subseteq P$ implies either $a \in P$ or $b \in P$ [13]. After that, in [4] Dasgupta studied on prime and primary hyperideals in hyperrings. A proper hyperideal Q of R is called a primary hyperideal if $a \circ b \subseteq Q$ then either $a \in Q$ or $b^n \subseteq Q$ for some $n \in \mathbb{N}$ [4]. For any given hyperideal I of R, $\sqrt{I}$ denotes the intersection of all prime hyperideals containing I, if there is no such prime hyperideal of R, we assume that $\sqrt{I} = R$. Also the hyperideal $\{r \in R : r^n \subseteq I \text{ for some } n \in \mathbb{N}\}$ will be designated by $D(I)$ and note that the inclusion $D(I) \subseteq \sqrt{I}$ always holds. In addition, if I is a C-ideal of R, other inclusion holds by Propositipon 3.2 of [4].

In this paper, our aim is to study union of primary hyperideals in hyperrings. Among many results in this paper we give primary hyperideal avodiance theorem with Theorem 2.6. Let $(R, +, \circ)$ and $(R', +, \circ)$ be two hyperrings and $f : R \to R'$ a function such that $f(a + b) = f(a) + f(b)$ and $f(a \circ b) \subseteq f(a) \circ f(b)$ for all $a, b \in R$. Then f is called a homomorphism. In particular f is called a good homomorphism in case $f(a \circ b) = f(a) \circ f(b)$ [5]. Furthermore, the kernel of a homorphism is defined as $Ker(f) = f^{-1}(\langle 0 \rangle) = \{r \in R : f(r) \in \langle 0 \rangle\}$ and note that $f(r)$ may not be a zero element. From Proposition 2.7 to Corollary 2.12 we investigate the prime, primary, C-hyperideals under good homomorphism and determine all prime, primary, C-hyperideal of any quotient hyperring. Finally we extend the primary hyperideal avodiance theorem to cosets of any quotient hyperring.

For more detail and terminology on hyperrings, the reader may consult [5].

2. Primary hyperideals in multiplicative hyperrings

Throughout, R denotes a commutative multiplicative hyperring.

Definition 2.1. *Let Q be a proper hyperideal of R. Then Q is called a primary hyperideal of R if for each $a, b \in R$ and whenever $a \circ b \subseteq Q$ then either $a \in Q$ or $b^n \subseteq Q$ for some $n \in \mathbb{N}$* [4].

Lemma 2.2. *Suppose that Q is a primary C-hyperideal of R. Then $\sqrt{Q}$ is a prime C-hyperideal of R.*

Proof. By [4, Proposition 3.6] we know that $\sqrt{Q}$ is a prime hyperideal. Now we show that $\sqrt{Q}$ is a C-ideal of R. Let $a_1 \circ a_2 \circ ... \circ a_n \cap \sqrt{Q} \neq \emptyset$ for some $a_1, a_2, ..., a_n \in R$. Then we have $x \in a_1 \circ a_2 \circ ... \circ a_n$ such that $x \in \sqrt{Q}$. This implies that $x^t \subseteq (a_1 \circ a_2 \circ ... \circ a_n)^t$ and $x^t \subseteq Q$ for some $t \in \mathbb{N}$. Since Q is a C-ideal and $(a_1 \circ a_2 \circ ... \circ a_n)^t \cap Q \neq \emptyset$, we get $(a_1 \circ a_2 \circ ... \circ a_n)^t \subseteq Q$ and so $(a_1 \circ a_2 \circ ... \circ a_n) \subseteq \sqrt{Q}$. Hence $\sqrt{Q}$ is a C-ideal of R.

In [2], author proved prime hyperideal avodiance theorem for C-ideals: Let $P_1, P_2, ..., P_n$ be prime C-hyperideals such that $I \subseteq P_1 \cup P_2 \cup ... \cup P_n$ then $I \subseteq P_i$ for some $1 \leq i \leq n$. Actually

the condition of C-ideal is not necessary. Further if at most two of P_i's are not prime hyperideal, then the result is also valid. Now we prove the primary avodiance thoerem but first we give some results we need.

Lemma 2.3. *Let Q be a proper hyperideal of R. Then Q is primary hyperideal if and only if $J \circ K \subseteq Q$ implies either $J \subseteq Q$ or $K \subseteq D(Q)$, where J, K are hyperideals of R.*

Proof. Suppose that Q is primary hyperideal of R such that $J \circ K \subseteq Q$ and $J \nsubseteq Q$. Then there exists an element $j \notin Q$ for some $j \in J$. Take any $k \in K$. Then we have $j \circ k \subseteq J \circ K \subseteq Q$. Since Q is primary hyperideal, we have $k^n \subseteq Q$ and so $k \in D(Q)$. This implies that $K \subseteq D(Q)$. Conversely, let $a \circ b \subseteq Q$ for some $a, b \in R$. Then we have $\langle a \circ b \rangle \subseteq Q$. Take any $t \in \langle a \rangle$ and $s \in \langle b \rangle$. Then $t = \sum_{i=1}^{n_t} x_i + n'_t a$ for some $n'_t \in \mathbb{Z}$ and $x_i \in r_i \circ a$ and also $s = \sum_{i=1}^{s_t} y_i + s'_t a$ for some $s'_t \in \mathbb{Z}$ and $y_i \in r'_i \circ b$. This implies that $t \circ s = (\sum_{i=1}^{n_t} x_i + n'_t a) \circ (\sum_{i=1}^{s_t} y_i + s'_t b) \subseteq \sum_{i=1}^{n_t} \sum_{j=1}^{s_t} x_i \circ y_j + n'_t \sum_{i=1}^{s_t} a \circ y_i + s'_t \sum_{i=1}^{n_t} x_i \circ b + n'_t s'_t (a \circ b) \subseteq \langle a \circ b \rangle \subseteq Q$ and so $\langle a \rangle \circ \langle b \rangle \subseteq Q$. Then we have either $\langle a \rangle \subseteq Q$ or $\langle b \rangle \subseteq D(Q)$. Thus we get $a \in Q$ or $b^n \subseteq Q$ for some $n \in \mathbb{N}$.

By induction hypothesis one can easily obtain following result:

Corollary 2.4. *If Q is a primary hyperideal of R such that $J_1 \circ J_2 \circ ... \circ J_n \subseteq Q$ then either $J_1 \subseteq Q$ or $J_i \subseteq D(Q)$ for some $2 \leq i \leq n$.*

Let $I \subseteq Q_1 \cup ... \cup Q_n$ be a covering of hyperideals of R. Then this covering is called efficient if none of the Q_i's are superfluous. Note that a covering by two hyperideals can not be efficient.

Proposition 2.5. *Suppose that $I \subseteq Q_1 \cup ... \cup Q_n$ is an efficient covering of hyperideals of R, where I is a hyperideal of R. If $\sqrt{Q_i} \nsubseteq \sqrt{Q_j}$ for each $i \neq j$, then any of Q_i's are not primary hyperideal of R.*

Proof. First we show that $\sqrt{J} = \sqrt{D(J)}$ for any hyperideal J of R. The inclusion $\sqrt{J} \subseteq \sqrt{D(J)}$ always holds. Take a prime hyperideal P containing J. Then it is sufficient to show that P contains $D(J)$. Let $x \in D(J)$. Then we have $x^n = x \circ x \circ ... \circ x \subseteq J \subseteq P$ for some $n \in \mathbb{N}$. This yields $x \in P$ and thus $\sqrt{D(J)} \subseteq \sqrt{J}$. Since covering is efficient, we have $n > 2$. Assume that Q_1 is a primary hyperideal of R. Again since the covering is efficient, we have $I \cap Q_2 \cap Q_3 \cap ... \cap Q_n \subseteq I \cap Q_1 \subseteq Q_1$ by [10]. As $I \nsubseteq Q_1$ and $I \circ Q_2 \circ ... \circ Q_n \subseteq Q_1$, by previous corollary, there exists $2 \leq j \leq n$ such that $Q_j \subseteq D(Q_1)$ and so $\sqrt{Q_j} \subseteq \sqrt{D(Q_1)} = \sqrt{Q_1}$ which is a contradiction.

By the following theorem we can obtain prime hyperideal avodiance theorem in [4, Proposition 2.19] without assumption C-ideal.

Theorem 2.6. *Suppose that $I \subseteq Q_1 \cup ... \cup Q_n$ is a covering and at most two of Q_i's are not primary hyperideal of R. If $\sqrt{Q_i} \nsubseteq \sqrt{Q_j}$ for each $i \neq j$, then $I \subseteq Q_i$ for some $1 \leq i \leq n$.*

Proof. If $n = 2$, then the result is valid. Also we may assume that the covering is efficient so $n \neq 2$. Suppose $n > 2$. But in this case, there exists a primary hyperideal Q_j of covering and this contradicts by Proposition 2.5. So we have $n = 1$ and this completes the proof.

Proposition 2.7. *Let $f : R \to S$ be a good homorphism and I, J be hyperideals of R and S, respectively. Then the followings are satisfied:*

(i) If I is a prime hyperideal containing $Ker(f)$ and f is a epimorphsim, then $f(I)$ is a prime hyperideal of S.

(ii) If J is a prime hyperideal of S, then $f^{-1}(J)$ is a prime hyperideal of R.

Proof. (i) It is easy to see that $f(I)$ is a hyperideal of S since f is good epimorphism. Let $f(a) \circ f(b) \subseteq f(I)$ for $a, b \in R$. Since f is homomorphsim, $f(a \circ b) \subseteq f(a) \circ f(b) \subseteq f(I)$. Now take any $t \in a \circ b$. Then we have $f(t) \subseteq f(a \circ b) \subseteq f(I)$ and so $f(t) = f(x)$ for some $x \in I$. This implies that $f(t - x) = 0 \in \langle 0 \rangle$, that is, $t - x \in Ker(f) \subseteq I$ and so $t \in I$. Thus we conclude that $a \circ b \subseteq I$. Since I is prime hyperideal of R, we get $a \in I$ or $b \in I$ and so either $f(a) \in f(I)$ or $f(b) \in f(I)$. Consequently $f(I)$ is prime hyperideal of S.

(ii) First note that $f^{-1}(J)$ is a hyperideal of R. Let $a \circ b \subseteq f^{-1}(J)$ for $a, b \in R$. Then we have $f(a \circ b) = f(a) \circ f(b) \subseteq J$. Since J is prime hyperideal of S, we have $f(a) \in J$ or $f(b) \in J$ and so either $a \in f^{-1}(J)$ or $b \in f^{-1}(J)$.

Proposition 2.8. *Let $f : R \to S$ be a good homorphism and I, J be hyperideals of R and S, respectively. Then the followings are satisfied:*

(i) If I is a C-hyperideal of R containing $Ker(f)$ and f is a epimorphsim, then $f(I)$ is a C-hyperideal of S.

(ii) If J is a C-hyperideal of S, then $f^{-1}(J)$ is a C-hyperideal of R.

Proof. (i) Let $s_1 \circ s_2 \circ ... \circ s_n \cap f(I) \neq \emptyset$ for some $s_1, s_2, ..., s_n \in S$. Since f is epimorphsim, we have $f(a_i) = s_i$ for some $a_i \in R, 1 \leq i \leq n$. Then we have $(f(a_1) \circ f(a_2) \circ ... \circ f(a_n)) \cap f(I) = f(a_1 \circ a_2 \circ ... \circ a_n) \cap f(I) \neq \emptyset$ because f is good homomorphism. So there exists $t \in a_1 \circ a_2 \circ ... \circ a_n$ such that $f(t) \in f(I)$. Since $Ker(f) \subseteq I$, we conclude that $t \in I$ and so $a_1 \circ a_2 \circ ... \circ a_n \cap I \neq \emptyset$. As I is a C-ideal of R, $a_1 \circ a_2 \circ ... \circ a_n \subseteq I$ and so $f(a_1) \circ f(a_2) \circ ... \circ f(a_n) = f(a_1 \circ a_2 \circ ... \circ a_n) \subseteq f(I)$ that is $s_1 \circ s_2 \circ ... \circ s_n \subseteq f(I)$.

(ii) Let $a_1 \circ a_2 \circ ... \circ a_n \cap f^{-1}(J) \neq \emptyset$ for some $a_1, a_2, ..., a_n \in R$. This implies that $t \in f^{-1}(J)$ for some $t \in a_1 \circ a_2 \circ ... \circ a_n$ and so $f(t) \in J \cap f(a_1 \circ a_2 \circ ... \circ a_n)$. Then we have $J \cap (f(a_1) \circ f(a_2) \circ ... \circ f(a_n)) \neq \emptyset$. Since J is a C-ideal of S, we conclude that $f(a_1) \circ f(a_2) \circ ... \circ f(a_n) = f(a_1 \circ a_2 \circ ... \circ a_n) \subseteq J$ and so $a_1 \circ a_2 \circ ... \circ a_n \subseteq f^{-1}(J)$.

Suppose that I is a hyperideal of R. Then quotient abelian group $R/I = \{a + I : a \in R\}$ becomes a hyperring with the multiplication $(a + I) \circ (b + I) = \{r + I : r \in a \circ b\}$. In this case R/I is called quotient hyperring. One can easily prove that all hyperideal of R/I is of the form J/I, where J is a hyperideal of R containing I, since the natural homomorphism $\pi : R \to R/I,\ \pi(r) = r + I$ is a good epimorphism.

Proposition 2.9. *Let $I \subseteq P$ be hyperideals of R. Then the followings are satisfied.*

(i) P is prime hyperideal of R iff P/I is a prime hyperideal of R/I. In particular all prime hyperideal of R/I is of the form P/I, where P is a prime hyperideal of R containing I.

(ii) P is C-hyperideal of R iff P/I is a C-hyperideal of R/I. In particular all C-hyperideal of R/I is of the form P/I, where P is a C-hyperideal of R containing I.

Proof. (i) Consider the natural homomorphism $\pi : R \to R/I$ defined by $\pi(r) = r + I$. Since f is good epimorphism, the claim follows from Proposition 2.7.

(ii) It follows from Proposition 2.8.

Corollary 2.10. *Suppose that $I \subseteq J$ are hyperideals of R. Then the followings are satisfied:*

(i) $\sqrt{J/I} = \sqrt{J}/I$.

(ii) $D(J/I) = D(J)/I$.

(iii) If J is a C-hyperideal of R. Then $D(J/I) = \sqrt{J/I}$.

Proof. (i) It follows from Proposition 2.9.

(ii) Let $a + I \in D(J/I)$ for some $a \in R$. Then we have $(a + I)^n \subseteq J/I$. Take any $t \in a^n$. Since $t + I \in (a + I)^n$, we have $t + I \in J/I$ and so $t \in J$. Thus we have $a^n \subseteq J$ and this implies that $a+I \in D(J)/I$. Conversely assume that $a+I \in D(J)/I$ for $a \in R$. Then we have $a \in D(J)$ and so $a^n \subseteq J$ for some $n \in \mathbb{N}$. Take any $t+I \in (a+I)^n$, then we have $t+I = s+I$ for some $s \in a^n$. This implies that $t - s \in I \subseteq J$ and so $t = (t - s) + s \in I + a^n \subseteq J$ and thus $t + I \in J/I$. This implies that $(a + I)^n \subseteq (J/I)$ that is $a + I \in D(J/I)$.

(iii) It follows from [4, Proposition 3.2] and Proposition 2.9.

Proposition 2.11. *Let $f : R \to S$ be a good homorphism and I, J be hyperideals of R and S, respectively. Then the followings are satisfied:*

(i) If I is a primary hyperideal containing $Ker(f)$ and f is a epimorphsim, then $f(I)$ is a primary hyperideal of S.

(ii) If J is a primary hyperideal of S, then $f^{-1}(J)$ is a primary hyperideal of R.

Proof. (i),(ii): The proof is similar to Proposition 2.8.

Corollary 2.12. *Let $I \subseteq Q$ be hyperideals of R. Then,*

(i) Q is primary hyperideal of R iff Q/I is a primary hyperideal of R/I. In particular all primary hyperideal of R/I is of the form Q/I, where Q is a primary hyperideal of R containing I.

(ii) Suppose that Q is a primary C-hyperideal of R containing I. Then $\sqrt{Q/I}$ is a prime C-hyperideal of R/I.

Proof. (i): Follows from Proposition 2.11.

(ii): Follows from (i), Lemma 2.2, Proposition 2.9 and Corollary 2.10.

Corollary 2.13. *Suppose that I is a hyperideal of R. Then primary hyperideal avodiance theorem holds for quotient hyperring R/I.*

Proof. It follows from Corollary 2.12 and Theorem 2.6.

Now we extend the primary hyperideal avodiance theorem to cosets of any quotient hyperring. Suppose that $J, J_1, J_2, ..., J_n$ are hyperideals of R and $a_i \in R$ for each $i = 1, 2, ..., n$. Then we say a covering of cosets $J \subseteq (J_1 + a_1) \cup (J_2 + a_2) \cup ... \cup (J_n + a_n)$ is efficient if no cosets is unnecessary. Now we take such an efficient covering of cosets and assume that $a_k = a$ for all $k = 1, 2, ..., n$. Then we obtain an efficient covering $J - a \subseteq J_1 \cup J_2 \cup ... \cup J_n$. If $n = 1$, then we have $J - a \subseteq J_1$ and so $J \subseteq J_1 + a$. Since $0 \in J$, we get $a \in J_1$, that is, $J \subseteq J_1$.

Lemma 2.14. *Suppose that $J \subseteq (J_1 + a_1) \cup (J_2 + a_2) \cup ... \cup (J_n + a_n)$ is an efficient covering of cosets, where $J, J_1, J_2, ..., J_n$ are hyperideals of R and $a_i \in R$ for each $i = 1, 2, ..., n$. Then $J \cap \bigcap_{i \neq k} J_i \subseteq J_k$ but $J \nsubseteq J_k$.*

Proof. It is similar to proof of Lemma 4 in [7].

Proposition 2.15. *Suppose that $Q_1, Q_2, ..., Q_n$ are hyperideals of R and J is a hyperideal such that $J + a \subseteq Q_1 \cup Q_2 \cup ... \cup Q_n$ is an efficient covering. If $\sqrt{Q_i} \nsubseteq \sqrt{Q_k}$ for each $i \neq k$, then no Q_i is primary hyperideal of R.*

Proof. Assume that Q_1 is primary hyperideal of R. Then by previous lemma, we get $J \cap Q_2 \cap Q_3 \cap ... \cap Q_n \subseteq Q_1$ and also $J \nsubseteq Q_1$. Note that $J \circ Q_2 \circ ... \circ Q_n \subseteq J \cap Q_2 \cap Q_3 \cap ... \cap Q_n \subseteq Q_1$. Then by Corollary 2.4 we get $Q_i \subseteq D(Q_1)$and so $\sqrt{Q_i} \subseteq \sqrt{Q_1}$ which is a contradiction.

Theorem 2.16. *Suppose that $Q_1, Q_2, ..., Q_n$ are hyperideals and at most $n - 1$ of them are not primary hyperideal of R. If $\sqrt{Q_i} \nsubseteq \sqrt{Q_k}$ for $i \neq k$ and $J + a \subseteq Q_1 \cup Q_2 \cup ... \cup Q_n$, then $J + a \subseteq Q_i$. Further, $J + \langle a \rangle \subseteq Q_i$ for some $i \in \{1, 2, ..., n\}$.*

Proof. We may assume that covering is efficient. If $n > 1$, there exists a primary C-hyperideal Q_i of R which contradicts by previous proposition. So we have $n = 1$, $J + a \subseteq Q_i$ for some $i \in \{1, 2, ..., n\}$. Since $0 \in J$, we have $a \in Q_i$ and so $\langle a \rangle \subseteq Q_i$. As $a \in Q_i$ and $J + a \subseteq Q_i$, we get $J \subseteq Q_i$ and hence $J + \langle a \rangle \subseteq Q_i$.

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