Appl. Comput. Math., Vol.24, No.2, 2025, pp.326
DOI: 10.30546/1683-6154.24.2.2025.326

HERMITE-HADAMARD TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS WITH VARIABLE-ORDER FRACTIONAL INTEGRALS

A. YALÇIN[1,*], E. GÜL[1], A.O. AKDEMIR[2]

Abstract. This study develops a novel framework for Hermite-Hadamard-type inequalities by employing multivariate variable-order Riemann-Liouville fractional integral operators. These operators, which extend classical fractional calculus, allow fractional orders to vary dynamically, providing a powerful tool for capturing spatially and temporally dependent behaviors in multidimensional systems. We rigorously define the variable-order fractional integrals with new formulations of lower and upper bounds tailored for Hermite-Hadamard-type inequalities. By analyzing the properties and well-posedness of the proposed operators, we establish generalized Hermite-Hadamard inequalities for coordinated convex functions. These results represent a significant advancement in fractional analysis, bridging the gap between classical results and the more flexible, dynamic nature of variable-order systems. This work lays the foundation for further exploration of fractional inequalities and their application in systems governed by varying memory effect.

Keywords: Variable-Order Fractional Integrals, Co-Ordinated Convex Mapping, Hermite-Hadamard Inequality, Convex Functions, Riemann-Liouville Fractional Integrals, Mittag-Leffler Function.

AMS Subject Classification: 26A33, 26A51, 26D10.

1. Introduction

Fractional calculus, as an extension of classical calculus, has emerged as a powerful mathematical tool for modeling complex real-world phenomena that exhibit memory effects and non-local behavior. Its applications span across various scientific and engineering disciplines, particularly due to its ability to accurately describe hereditary properties and anomalous diffusion processes. Moreover, the integration of fractional calculus with inequality theory has further broadened its applicability in mathematical modeling and optimization problems. Classical fractional operators, such as the Riemann-Liouville and Caputo derivatives with constant fractional orders, have been extensively studied and applied in numerous fields.

The study of integral inequalities has provided a foundational framework for understanding various integral operators within fractional calculus [14]. Additionally, fractional differential equations have been extensively analyzed, emphasizing their significant role in mathematical physics and engineering applications [27]. In recent years, fractional calculus has expanded into new domains, demonstrating its relevance in solving complex real-world problems. The role of fractional modeling and control in addressing dynamic systems has been emphasized in various studies [62]. For example, the application of fractional derivatives in biomedical sciences has been explored through mathematical models of diseases, such as breast cancer, where the

[1]Department of Mathematics, Faculty of Arts and Science, Yıldız Technical University, Istanbul, Türkiye
e-mail: gul@yildiz.edu.tr
[2]Department of Mathematics, Faculty of Science and Letters, Ağrı Ibrahim Çeçen University, Ağrı, Türkiye
e-mail: aocakakdemir@gmail.com
*Corresponding author e-mail: latif.yalcin012@gmail.com
Manuscript received xx.

Caputo derivative and Mittag-Leffler functions have been effectively utilized to enhance numerical accuracy and predictive capabilities [56]. Furthermore, the introduction of a fractional derivative without a singular kernel has provided a more generalized framework for fractional dynamics, addressing limitations in classical formulations [16]. Advancements in fractional integral operators have also led to significant progress in inequality theory. The development of new integral inequalities for differentiable convex functions using Atangana-Baleanu fractional integral operators has enriched the mathematical structure of fractional analysis and strengthened its theoretical foundations [51]. Some further findings related to fractional calculus, inequality theory and applied mathematics can be found in the papers [1–7, 13, 15, 17, 52, 58]. However, despite these advances, the assumption of constant fractional orders often proves inadequate in describing systems where scaling properties or memory effects vary dynamically with respect to spatial or temporal variables. This limitation has prompted the introduction of variable-order fractional calculus (VOFC), which allows fractional orders to be defined as functions of independent variables, making it a more flexible and accurate modeling tool [32]. The concept of variable-order fractional operators has been explored in various studies, leading to the development of new mathematical approaches for solving fractional differential equations with dynamic properties. Methods such as integro-quadratic spline techniques have been proposed to efficiently handle variable-order fractional initial value problems, improving numerical accuracy and computational performance [36]. Additionally, the application of variable-order fractional synchronization in chaotic and reverse chaotic systems has demonstrated its potential in secure communication and cryptographic systems [37]. In viscoelasticity, variable-order models have been shown to more accurately describe time-dependent material behavior compared to constant fractional-order approaches [44].

The theoretical foundations of VOFC were first introduced in the early 1990s, providing a rigorous mathematical framework for defining and analyzing variable-order fractional integrals and derivatives [47]. Subsequent refinements extended these initial formulations, broadening their applicability across multiple scientific disciplines [46]. To further enhance the modeling of systems with non-stationary power-law kernels, VOFC has been utilized to capture the memory and hereditary properties of various physical processes [63]. Further developments have consolidated the theoretical background of VOFC, establishing its potential as a fundamental tool in mathematical modeling [33]. In the context of inequality theory, researchers have recently turned their attention to fractional Hermite-Hadamard-type inequalities due to their significance in estimating integral bounds for convex functions. While these inequalities have been well-established for classical fractional operators, their extension to variable-order fractional operators particularly in two-variable systems remains a challenging and open problem.

New advancements in this field have provided different approaches to addressing this challenge. For example, recent studies have presented novel estimations of Hadamard-type inequalities for various convex functions using tempered fractional integral operators, broadening the applicability of fractional inequalities in mathematical analysis [24]. Further investigations have introduced new Hermite-Hadamard-Mercer-type inequalities using a general family of fractional integral operators, offering refined integral bounds for various classes of convex functions [51]. Moreover, this work has contributed to significant improvements in the Hermite-Hadamard inequalities, especially for functions with layered structures, due to advances in conformable fractional integrals [26]. Additionally, researchers have derived integral inequalities through tempered fractional integral operators, further strengthening the theoretical framework for fractional inequalities in convex analysis [25]. Finally, new estimations based on exponentially convex functions have been explored using conformable fractional operators, opening new directions for fractional inequality research [21]. Addressing this gap requires the introduction of novel formulations for fractional integrals and the development of new bounds that account for the dynamic nature of variable orders.

Convexity theory plays a crucial role in numerous domains of pure and applied sciences. Consequently, classical concepts related to convex sets and convex functions have been extended in multiple directions. Functions form one of the fundamental structures in mathematics, and extensive research has been conducted to classify and generalize different function spaces.

One of the most significant classes of functions emerging from these studies is convex functions, which play a fundamental role in statistics, inequality theory, convex optimization, and numerical analysis. Formally, convex functions can be defined as follows:

Let $\mathcal{H}$ be an interval in $\mathbb{R}$. Then, $\psi : \mathcal{H} \to \mathbb{R}$, $\emptyset \neq \mathcal{H} \subseteq \mathbb{R}$ is said to be convex if

$$\psi(t\kappa_1 + (1-t)\kappa_2) \leq t\psi(\kappa_1) + (1-t)\psi(\kappa_2), \tag{1}$$

for all $\kappa_1, \kappa_2 \in \mathcal{H}$ and $t \in [0,1]$.

For additional insights, refer to the works in [42, 45, 59]. The concept of convex functions has a strong connection with the theory of inequalities, as many well-known inequalities are derived using this concept. Readers interested in exploring the fundamentals of convexity are encouraged to consult [11, 12, 20, 22, 57]. Among these inequalities, the Hermite-Hadamard inequality stands out as one of the most extensively studied results. This inequality provides a necessary and sufficient condition for a function to be convex. The classical Hermite-Hadamard inequality can be stated as follows:

Let $\psi : \mathcal{H} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined in the interval $\mathcal{H}$ of the real numbers $\kappa_1, \kappa_2 \in \mathcal{H}$, with $\kappa_1 < \kappa_2$. Then, one has

$$\psi\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\psi(x)\,dx \leq \frac{\psi(\kappa_1)+\psi(\kappa_2)}{2}. \tag{2}$$

In addition, it has been extended and transformed into various other integral inequalities through the use of different types of fractional operators (see [23,34,49,60]). In [30], the authors proposed the concept of coordinated convex functions defined over a rectangular region in the plane $\mathbb{R}^2$.

Definition 1. A function $\psi : \Delta \to \mathbb{R}$ is said to be coordinated convex on Δ if, for all $(\kappa_1, \kappa_2), (\rho_1, \rho_2) \in \Delta$ and $t, s \in [0,1]$, the following inequality holds:

$$\begin{aligned}\psi(t\kappa_1 + (1-t)\kappa_2, s\rho_1 + (1-s)\rho_2) \leq\ & ts\psi(\kappa_1,\rho_1) + s(1-t)\psi(\kappa_2,\rho_1) \\ & + t(1-s)\psi(\kappa_1,\rho_2) + (1-t)(1-s)\psi(\kappa_2,\rho_2).\end{aligned} \tag{3}$$

In [19], the author derived the following Hermite-Hadamard-type inequalities, analogous to those in the one-dimensional case.

Theorem 1. *Suppose that $\psi : \Delta \longrightarrow \mathbb{R}$ is co-ordinated convex on Δ. Then, one has the inequalities:*

$$\begin{aligned}&\psi\left(\frac{\kappa_1+\kappa_2}{2}, \frac{\rho_1+\rho_2}{2}\right) \\ &\leq \frac{1}{2}\left[\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\psi\left(x, \frac{\rho_1+\rho_2}{2}\right)dx + \frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\psi\left(y, \frac{\kappa_1+\kappa_2}{2}\right)dy\right] \\ &\leq \frac{1}{(\kappa_2-\kappa_1)(\rho_2-\rho_1)}\int_{\kappa_1}^{\kappa_2}\int_{\rho_1}^{\rho_2}\psi(x,y)\,dy\,dx \\ &\leq \frac{1}{4}\left[\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\psi(x,\rho_1)\,dx + \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\psi(x,\rho_2)\,dx\right.\end{aligned} \tag{4}$$

$$+\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\psi(y,\kappa_1)\,dy+\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\psi(y,\kappa_2)\,dy\Bigg] \tag{5}$$
$$\leq \frac{1}{4}\left[\psi(\kappa_1,\rho_1)+\psi(\kappa_1,\rho_2)+\psi(\kappa_2,\rho_1)+\psi(\kappa_2,\rho_2)\right].$$

These inequalities are sharp.

Numerous related results have been established in this area for various classes of convex functions; see [8–10, 28, 31, 38, 39] and the references cited therein.

The concept of variable-order (VO) fractional integrals and derivatives, first introduced in [46], has been increasingly developed and explored in recent years. Notably, there are three primary approaches to establishing the mathematical foundation of VO fractional integrals and derivatives: direct extensions from classical fractional operators [63], definitions based on Laplace transforms [18], and formulations driven by physical motivations [22, 35, 43, 53–55].

In this section, we present the existing definitions of fractional integral operators.

Definition 2. Let $0<\alpha(x)<1$ for all $x\in[\kappa_1,\kappa_2]$ and $\psi\in\mathcal{L}^1(\kappa_1,\rho_1)$. Then, the left and right-sided Riemann-Liouville fractional integrals of variable fractional order $\alpha(x)$ are defined by

$$J_{\kappa_1^+}^{\alpha(x)}\psi(x)=\frac{1}{\Gamma[\alpha(x)]}\int_{\kappa_1}^{x}\psi(t)(x-t)^{\alpha(x)-1}\,dt,\quad Re\ \alpha(x)>0 \tag{6}$$

and

$$J_{\rho_1^-}^{\alpha(x)}\psi(x)=\frac{1}{\Gamma[\alpha(x)]}\int_{x}^{\rho_1}\psi(t)(t-x)^{\alpha(x)-1}\,dt,\quad Re\ \alpha(x)>0. \tag{7}$$

respectively. Here, Γ is the Gamma function.

The classical Riemann-Liouville fractional integral operates with a fixed constant order of integration α, whereas the variable-order definition allows the integration order $\alpha(x)$ to change as a function of the variable x. In our case, $\alpha(x)$ varies across the integration domain, taking different values at different points, while both α and the interval points change simultaneously. This flexibility enables a graphical analysis that highlights the validity of the newly generalized inequalities.

It is noteworthy that in [61] the authors was introduced the following intriguing Hermite-Hadamard-type integral inequalities involving variable-order Riemann-Liouville fractional integrals.

Theorem 2. *$\psi:[\kappa_1,\kappa_2]\longrightarrow\mathbb{R}$ be a positive function with $0\leq\kappa_1<\kappa_2$ and $\psi\in\mathcal{L}^1[\kappa_1,\kappa_2]$. If it is a convex function on $[\kappa_1,\kappa_2]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned}&\psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\\ &\leq\frac{2^{\alpha(x,y)-1}\Gamma(\alpha(x,y)+1)}{(\kappa_2-\kappa_1)^{\alpha(x,y)}}\left[J_{\kappa_1^+}^{\alpha(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2}\right)+J_{\kappa_2^-}^{\alpha(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\ &\leq\frac{\psi(\kappa_1)+\psi(\kappa_2)}{2}.\end{aligned} \tag{8}$$

Based on Definition 2, the variable-order fractional Riemann-Liouville integral on the coordinates can be defined as follows.

Definition 3. Let $\psi \in \mathcal{L}^1([\kappa_1,\kappa_2]\times[\rho_1,\rho_2])$ be a function. The variable-order Riemann-Liouville fractional integral operators are defined as follows:

$$J_{\kappa_1^+,\rho_1^+}^{\alpha(x,y),\beta(x,y)}\ \psi(x,y)= \frac{1}{\Gamma[\alpha(x,y)]\,\Gamma[\beta(x,y)]}\int_{\kappa_1}^{x}\int_{\rho_1}^{y}(x-t)^{\alpha(x,y)-1}(y-s)^{\beta(x,y)-1}\,\psi(t,s)\,dsdt, \tag{9}$$

$x>\kappa_1,\ y>\rho_1.$

$$J_{\kappa_1^+,\rho_2^-}^{\alpha(x,y),\beta(x,y)}\psi(x,y) = \frac{1}{\Gamma(\alpha(x,y))\,\Gamma(\beta(x,y))}\int_{\kappa_1}^{x}\int_{y}^{\rho_2}(x-t)^{\alpha(x,y)-1}(s-y)^{\beta(x,y)-1}\,\psi(t,s)\,ds\,dt, \tag{10}$$

for $\quad x>\kappa_1, \quad y<\rho_2.$

and

$$J_{\kappa_2^-,\rho_1^+}^{\alpha(x,y),\beta(x,y)}\psi(x,y) = \frac{1}{\Gamma(\alpha(x,y))\,\Gamma(\beta(x,y))}\int_{x}^{\kappa_2}\int_{\rho_1}^{y}(t-x)^{\alpha(x,y)-1}(y-s)^{\beta(x,y)-1}\,\psi(t,s)\,ds\,dt, \tag{11}$$

for $x<\kappa_2,\ y>\rho_1.$

$$J_{\kappa_2^-,\rho_2^-}^{\alpha(x,y),\beta(x,y)}\psi(x,y) = \frac{1}{\Gamma(\alpha(x,y))\,\Gamma(\beta(x,y))}\int_{x}^{\kappa_2}\int_{y}^{\rho_2}(t-x)^{\alpha(x,y)-1}(s-y)^{\beta(x,y)-1}\,\psi(t,s)\,ds\,dt, \tag{12}$$

for $\quad x<\kappa_2, \quad y<\rho_2.$

Here, $\alpha(x,y),\beta(x,y)>0$ and Γ is the Gamma function.

Similar to Definition 2 and Definition 3, for a given function $\psi\in\mathcal{L}^1([\kappa_1,\kappa_2]\times[\rho_1,\rho_2])$, the special variable-order Riemann-Liouville fractional integral operators are defined as follows:

i. Variable-Order Fractional Integrals Along the x -Direction (Fixed y).

$$J_{\kappa_1^+}^{\alpha(x,y)}\psi\left(x,\frac{\rho_1+\rho_2}{2}\right)=\frac{1}{\Gamma(\alpha(x,y))}\int_{\kappa_1}^{x}\psi\left(t,\frac{\rho_1+\rho_2}{2}\right)(x-t)^{\alpha(x,y)-1}\,dt, \tag{13}$$

for $\quad x>\kappa_1.$

$$J_{\kappa_2^-}^{\alpha(x,y)}\psi\left(x,\frac{\rho_1+\rho_2}{2}\right)=\frac{1}{\Gamma(\alpha(x,y))}\int_{x}^{\kappa_2}\psi\left(t,\frac{\rho_1+\rho_2}{2}\right)(t-x)^{\alpha(x,y)-1}\,dt, \tag{14}$$

for $\quad x<\kappa_2.$

ii. Variable-Order Fractional Integrals Along the y -Direction (Fixed x).

$$J_{\rho_1^+}^{\beta(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},y\right)=\frac{1}{\Gamma(\beta(x,y))}\int_{\rho_1}^{y}\psi\left(\frac{\kappa_1+\kappa_2}{2},s\right)(y-s)^{\beta(x,y)-1}\,ds, \tag{15}$$

for $\quad y>\rho_1.$

$$J_{\rho_2^+}^{\beta(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},y\right)=\frac{1}{\Gamma\left(\beta(x,y)\right)}\int_{\rho_2}^{y}\psi\left(\frac{\kappa_1+\kappa_2}{2},s\right)(y-s)^{\beta(x,y)-1}\,ds, \tag{16}$$

for $y<\rho_2$.

These operators allow fractional integration to be applied along a single variable while keeping the other variable fixed. Such special fractional integrals play a crucial role in modeling asymmetric systems, anisotropic diffusion processes, and directional memory effects (see [29, 40, 41]), where variable-order operators provide a more flexible framework for capturing spatially and temporally varying behaviors.

Hermite-Hadamard-type inequalities for fixed-order Riemann-Liouville fractional integrals using convex functions of two variables on the coordinates, as presented in [48].

Theorem 3. *Suppose that $\psi:\Delta\longrightarrow\mathbb{R}$ is coordinated convex on $\Delta=[\kappa_1,\kappa_2]\times[\rho_1,\rho_2]$. In $\mathbb{R}^2$ and $\psi\in\mathcal{L}^1([\kappa_1,\kappa_2]\times[\rho_1,\rho_2])$, then one has the inequalities:*

$$\begin{aligned}&\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\leq\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{4\left(\kappa_2-\kappa_1\right)^{\alpha}\left(\rho_2-\rho_1\right)^{\beta}}\\&\times\left[J_{\kappa_1^+,\rho_1^+}^{\alpha,\beta}\psi(\kappa_2,\rho_2)+J_{\kappa_1^+,\rho_2^-}^{\alpha,\beta}\psi(\kappa_2,\rho_1)+J_{\kappa_2^-,\rho_1^+}^{\alpha,\beta}\psi(\kappa_1,\rho_2)+J_{\kappa_2^-,\rho_2^-}^{\alpha,\beta}\psi(\kappa_1,\rho_1)\right]\\&\leq\frac{\psi(\kappa_1,\rho_1)+\psi(\kappa_1,\rho_2)+\psi(\kappa_2,\rho_1)+\psi(\kappa_2,\rho_2)}{4}.\end{aligned} \tag{17}$$

These inequalities are sharp.

In this paper, we introduce multivariate variable-order Riemann-Liouville fractional integral operators and establish newly formulated lower and upper bounds tailored for Hermite-Hadamard-type inequalities. These operators not only generalize classical fractional operators, but also provide a rigorous mathematical framework for analyzing systems with multidimensional variable-order dynamics. By systematically defining these operators and exploring their properties, we derive generalized Hermite-Hadamard inequalities for coordinated convex functions. The novelty of this work lies in the formulation of new integral bounds and the application of variable-order fractional operators to Hermite-Hadamard inequalities, bridging the gap between classical and dynamic fractional systems. These results establish a versatile mathematical foundation for the analysis of multidimensional systems with spatially and temporally varying behaviors. Additionally, this study paves the way for further exploration of variable-order fractional calculus and its applications in diverse fields, including physics, engineering, and applied mathematics.

2. New version of Hermite-Hadamard-type integral inequalities involving variable-order fractional integrals

In this section, we establish Hermite-Hadamard-type inequalities for variable-order Riemann-Liouville fractional integrals using convex functions of two variables on the coordinates. Furthermore, we derive a new version of the Hermite-Hadamard inequality adapted to the variable-order fractional integral framework.

Theorem 4. *Suppose that $\psi:\Delta\longrightarrow\mathbb{R}$ is coordinated convex on $\Delta=[\kappa_1,\kappa_2]\times[\rho_1,\rho_2]$ in $\mathbb{R}^2$ and also assume that $\psi\in\mathcal{L}^1([\kappa_1,\kappa_2]\times[\rho_1,\rho_2])$. Furthermore, let the variable-order parameters satisfy $Re\ (\alpha)>0$ and $Re\ (\beta)>0$. Under these conditions, the following inequality holds for the variable-order Riemann-Liouville fractional integrals:*

$$
\begin{aligned}
&\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\\
&\leq \frac{2^{\alpha(x,y)+\beta(x,y)-2}\Gamma\left(\alpha(x,y)+1\right)\Gamma\left(\beta(x,y)+1\right)}{\left(\kappa_2-\kappa_1\right)^{\alpha(x,y)}\left(\rho_2-\rho_1\right)^{\beta(x,y)}}\\
&\quad\times\left[J^{\alpha(x,y),\beta(x,y)}_{\kappa_1^+,\rho_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha(x,y),\beta(x,y)}_{\kappa_1^+,\rho_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right.\\
&\quad\left.+J^{\alpha(x,y),\beta(x,y)}_{\kappa_2^-,\rho_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha(x,y),\beta(x,y)}_{\kappa_2^-,\rho_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right]\\
&\leq \frac{\psi\left(\kappa_1,\rho_1\right)+\psi\left(\kappa_1,\rho_2\right)+\psi\left(\kappa_2,\rho_1\right)+\psi\left(\kappa_2,\rho_2\right)}{4}.
\end{aligned}
\tag{18}
$$

Proof. Since $\psi:\Delta\longrightarrow\mathbb{R}$ is co-ordinated convex on $\Delta=[\kappa_1,\kappa_2]\times[\rho_1,\rho_2]$ in $\mathbb{R}^2$ and $t,s\in[0,1]$, then we can write:

$$
\begin{aligned}
\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\leq\frac{1}{4}\Bigg(&\psi\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2,\frac{s}{2}\rho_1+\frac{2-s}{2}\rho_2\right)\\
&+\psi\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2,\frac{2-s}{2}\rho_1+\frac{s}{2}\rho_2\right)\\
&+\psi\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2,\frac{s}{2}\rho_1+\frac{2-s}{2}\rho_2\right)\\
&+\psi\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2,\frac{2-s}{2}\rho_1+\frac{s}{2}\rho_2\right)\Bigg).
\end{aligned}
\tag{19}
$$

According to Definition 1, we obtain the following result:

$$
\begin{aligned}
&\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\\
&\leq\frac{1}{4}\left[\psi\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2,\frac{s}{2}\rho_1+\frac{2-s}{2}\rho_2\right)+\psi\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2,\frac{2-s}{2}\rho_1+\frac{s}{2}\rho_2\right)\right.\\
&\quad\left.+\psi\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2,\frac{s}{2}\rho_1+\frac{2-s}{2}\rho_2\right)+\psi\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2,\frac{2-s}{2}\rho_1+\frac{s}{2}\rho_2\right)\right]\\
&\leq\frac{\psi\left(\kappa_1,\rho_1\right)+\psi\left(\kappa_1,\rho_2\right)+\psi\left(\kappa_2,\rho_1\right)+\psi\left(\kappa_2,\rho_2\right)}{4}.
\end{aligned}
\tag{20}
$$

Thus, by multiplying both sides of (19) by $(1-t)^{\alpha(x,y)-1}.(1-s)^{\beta(x,y)-1}$ and then integrating over $[0,1]\times[0,1]$, we obtain:

$$
\begin{aligned}
&\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\times\int_0^1\int_0^1(1-t)^{\alpha(x,y)-1}(1-s)^{\beta(x,y)-1}\psi(t,s)\,ds\,dt\\
&\leq\frac{1}{4}\left\{\int_0^1\int_0^1(1-t)^{\alpha(x,y)-1}(1-s)^{\beta(x,y)-1}\psi\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2,\frac{s}{2}\rho_1+\frac{2-s}{2}\rho_2\right)ds\,dt\right.\\
&\quad+\int_0^1\int_0^1(1-t)^{\alpha(x,y)-1}(1-s)^{\beta(x,y)-1}\psi\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2,\frac{2-s}{2}\rho_1+\frac{s}{2}\rho_2\right)ds\,dt\\
&\quad\left.+\int_0^1\int_0^1(1-t)^{\alpha(x,y)-1}(1-s)^{\beta(x,y)-1}\psi\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2,\frac{s}{2}\rho_1+\frac{2-s}{2}\rho_2\right)ds\,dt\right.
\end{aligned}
$$

$$+\int_0^1\int_0^1 (1-t)^{\alpha(x,y)-1}(1-s)^{\beta(x,y)-1}\psi\left(\frac{2-t}{2}\kappa_1+\frac{t}{2}\kappa_2,\frac{2-s}{2}\rho_1+\frac{s}{2}\rho_2\right)ds\,dt\Bigg\}$$
$$\leq \frac{\psi(\kappa_1,\rho_1)+\psi(\kappa_1,\rho_2)+\psi(\kappa_2,\rho_1)+\psi(\kappa_2,\rho_2)}{4} \tag{21}$$
$$\times\int_0^1\int_0^1 (1-t)^{\alpha(x,y)-1}(1-s)^{\beta(x,y)-1}\,ds\,dt.$$

By changing the variables in both sides of the integrals, we obtain

$$\begin{aligned}
&\int_0^1\int_0^1 (1-t)^{\alpha(x,y)-1}(1-s)^{\beta(x,y)-1}\times\psi\left(\frac{t}{2}\kappa_1+\frac{2-t}{2}\kappa_2,\frac{s}{2}\rho_1+\frac{2-s}{2}\rho_2\right)ds\,dt\\
&=\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha(x,y)-1}\left(\frac{2}{\rho_2-\rho_1}\right)^{\beta(x,y)-1}\\
&\times\left\{\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\alpha(x,y)-1}\left(\frac{\rho_1+\rho_2}{2}-y\right)^{\beta(x,y)-1}\psi(x,y)\,dx\,dy\right\} \qquad (22)\\
&=\frac{2^{\alpha(x,y)}2^{\beta(x,y)}\Gamma[\alpha(x,y)]\Gamma[\beta(x,y)]}{(\kappa_2-\kappa_1)^{\alpha(x,y)}(\rho_2-\rho_1)^{\beta(x,y)}}\\
&\times\left[J^{\alpha(x,y),\beta(x,y)}_{\kappa_1^+,\rho_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right].
\end{aligned}$$

On the other side, we have

$$\int_0^1\int_0^1 (1-t)^{\alpha(x,y)-1}(1-s)^{\beta(x,y)-1}\,dsdt=\frac{1}{\alpha(x,y)\beta(x,y)}. \tag{23}$$

This can be rewritten as

$$\begin{aligned}
&\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\frac{1}{\alpha(x,y)\beta(x,y)}\\
&\leq\frac{2^{\alpha(x,y)}2^{\beta(x,y)}\Gamma[\alpha(x,y)]\Gamma[\beta(x,y)]}{4(\kappa_2-\kappa_1)^{\alpha(x,y)}(\rho_2-\rho_1)^{\beta(x,y)}}\left[J^{\alpha(x,y),\beta(x,y)}_{\kappa_1^+,\rho_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right.\\
&+J^{\alpha(x,y),\beta(x,y)}_{\kappa_1^+,\rho_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha(x,y),\beta(x,y)}_{\kappa_2^-,\rho_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right) \qquad (24)\\
&\left.+J^{\alpha(x,y),\beta(x,y)}_{\kappa_2^-,\rho_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right]\\
&\leq\frac{\psi(\kappa_1,\rho_1)+\psi(\kappa_1,\rho_2)+\psi(\kappa_2,\rho_1)+\psi(\kappa_2,\rho_2)}{4\alpha(x,y)\beta(x,y)}.
\end{aligned}$$

The proof is thus concluded. □

Remark 1. In Theorem 4, if we choose $\alpha(x,y)=1$ and $\beta(x,y)=1$ then the inequalities (17) become the inequalities (4).

Corollary 1. *In Theorem 4, if we choose $\alpha(x,y)=\alpha$ and $\beta(x,y)=\beta$, the following inequalities are valid:*

$$
\begin{aligned}
&\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\\
&\leq \frac{2^{\alpha-1}2^{\beta-1}\Gamma\left[\alpha+1\right]\Gamma\left[\beta+1\right]}{\left(\kappa_2-\kappa_1\right)^{\alpha}\left(\rho_2-\rho_1\right)^{\beta}}\\
&\times\left[J^{\alpha,\beta}_{\kappa_1^+,\rho_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha,\beta}_{\kappa_1^+,\rho_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right.\\
&\quad\left.+J^{\alpha,\beta}_{\kappa_2^-,\rho_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha,\beta}_{\kappa_2^-,\rho_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right]\\
&\leq \frac{\psi\left(\kappa_1,\rho_1\right)+\psi\left(\kappa_1,\rho_2\right)+\psi\left(\kappa_2,\rho_1\right)+\psi\left(\kappa_2,\rho_2\right)}{4\alpha\beta}.
\end{aligned}
\tag{25}
$$

Theorem 5. *Suppose that $\psi:\Delta\longrightarrow\mathbb{R}$ is coordinated convex on $\Delta=[\kappa_1,\kappa_2]\times[\rho_1,\rho_2]$ in $\mathbb{R}^2$, and assume that $\psi\in\mathcal{L}^1\left([\kappa_1,\kappa_2]\times[\rho_1,\rho_2]\right)$. Furthermore, let the variable-order parameters satisfy $Re\ (\alpha)>0$ and $Re\ (\beta)>0$. Under these assumptions, the following inequality holds for the variable-order Riemann-Liouville fractional integrals:*

$$
\begin{aligned}
&\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\\
&\leq \frac{2^{\beta(x,y)-2}\Gamma\left[\beta\left(x,y\right)+1\right]}{\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}\\
&\times\left[J^{\beta(x,y)}_{\kappa_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J^{\beta(x,y)}_{\kappa_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right]\\
&+\frac{2^{\alpha(x,y)-2}\Gamma\left[\alpha\left(x,y\right)+1\right]}{\left(\rho_2-\rho_1\right)^{\alpha(x,y)}}\\
&\times\left[J^{\alpha(x,y)}_{\rho_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha(x,y)}_{\rho_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right]\\
&\leq \frac{2^{\beta(x,y)-2}2^{\alpha(x,y)-2}\Gamma\left[\alpha\left(x,y\right)+1\right]\Gamma\left[\beta\left(x,y\right)+1\right]}{\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}\left(\rho_2-\rho_1\right)^{\alpha(x,y)}}\\
&\quad\times\left[J^{\alpha(x,y),\beta(x,y)}_{\kappa_1^+,\rho_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha(x,y),\beta(x,y)}_{\kappa_1^+,\rho_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right.\\
&\quad\left.+J^{\alpha(x,y),\beta(x,y)}_{\kappa_2^-,\rho_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J^{\alpha(x,y),\beta(x,y)}_{\kappa_2^-,\rho_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right]\\
&\leq \frac{2^{\alpha(x,y)-3}\Gamma\left[\alpha\left(x,y\right)+1\right]}{\left(\kappa_2-\kappa_1\right)^{\alpha(x,y)}}\left[J^{\alpha(x)}_{\kappa_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\rho_1\right)+J^{\alpha(x)}_{\kappa_1^+}\psi\left(\frac{\kappa_1+\kappa_2}{2},\rho_2\right)\right.\\
&\quad\left.+J^{\alpha(x)}_{\kappa_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\rho_1\right)+J^{\alpha(x)}_{\kappa_2^-}\psi\left(\frac{\kappa_1+\kappa_2}{2},\rho_2\right)\right]\\
&+\frac{2^{\beta(x,y)-3}\Gamma\left[\beta\left(x,y\right)+1\right]}{\left(\rho_2-\rho_1\right)^{\beta(x,y)}}\left[J^{\beta(x)}_{\rho_1^+}\psi\left(\kappa_1,\frac{\rho_1+\rho_2}{2}\right)+J^{\beta(x)}_{\rho_1^+}\psi\left(\kappa_2,\frac{\rho_1+\rho_2}{2}\right)\right.\\
&\quad\left.+J^{\beta(x)}_{\rho_2^-}\psi\left(\kappa_1,\frac{\rho_1+\rho_2}{2}\right)+J^{\beta(x)}_{\rho_2^-}\psi\left(\kappa_2,\frac{\rho_1+\rho_2}{2}\right)\right]\\
&\leq \frac{\psi\left(\kappa_1,\rho_1\right)+\psi\left(\kappa_1,\rho_2\right)+\psi\left(\kappa_2,\rho_1\right)+\psi\left(\kappa_2,\rho_2\right)}{4}.
\end{aligned}
\tag{26}
$$

Proof. Let $\psi:\Delta\longrightarrow\mathbb{R}$ be coordinated convex on $\Delta=[\kappa_1,\kappa_2]\times[\rho_1,\rho_2]\subset\mathbb{R}^2$. It follows that the mapping $\varphi_x(y)=(x,y)$, where $\varphi_x:[\rho_1,\rho_2]\longrightarrow\mathbb{R}$, is convex on $[\rho_1,\rho_2]$ for all $x\in[\kappa_1,\kappa_2]$.

Then, by using inequalities (7), we can write

$$\begin{aligned}\varphi_x\left(\frac{\rho_1+\rho_2}{2}\right) &\leq \frac{2^{\alpha(x,y)-1}\Gamma\left[\alpha\left(x,y\right)+1\right]}{\left(\rho_2-\rho_1\right)^{\alpha(x,y)}}\left[J_{\rho_1^+}^{\alpha(x,y)}\varphi_x\left(\frac{\rho_1+\rho_2}{2}\right)\right.\\ &\left.+\ J_{\rho_2^-}^{\alpha(x,y)}\varphi_x\left(\frac{\rho_1+\rho_2}{2}\right)\right]\\ &\leq \frac{\varphi_x\left(\rho_1\right)+\varphi_x\left(\rho_2\right)}{2},\quad x\in\left[\kappa_1,\kappa_2\right].\end{aligned} \tag{27}$$

That is,

$$\begin{aligned}&\psi\left(x,\frac{\rho_1+\rho_2}{2}\right)\\ &\leq \frac{2^{\alpha(x,y)-1}\alpha\left(x,y\right)}{\left(\rho_2-\rho_1\right)^{\alpha(x,y)}}\left[\int_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}\left(\frac{\rho_1+\rho_2}{2}-y\right)^{\alpha(x,y)-1}\psi\left(x,y\right)\,dy\right.\\ &\left.+\int_{\frac{\rho_1+\rho_2}{2}}^{\rho_2}\left(y-\frac{\rho_1+\rho_2}{2}\right)^{\alpha(x,y)-1}\psi\left(x,y\right)\,dy\right]\\ &\leq \frac{\psi\left(x,\rho_1\right)+\psi\left(x,\rho_2\right)}{2}.\end{aligned} \tag{28}$$

By multiplying both sides of (27) by $\frac{2^{\beta(x,y)-1}\beta(x,y)}{\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}$ and integrating with respect to x over $\left[\kappa_1,\frac{\kappa_1+\kappa_2}{2}\right]$, we obtain

$$\begin{aligned}&\frac{2^{\beta(x,y)-1}\beta\left(x,y\right)}{\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\psi\left(x,\frac{\rho_1+\rho_2}{2}\right)\,dx\\ &\leq \frac{2^{\alpha(x,y)-1}\alpha\left(x,y\right)2^{\beta(x,y)-1}\beta\left(x,y\right)}{\left(\rho_2-\rho_1\right)^{\alpha(x,y)}\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}\\ &\times\left[\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\right.\\ &\times\left(\frac{\rho_1+\rho_2}{2}-y\right)^{\beta(x,y)-1}\psi\left(x,y\right)\,dx\,dy\\ &+\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\frac{\rho_1+\rho_2}{2}}^{\rho_2}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\\ &\left.\times\left(y-\frac{\rho_1+\rho_2}{2}\right)^{\alpha(x,y)-1}\psi\left(x,y\right)\,dx\,dy\right]\end{aligned}$$

$$\leq \frac{2^{\beta(x,y)-1}\beta}{(\kappa_2-\kappa_1)^{\beta(x,y)}}\left[\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\psi\left(x,\rho_1\right)\,dx\right. \tag{29}$$

$$\left.+\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\psi\left(x,\rho_2\right)\,dx\right].$$

Similarly, multiplying (27) by $\frac{2^{\beta(x,y)-1}\beta(x,y)}{(\kappa_2-\kappa_1)^{\beta(x,y)}}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\beta(x,y)-1}$ and integrating with respect to x over $\left[\frac{\kappa_1+\kappa_2}{2},\kappa_2\right]$ respectively, we have

$$\begin{aligned}
&\frac{2^{\beta(x,y)-1}\beta\left(x,y\right)}{(\kappa_2-\kappa_1)^{\beta(x,y)}}\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\beta(x,y)-1}\psi\left(x,\frac{\rho_1+\rho_2}{2}\right)\,dx\\
&\leq \frac{2^{\alpha(x,y)-1}\alpha\left(x,y\right)2^{\beta(x,y)-1}\beta\left(x,y\right)}{(\rho_2-\rho_1)^{\alpha(x,y)}(\kappa_2-\kappa_1)^{\beta(x,y)}}\left[\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\int_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\beta(x,y)-1}\right.\\
&\times\left(\frac{\rho_1+\rho_2}{2}-y\right)^{\beta(x,y)-1}\psi\left(x,y\right)\,dx\,dy+\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1}\int_{\frac{\rho_1+\rho_2}{2}}^{\rho_1}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\beta(x,y)-1}\\
&\left.\times\left(y-\frac{\rho_1+\rho_2}{2}\right)^{\alpha(x,y)-1}\psi\left(x,y\right)\,dx\,dy\right]\\
&\leq \frac{2^{\beta(x,y)-1}\beta\left(x,y\right)}{(\kappa_2-\kappa_1)^{\beta(x,y)}}\left[\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\beta(x,y)-1}\psi\left(x,\rho_1\right)\,dx\right.\\
&\left.+\int_{\kappa_2}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\psi\left(x,\rho_2\right)\,dx\right].
\end{aligned} \tag{30}$$

In the same way, from the inequality of (7) as $\varphi_y\left(x\right)=\left(y,x\right)$, $\varphi_y:[\kappa_1,\kappa_2]\longrightarrow\mathbb{R}$ is convex on $[\kappa_1,\kappa_2]$ for all $y\in[\rho_1,\rho_2]$, we have

$$\begin{aligned}
\varphi_y\left(\frac{\kappa_1+\kappa_2}{2}\right)&\leq\frac{2^{\beta(x,y)-1}\Gamma\left[\beta\left(x,y\right)+1\right]}{(\kappa_2-\kappa_1)^{\alpha(x,y)}}\left[J_{\rho_1^+}^{\beta(x,y)}\varphi_y\left(\frac{\kappa_1+\kappa_2}{2}\right)\right.\\
&\left.+J_{\rho_1^-}^{\beta(x,y)}\varphi_y\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\\
&\leq\frac{\varphi_y\left(\kappa_1\right)+\varphi_y\left(\kappa_2\right)}{2},\quad y\in[\rho_1,\rho_2].
\end{aligned} \tag{31}$$

That is,

$$\psi\left(\frac{\kappa_1+\kappa_2}{2},y\right)\leq\frac{2^{\beta(x,y)-1}\beta\left(x,y\right)}{(\kappa_2-\kappa_1)^{\beta(x,y)}}\left[\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\psi\left(x,y\right)\,dx\right.$$

$$+\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\alpha(x,y)-1}\psi\left(x,y\right)\,dx\Bigg] \tag{32}$$
$$\leq \frac{\psi\left(y,\kappa_1\right)+\psi\left(y,\kappa_2\right)}{2}.$$

Then, multiplying both sides of (31) by $\frac{2^{\alpha(x,y)-1}\alpha(x,y)}{(\kappa_2-\kappa_1)^{\alpha(x,y)}}\left(\frac{\rho_1+\rho_2}{2}-y\right)^{\alpha(x,y)-1}$ and, integrating with respect to y over $\left[\rho_1,\frac{\rho_1+\rho_2}{2}\right]$ respectively, we have,

$$\frac{2^{\alpha(x,y)-1}\alpha\left(x,y\right)}{\left(\kappa_2-\kappa_1\right)^{\alpha(x,y)}}\int_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}\left(\frac{\rho_1+\rho_2}{2}-x\right)^{\alpha(x,y)-1}\psi\left(\frac{\rho_1+\rho_2}{2},y\right)\,dy$$
$$\leq \frac{2^{\alpha(x,y)-1}\alpha\left(x,y\right)2^{\beta(x,y)-1}\beta\left(x,y\right)}{\left(\rho_2-\rho_1\right)^{\alpha(x,y)}\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}$$
$$\Bigg[\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\left(\frac{\rho_1+\rho_2}{2}-y\right)^{\beta(x,y)-1}\psi\left(x,y\right)\,dx\,dy$$
$$+\int_{\kappa_2}^{\frac{\kappa_1+\kappa_2}{2}}\int_{\frac{\rho_1+\rho_2}{2}}^{\rho_1}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\left(y-\frac{\rho_1+\rho_2}{2}\right)^{\alpha(x,y)-1}\psi\left(x,y\right)\,dx\,dy\Bigg] \tag{33}$$
$$\leq \frac{2^{\beta(x,y)-1}\beta\left(x,y\right)}{\left(\rho_2-\rho_1\right)^{\beta(x,y)}}\Bigg[\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\psi\left(x,\kappa_1\right)\,dy$$
$$+\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\psi\left(x,\kappa_2\right)\,dy\Bigg].$$

Similarly, if we multiply (31) by $\frac{2^{\alpha(x,y)-1}\alpha(x,y)}{(\kappa_2-\kappa_1)^{\alpha(x,y)}}\left(y-\frac{\rho_1+\rho_2}{2}\right)^{\alpha(x,y)-1}$ and integrating with respect to y over $\left[\frac{\rho_1+\rho_2}{2},\ \rho_2\right]$ respectively, we have,

$$\frac{2^{\alpha(x,y)-1}\alpha\left(x,y\right)}{\left(\rho_2-\rho_1\right)^{\alpha(x,y)}}\int_{\frac{\rho_1+\rho_2}{2}}^{\rho_2}\left(x-\frac{\rho_1+\rho_2}{2}\right)^{\alpha(x,y)-1}\psi\left(y,\frac{\kappa_1+\kappa_2}{2}\right)\,dy$$
$$\leq \frac{2^{\alpha(x,y)-1}\alpha\left(x,y\right)2^{\beta(x,y)-1}\beta\left(x,y\right)}{\left(\rho_2-\rho_1\right)^{\alpha(x,y)}\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}$$
$$\Bigg[\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1}\int_{\rho_2}^{\frac{\rho_1+\rho_2}{2}}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\beta(x,y)-1}\left(\frac{\rho_1+\rho_2}{2}-y\right)^{\beta(x,y)-1}\psi\left(x,y\right)\,dx\,dy$$
$$+\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\int_{\frac{\rho_1+\rho_2}{2}}^{\rho_2}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\beta(x,y)-1}\left(y-\frac{\rho_1+\rho_2}{2}\right)^{\alpha(x,y)-1}\psi\left(x,y\right)\,dx\,dy\Bigg]$$

$$\leq \frac{2^{\alpha(x,y)-1}\alpha\left(x,y\right)}{\left(\rho_2-\rho_1\right)^{\alpha(x,y)}}\left[\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\alpha(x,y)-1}\psi\left(y,\kappa_1\right)\,dy\right.$$
$$\left.+\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\alpha(x,y)-1}\psi\left(y,\kappa_2\right)\,dy\right]. \tag{34}$$

By adding inequalities (28), (29), (32), and (33), and then dividing by 2, we obtain

$$\begin{aligned}
&\frac{2^{\beta(x,y)-2}\Gamma\left[\beta\left(x,y\right)+1\right]}{\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}\\
&\times\left[J_{\kappa_1^+}^{\beta(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J_{\kappa_2^-}^{\beta(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right]\\
&+\frac{2^{\alpha(x,y)-2}\Gamma\left[\alpha\left(x,y\right)+1\right]}{\left(\rho_2-\rho_1\right)^{\alpha(x,y)}}\\
&\times\left[J_{\rho_1^+}^{\alpha(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J_{\rho_2^-}^{\alpha(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right]\\
&\leq\frac{2^{\alpha(x,y)-1}2^{\beta(x,y)-1}\Gamma\left[\alpha\left(x,y\right)\right]\Gamma\left[\beta\left(x,y\right)\right]}{\left(\kappa_2-\kappa_1\right)^{\alpha(x,y)}\left(\rho_2-\rho_1\right)^{\beta(x,y)}}\\
&\left[J_{\kappa_1^+,\rho_1^+}^{\alpha(x,y),\beta(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J_{\kappa_1^+,\rho_2^-}^{\alpha(x,y),\beta(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right.\\
&\left.+J_{\kappa_2^-,\rho_1^+}^{\alpha(x,y),\beta(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)+J_{\kappa_2^-,\rho_2^-}^{\alpha(x,y),\beta(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\right]\\
&\leq\frac{2^{\alpha(x,y)-3}\Gamma\left[\alpha\left(x,y\right)+1\right]}{\left(\kappa_2-\kappa_1\right)^{\alpha(x,y)}}\\
&\left[J_{\kappa_1^+}^{\alpha(x)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\rho_1\right)+J_{\kappa_1^+}^{\alpha(x)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\rho_2\right)+J_{\kappa_2^-}^{\alpha(x)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\rho_1\right)\right.\\
&\left.+J_{\kappa_2^-}^{\alpha(x)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\rho_2\right)\right]\\
&+\frac{2^{\beta(x,y)-3}\Gamma\left[\beta\left(x,y\right)+1\right]}{\left(\rho_2-\rho_1\right)^{\beta(x,y)}}\\
&\left[J_{\rho_1^+}^{\beta(x)}\psi\left(\kappa_1,\frac{\rho_1+\rho_2}{2}\right)+J_{\rho_1^+}^{\beta(x)}\psi\left(\kappa_2,\frac{\rho_1+\rho_2}{2}\right)+J_{\rho_2^-}^{\beta(x)}\psi\left(\kappa_1,\frac{\rho_1+\rho_2}{2}\right)\right.\\
&\left.+J_{\rho_2^-}^{\beta(x)}\psi\left(\kappa_2,\frac{\rho_1+\rho_2}{2}\right)\right].
\end{aligned}$$

This yields the second and third inequalities in (25). Now, substituting $x=\frac{\kappa_1+\kappa_2}{2}$ into the left-hand side of inequality (27), we obtain:

$$\begin{aligned}
&\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\\
&\leq\frac{2^{\beta(x,y)-1}\beta\left(x,y\right)}{\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\psi\left(x,\frac{\rho_1+\rho_2}{2}\right)dx
\end{aligned}$$

$$+\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\beta(x,y)-1}\psi\left(x,\frac{\rho_1+\rho_2}{2}\right)dx. \tag{35}$$

Next, by substituting $y=\frac{\rho_1+\rho_2}{2}$ into the left-hand side of inequality (31), we obtain:

$$\begin{aligned}&\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\\&\leq\frac{2^{\alpha(x,y)-1}\alpha\left(x,y\right)}{\left(\kappa_2-\kappa_1\right)^{\alpha(x,y)}}\int_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}\left(\frac{\rho_1+\rho_2}{2}-y\right)^{\alpha(x,y)-1}\psi\left(y,\frac{\kappa_1+\kappa_2}{2}\right)dy\\&+\int_{\frac{\rho_1+\rho_2}{2}}^{\rho_2}\left(y-\frac{\rho_1+\rho_2}{2}\right)^{\alpha(x,y)-1}\psi\left(y,\frac{\kappa_1+\kappa_2}{2}\right)dy.\end{aligned} \tag{36}$$

By adding inequalities (35) and (36) side by side and then dividing by 2, we obtain the following inequality:

$$\begin{aligned}&\psi\left(\frac{\kappa_1+\kappa_2}{2},\frac{\rho_1+\rho_2}{2}\right)\\&\leq\frac{2^{\beta(x,y)-2}\Gamma\left[\beta\left(x,y\right)+1\right]}{\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}\left[J_{\kappa_1^+}^{\beta(x,y)}\psi\left(\kappa_2,\frac{\rho_1+\rho_2}{2}\right)+J_{\kappa_2^-}^{\beta(x,y)}\psi\left(\kappa_1,\frac{\rho_1+\rho_2}{2}\right)\right]\\&+\frac{2^{\alpha(x,y)-2}\Gamma\left[\alpha\left(x,y\right)+1\right]}{\left(\rho_2-\rho_1\right)^{\alpha(x,y)}}\left[J_{\kappa_1^+}^{\alpha(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\rho_2\right)+J_{\kappa_2^-}^{\alpha(x,y)}\psi\left(\frac{\kappa_1+\kappa_2}{2},\rho_1\right)\right].\end{aligned} \tag{37}$$

The inequality in (37) corresponds to the first inequality in Theorem 5.

Finally, by utilizing the second inequality in (7) and substituting $y=\rho_1$ into the right-hand side of inequality (31), we obtain:

$$\begin{aligned}&\frac{2^{\beta(x,y)-1}\beta}{\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}\left[\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\psi\left(x,\rho_1\right)\,dx\right.\\&\left.+\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\beta(x,y)-1}\psi\left(x,\rho_1\right)\,dx\right]\\&\leq\frac{\psi\left(\kappa_1,\rho_1\right)+\psi\left(\kappa_2,\rho_1\right)}{2}.\end{aligned} \tag{38}$$

And similarly, by assuming, $y=\rho_2$

$$\begin{aligned}&\frac{2^{\beta(x,y)-1}\beta}{\left(\kappa_2-\kappa_1\right)^{\beta(x,y)}}\left[\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\kappa_1+\kappa_2}{2}-x\right)^{\beta(x,y)-1}\psi\left(x,\rho_2\right)\,dx\right.\\&\left.+\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_1}\left(x-\frac{\kappa_1+\kappa_2}{2}\right)^{\beta(x,y)-1}\psi\left(x,\rho_2\right)\,dx\right]\\&\leq\frac{\psi\left(\kappa_1,\rho_2\right)+\psi\left(\kappa_2,\rho_2\right)}{2}.\end{aligned} \tag{39}$$

Likewise, by assuming $x = \kappa_1$ on the right-hand side of inequality (27), we obtain:

$$\begin{aligned} &\frac{2^{\alpha(x,y)-1}\alpha}{(\rho_2-\rho_1)^{\alpha(x,y)}}\left[\int_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}\left(\frac{\rho_1+\rho_2}{2}-y\right)^{\alpha(x,y)-1}\psi(\kappa_1,y)\,dy\right. \\ &\left.+\int_{\frac{\rho_1+\rho_2}{2}}^{\rho_2}\left(y-\frac{\rho_1+\rho_2}{2}\right)^{\alpha(x,y)-1}\psi(\kappa_1,y)\,dy\right] \\ &\leq \frac{\psi(\kappa_1,\rho_1)+\psi(\kappa_1,\rho_2)}{2} \end{aligned} \tag{40}$$

and by assuming $x = \kappa_2$ on the right-hand side of inequality (27), we obtain:

$$\begin{aligned} &\frac{2^{\alpha(x,y)-1}\alpha}{(\rho_2-\rho_1)^{\alpha(x,y)}}\left[\int_{\rho_1}^{\frac{\rho_1+\rho_2}{2}}\left(\frac{\rho_1+\rho_2}{2}-y\right)^{\alpha(x,y)-1}\psi(\kappa_2,y)\,dy\right. \\ &\left.+\int_{\frac{\rho_1+\rho_2}{2}}^{\rho_2}\left(y-\frac{\rho_1+\rho_2}{2}\right)^{\alpha(x,y)-1}\psi(\kappa_1,y)\,dy\right] \\ &\leq \frac{\psi(\kappa_2,\rho_1)+\psi(\kappa_2,\rho_2)}{2}. \end{aligned} \tag{41}$$

When the inequalities from (38) to (41) are added side by side and then divided by 4, the fourth inequality of Theorem 5 is obtained. □

Remark 2. In Theorem 5 if we choose $\alpha(x,y) = 1$ and $\beta(x,y) = 1$, then the inequalities (25) become the inequalities (4).

3. Conclusion

In this study, we established new Hermite-Hadamard-type inequalities for coordinated convex functions using multivariate variable-order Riemann-Liouville fractional integrals. By incorporating variable-order operators, we provided a more flexible and generalized framework that extends classical fractional inequalities to systems with dynamically changing fractional orders. Future research may explore new versions of the inequalities derived in this work by utilizing different convexity definitions in various coordinate systems. Additionally, novel and more refined inequalities can be obtained by employing alternative formulations of variable-order fractional integrals, further expanding the applicability of fractional calculus in mathematical analysis and applied sciences.

4. Acknowledgements

This study is produced from the findings of the Ph.D. thesis of Abdüllatif Yalçın.

References

[1] Ahmad, B., Alsaedi, A., Ntouyas, S.K., Lotaibi, F.M. A coupled Hilfer-Hadamard fractional differential system with non-local fractional integral boundary conditions, *TWMS J. Pure Appl. Math.,* Vol.15, No.1, 2024, pp.95-114.

[2] Agarwal, G., Yadav, L.K., Nisar, K.S., Alqarni, M.M., Mahmoud, E.E. A Hybrid method for the analytical solution of time fractional Whitham-Broer-Kaup equations, *Appl. Comput. Math.,* Vol.23, No.1, 2024, pp.3-17.

[3] Aliev, F.A., Aliev, N., Hajiyeva, N., Safarova, N., Aliyeva, R. Asymptotic method for solution of oscillatory fractional derivative, *Comput. Meth. Diff. Equat.,* Vol.10, No.4, 2022, pp.1123-1130.

[4] Aliev, F.A., Aliev, N. A., Safarova, N.A., Gasimova, K,G. Velieva, N. I. Solution of linear fractional-derivative ordinary differential equations with constant matrix coefficients, *Appl. Comput. Math.,* Vol.17, No.2, 2018, pp.317-322.

[5] Aliev, F.A., Aliev, N.A., Safarova, N.A., Mamedova, Y.V. Solution of the problem of analytical construction of optimal regulators for a fractional order oscillatory system in the general case, *J . Appl. Comput. Mech.,* Vol.7, No.2, 2020, pp.970-976.

[6] Aliev, F.A., Aliev, N.A., Safarova, N.A. Transformation of the Mittag-Leffler function to an exponential function and some of its applications to problems with a fractional derivative, *Appl. Comput. Math.,* Vol.18, No.3, 2019, pp.316-325.

[7] Aliev, F.A., Aliev, N.A., Velieva, N.I., Safarova, N.A. Larin parameterization to solve the problem of analytical construction of the optimal regulator of oscillatory systems with liquid dampers, *J. Appl. Comput. Mech.,* Vol.6, SI, 2020, pp. 1426-1430.

[8] Alomari, M., Darus, M. On the Hadamard's inequality for log-convex functions on the coordinates, *J. Inequal. Appl.*, Vol.2009, 2009, pp.1-13.

[9] Alp, N., Sarikaya, M.Z. Quantum Hermite-Hadamard's type inequalities for co-ordinated convex functions, *Appl. Math. E-Notes*, Vol.20, 2020, pp.341-356.

[10] Alqudah, M.A., Kashuri, A., Mohammed, P.O., Abdeljawad, T., Raees, M., Anwar, M., Hamed, Y.S. Hermite-Hadamard integral inequalities on coordinated convex functions in quantum calculus, *Adv. Diff. Equ.*, Vol.2021, No.1, 2021, Article number 264.

[11] Alshehri, M.G.A., Hyder, A., Budak, H., Barakat, M.A. Some new improvements for fractional Hermite-Hadamard inequalities by Jensen-Mercer inequalities, *J. Function Spaces,* Vol.2024, No.1, 2024, Article number 6691058.

[12] Anastassiou, G.A. Ostrowski type inequalities, *Proc. Am. Math. Soc.*, Vol.123, No.12, 1995, pp.3775-3781.

[13] Aydin, M., Mahmudov, N. Qualitative analysis of the Prabhakar-Caputo type fractional delayed equations, *TWMS J. Pure Appl. Math.,* Vol.16, No.1, 2025, pp.115-136.

[14] Bainov, D.D., Simeonov, P.S. *Integral Inequalities and Applications*, Vol.57, Springer Science & Business Media, 2013, 245 p.

[15] Butt, S.I., Inam, H., Dokuyucu, M.A. New fractal simpson estimates for twice local differentiable generalized convex mappings, *Appl. Comput. Math.,* Vol.23, No.4, 2024, pp.474-503.

[16] Caputo, M. Fabrizio, M. A new definition of fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.*, Vol.1, No.2, 2015, pp.73-85.

[17] Çelik, B., Akdemir, A.O., Set, E., Aslan, S. Ostrowski-Mercer type integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators, *TWMS J. Pure Appl. Math.,* Vol.15, No.2, 2024, pp.269-285.

[18] Coimbra, C.F.M. Mechanics with variable-order differential operators, *Ann. Der Phys.*, Vol.12, No.11-12, 2003, pp.692-703.

[19] Dragomir, S.S. On Hadamard's inequality for convex functions on the coordinates in the rectangle from the plane, *Taiwanese J. Math.*, Vol.4, No.5, 2001, pp.775-788.

[20] Dragomir, S.S., Agarwal, R.P., Cerone, P. On Simpson's inequality and applications, *RGMIA Res. Rep. Collect.*, Vol.2, No.3, 1999.

[21] Ekinci, A., Dragomir, S.S. New estimations for exponentially convexity via conformable fractional operators, *Fractional Order Analysis: Theory, Methods and Applications*, 2020, pp.113-131.

[22] Goodrich, C.S., Ragusa, M.A., Scapellato, A. Partial regularity of solutions to p(x)-laplacian PDEs with discontinuous coefficients, *J. Differ. Equ.,* Vol.268, No.9, 2020, pp.5440-5468.

[23] Gül, E., Akdemir, A.O., Yalçın, A. Hermite-Hadamard inequalities for a new class of generalized-(s,m) via fractional integral, *Turk. J. Earth Sci.*, Vol.1, 2024, pp.109-115.

[24] Gül, E., Yalçın, A. Some novel estimations of Hadamard type inequalities for different kinds of convex functions via tempered fractional integral operator, *Filomat*, Vol.38, No.10, 2024, pp.3683-3690.

[25] Gül, E., Yalçın, A. Some integral inequalities through tempered fractional integral operator, *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.*, Vol.73, No.2, 2024, pp.399-409.

[26] Haider, W., Budak, H., Shehzadi, A., Hezenci, F., Chen, H. Advancements in Hermite-Hadamard inequalities via conformable fractional integrals for subadditive functions, *Int. J. Geom. Methods Mod. Phys.*, 2025, 2550099.
[27] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*, Vol.204, Elsevier, 2006.
[28] Kiris, M.E., Bayrak, G. New version of Hermite-Hadamard inequality for coordinated convex function via generalized conformable integrals, *Filomat*, Vol.38, No.16, 2024, pp.5575-5589.
[29] Korbel, J., Luchko, Y. Modeling of financial processes with a space-time fractional diffusion equation of varying order, *Fract. Calc. Appl. Anal.*, Vol.19, No.6, 2016, pp.1509-1551.
[30] Latif, M.A., Alomari, M. Hadamard-type inequalities for product two convex functions on the co-ordinates, *Int. Math. Forum,* Vol.4, No.47, 2009, pp.2327-2338.
[31] Latif, M.A., Hussain, S., Dragomir, S.S. On some new Fejer-type inequalities for coordinated convex functions, *TJMM*, Vol.3, No.2, 2011, pp.57-80.
[32] Lorenzo, C.F., Hartley, T.T. Variable order and distributed order fractional operators, *Nonlinear Dyn.*, Vol.29, No.1, 2002, pp.57-98.
[33] Lorenzo, C.F., Hartley, T.T. Initialization, conceptualization, and application in the generalized fractional calculus, *Crit. Rev. Biomed. Eng.*, Vol.35, No.6, 2007, pp.477-553.
[34] Mateen, A., Zhang, Z., Özcan, S., Ali, M.A. Generalization of Hermite-Hadamard, trapezoid, and midpoint Mercer type inequalities for fractional integrals in multiplicative calculus, *Bound. Value Probl.*, Vol.2025, No.1, 2025, pp.1-30.
[35] Meerschaert, M.M., Tadjeran, C. Finite difference approximations for two-sided space-fractional partial differential equations, *Appl. Numer. Math.*, Vol.51, No.1, 2006, pp.80-90.
[36] Moghaddam, B.P., Machado, J. A. T., Behforooz, H. An integro-quadratic spline approach for a class of variable-order fractional initial value problems, *Chaos, Solitons & Fractals*, Vol.102, 2017, pp.354-360.
[37] Muthukumar, P., Balasubramaniam, P., Ratnavelu, K. Fast projective synchronization of fractional order chaotic and reverse chaotic systems with its application to an affine cipher using date of birth (DOB), *Nonlinear Dyn.*, Vol.80, No.4, 2015, pp.1883-1897.
[38] Özdemir, M.E., Akdemir, A.O., Ekinci, A. New integral inequalities for co-ordinated convex functions, *Fundam. Contempl. Math. Sci.*, Vol.2, No.1, 2021, pp.52-69.
[39] Özdemir, M.E., Yıldız, C., Akdemir, A.O. On the coordinated convex functions, *International Conference On Applied Analysis and Algebra*, Istanbul, Turkey, Vol.8, 2011, pp.1058-1091.
[40] Paskas, M.P. Improved image denoising through fractional anisotropic diffusion and resolution-tailored differentiation in the Fourier domain, *Phys. D: Nonlinear Phenom.*, Vol.456, 2024, 133257.
[41] Patnaik, S., Hollkamp, J.P., Semperlotti, F. Applications of variable-order fractional operators: A review, *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, Vol.476, No.2234, 2020, 20190498.
[42] Pecaric, J., Proschan, F., Tong, Y.L. *Convex Functions, Partial Orderings and Statistical Applications,* Academic Press, Inc., Boston, 1992, 469 p.
[43] Pinheiro Neto, J., Moura Coelho, R., Valerio, D., Vinga, S., Sierociuk, D., Malesza, W., Macias, M., Dzielinski, A. Variable order differential models of bone remodelling, *IFAC Int. Fed. Autom. Control*, Vol.50, No.1, 2017, pp.8066-8071.
[44] Ramirez, L.E.S., Coimbra, C.F.M. A variable order constitutive relation for viscoelasticity, *Ann. Phys.*, Vol.16, No.7-8, 2007, pp.543-552.
[45] Roberts, A.W., Varberg, D.E. *Convex Functions,* Academic Press, 1973, 162 p.
[46] Samko, S.G. Fractional integration and differentiation of variable order, *Anal. Math.*, Vol.21, No.3, 1995, pp.213-236.
[47] Samko, S.G., Ross, B. Integration and differentiation to a variable fractional order, *Integr. Transf. Spec. Funct.*, Vol.1, No.4, 1993, pp.277-300.
[48] Sarikaya, M.Z. On Hermite-Hadamard type inequalities for coordinated convex function via fractional integrals, *Integral Transforms Spec. Funct.*, Vol.25, No.2, 2014, pp.134-147.
[49] Sarikaya, M.Z., Ertugral, F. On the generalized Hermite-Hadamard inequalities, *Ann. Univ. Craiova Math. Comput. Sci. Ser.*, Vol.47, No.1, 2020, pp.193-213.
[50] Set, E., Butt, S.I., Akdemir, A.O., Karaoglan, A., Abdeljawad, T. New integral inequalities for differentiable convex functions via Atangana-Baleanu fractional integral operators, *Chaos, Solitons & Fractals*, Vol.143, 2021, 110554.
[51] Set, E., Çelik, B., Özdemir, M.E., Aslan, M. Some new results on Hermite-Hadamard-Mercer-type inequalities using a general family of fractional integral operators, *Fractal Fract.*, Vol.5, No.3, 2021, 68.
[52] Sharif, A.A., Hamoud, A.A., Hamood, M.M., Ghadle, K.P. New results on Caputo fractional Volterra-Fredholm integro-differential equations with nonlocal conditions, *TWMS J. Apll. Eng. Math.,* Vol.15, No.2, 2025, pp.459-472.

[53] Soon, C.M., Coimbra, C.F.M., Kobayashi, M.H. The variable viscoelasticity oscillator, *Ann. Der. Phys.*, Vol.517, No.6, 2005, pp.378-389.
[54] Straka, P. Variable order fractional Fokker-Planck equations derived from continuous time random walks, *Phys. A*, Vol.503, 2018, pp.451-463.
[55] Sun, H.G., Chen, W., Wei, H., Chen, Y. A comparative study of constant-order and variable-order fractional models in characterizing memory property of systems, *Eur. Phys. J. Spec. Top.*, Vol.193, No.1, 2011, pp.185-192.
[56] Sweilam, N.H., Al-Mekhlafi, S.M., Abdel Kareem, W.S., Alqurishi, G. Comparative study of crossover mathematical model of breast cancer based on Ψ-caputo derivative and mittag-leffler laws: Numerical treatments, *Symmetry*, Vol.16, No.9, 2024, pp.1172.
[57] Tatar, M.C., Yildiz, C. Hermite-Hadamard-Jensen-Mercer type some new estimates for inequalities, *Turk. J. Sci.,* Vol.9, No.3, 2024, pp.225-240.
[58] Thomas, S., Nadupuri, S.K. A Crank-Nicolson type higher order compact exponential scheme for solving vairable order time fractional mobile-immobile Advection-Dispersion model, *TWMS J. Apl. Eng. Math.,* Vol.14, No.4, 2024, pp.1609-1621.
[59] Ullah, S.Z., Khan, M.A., Chu, Y.-M. A note on generalized convex functions, *J. Inequal. Appl.*, Vol.2019, 2019, pp.291.
[60] Wang, J., Feckan, M. *Fractional Hermite-Hadamard Inequalities,* Walter de Gruyter GmbH, Co KG, Vol.5, 2018, 387 p.
[61] Yalçın, A., Dokuyucu, M.A., Akdemir, A.O., Gül, E. Bridging convexity and fractional calculus: Hermite-Hadamard inequalities for variable order integrals, *arxiv.org*, 2025.
[62] Yavuz, M, Dassios, I. Significance of mathematical modeling and control in real-world problems: New developments and applications, *Math. Comput. Appl.*, Vol.29, No.5, 2024, 82.
[63] Zhang, H., Liu, F., Phanikumar, M.S., Meerschaert, M.M. A novel numerical method for the time variable fractional order mobile-immobile advection-dispersion model, *Comput. Math. Appl.*, Vol.66, No.5, 2013, pp.693-701.



Abdüllatif Yalçın - is Ph.D. student at the Department of the Mathematics of Yıldız Technical University, Istanbul, Türkiye. He obtained his M.Sc. degree in Mathematics from Ağrı Ibrahim Çeçen University Ağrı, Türkiye, in 2016. His research interests include convex functions, measure and integration, integral inequalities and fractional analysis.



Erdal Gül - is currently a full Professor at the Department of the Mathematics of Yıldız Technical University, Istanbul, Türkiye. He obtained his M.Sc. and Ph.D. degrees in Mathematics from Yıldız Technical University in 1992 and 1997, respectively. He has M.Sc. degree from International Centre For Theoretical Physics, Italy in postgraduate program in 1995. His research interests include functional analysis, operator theory, measure and integration, integral inequalities and fractional analysis.

Ahmet Ocak Akdemir - for a photograph and biography, see Applied and Computational Mathematics, V.21, N.3, 2022, p.348.