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The second regularized trace of differential operator equation with the semi-periodic boundary conditions

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In this paper, we have obtained the second regularized trace formula for the differential operator equation with the semi-periodic boundary conditions. Copyright © 2012 John Wiley & Sons, Ltd.

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1. Introduction

Let H be a separable Hilbert space. In the Hilbert space $H_1 = L_2([0,\pi],H)$, we consider the self-adjoint operator L generated by the differential expression

$$\ell(y) = -y''(x) + Q(x)y(x) \tag{1.1}$$

with the semi-periodic boundary conditions

$$y(0) = -y(\pi), \qquad y'(0) = -y'(\pi). \tag{1.2}$$

Suppose that the operator function $Q(x)$ in the expression $\ell(y)$ satisfies the following conditions:

(1) For $\forall x \in [0,\pi], Q(x) : H \to H$ is a self-adjoint nuclear operator. Moreover, $Q(x)$ has continuous derivative of second order with respect to the norm in space $\sigma_1(H)$ in the interval $[0,\pi]$ and for $x \in [0,\pi], Q^{(i)}(x) : H \to H$ are self-adjoint operators $(i = 1,2)$. Here $\sigma_1(H)$ is the space of nuclear operators from H to H as in [1].
(2) $\sup\limits_{[0,\pi]} \|Q(x)\|_H < 4$.
(3) There is an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of the space H such that

$$\sum_{n=1}^{\infty} \|Q(x)\varphi_n\|_{H_1} < \infty.$$

(4) $Q'(0) = Q'(\pi) = 0$.

Let L_0 be the operator generated by the differential expression $\ell_0(y) = -y''(x)$ with the boundary conditions (1.2). The spectrum of the operator L_0 is the set $\{(2m-1)^2\}_{m=1}^{\infty}$ and every point of this set is an eigenvalue of L_0 with infinite multiplicity. The orthonormal eigenfunctions corresponding to the eigenvalue $(2m-1)^2$ have the form

$$\sqrt{\frac{2}{\pi}}\cos(2m-1)x \cdot \varphi_n, \quad \sqrt{\frac{2}{\pi}}\sin(2m-1)x \cdot \varphi_n, \quad n = 1,2,\cdots.$$

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For convenience, we denote

$$\psi_{1n} = \sqrt{\frac{2}{\pi}}\cos x \cdot \varphi_n, \quad \psi_{2n} = \sqrt{\frac{2}{\pi}}\sin x \cdot \varphi_n,$$
$$\psi_{3n} = \sqrt{\frac{2}{\pi}}\cos 3x \cdot \varphi_n, \quad \psi_{4n} = \sqrt{\frac{2}{\pi}}\sin 3x \cdot \varphi_n, \cdots \tag{1.3}$$
$$\mu_1 = 1, \quad \mu_2 = 1, \quad \mu_3 = 9, \quad \mu_4 = 9, \cdots$$

Let R_λ^0 and R_λ be resolvents of the operators L_0 and L, respectively. The following theorem has been proved in [2].

Theorem 1.1 ([2, Theorem 1])
Spectrum of the operator L is the subset of the intervals

$$\Omega = [(2m-1)^2 - \|Q|_{H_1},\ (2m-1)^2 + \|Q\|_{H_1}] \quad m = 1, 2, \cdots$$

which are pairwise disjoint and

1. Every point different from $(2m-1)^2$, belonging to the interval Ω_m of the spectrum of the operator L is a discrete eigenvalue, whose multiplicity is finite.
2. $(2m-1)^2$ may be the eigenvalue, whose multiplicity is finite or infinite of the operator L.
3. The equality $\lim_{n\to\infty} \lambda_{mn} = (2m-1)^2$ holds. Here $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues, belonging to the interval Ω_m of the operator L and each eigenvalue has been repeated according to its multiplicity.

Moreover, the first regularized trace formula

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn} - 2(2m-1)^2) - \frac{2}{\pi}\int_0^{\pi} \mathrm{tr}Q(x)dx\right] = 0$$

has been obtained for the self-adjoint operator L in [2]. Note that the first regularized trace formulas for the problem (1.1) and (1.2) was founded in the scalar case in [3, 4].

In this paper, we will obtain the second regularized trace formula

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn}^2 - 2(2m-1)^4) - \frac{4(2m-1)^2}{\pi}\int_0^{\pi}\mathrm{tr}Q(x)dx - \frac{1}{\pi}\int_0^{\pi}\mathrm{tr}Q^2(x)dx - \frac{1}{\pi^2}\mathrm{tr}\left(\int_0^{\pi}Q(x)dx\right)^2\right] = 0$$

for the operator L.

The trace formulas for the scalar differential operators have been found by Gelfand and Levitan [5], Dikii [6, 7], Halberg and Kramer [3], and many others. The current situation of this subject and studies related to it are presented in this comprehensive survey papers [8, 9].

The trace formulas of the abstract self-adjoint operators with continuous spectrum were first analyzed by Krein [10]. In this work, he also proved the formula, which had been obtained earlier [11] through physical theories in quantum statistics and crystal theory, mathematically. The trace formulas related to the Sturm–Liouville problem with bounded self-adjoint operator given an infinite interval and having a continuous spectrum were considered in [12, 13]. The Faddev's study of the regularized trace formula for the Sturm–Liouville equation with the matrix coefficient in [14] has been predecessor for [12, 13].

Note that the trace formulas are used in the inverse problems of spectral analysis of differential equations (see, for example, [8]) and have applications in the approximate calculation of eigenvalues of the related operator [6, 7].

2. Some formulas about the operator QR_λ^0

The formula

$$\mathrm{tr}(R_\lambda - R_\lambda^0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn}-\lambda} - \frac{2}{(2m-1)^2-\lambda}\right)$$

is true for every λ, which belongs to the resolvent set of L (see [2]). By multiplying with $\frac{\lambda^2}{2\pi i}$ both sides of this equality and integrating over the circle $|\lambda| = b_p = (2p-1)^2 + 4p$, we obtain

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda^2\mathrm{tr}(R_\lambda - R_\lambda^0)d\lambda = \sum_{m=1}^{p}\sum_{n=1}^{\infty}(2(2m-1)^4 - \lambda_{mn}^2) \tag{2.1}$$

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From $R_\lambda - R_\lambda^0 = -R_\lambda Q R_\lambda^0$ and (2.1), we obtain

$$\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left(\lambda_{mn}^2 - 2(2m-1)^4\right) = \sum_{j=1}^{N} M_p^j + M_{pN} \tag{2.2}$$

Here, N is an integer,

$$M_p^j = \frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_p} \lambda^2 \operatorname{tr}[R_\lambda^0 (QR_\lambda^0)^j]d\lambda$$

and

$$M_{pN} = \frac{(-1)^N}{2\pi i}\int_{|\lambda|=b_p} \lambda^2 \operatorname{tr}[R_\lambda (QR_\lambda^0)^{N+1}]d\lambda. \tag{2.3}$$

The following formula can be proved similarly to Theorem 2 in [2]

$$M_p^j = \frac{(-1)^j}{\pi i j}\int_{|\lambda|=b_p} \lambda \operatorname{tr}(QR_\lambda^0)^j d\lambda. \tag{2.4}$$

From relations (1.3) and (2.4), we write

$$\begin{aligned} M_p^1 &= -\frac{1}{\pi i}\int_{|\lambda|=b_p} \lambda \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(QR_\lambda^0\psi_{mn},\ \psi_{mn})_{H_1}\, d\lambda \\ &= 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(Q\psi_{mn},\ \psi_{mn})_{H_1}\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-\mu_m}d\lambda \\ &= \frac{4}{\pi}\sum_{m=1}^{p}\sum_{n=1}^{\infty}(2m-1)^2\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_n)_H[\cos^2(2m-1)x + \sin^2(2m-1)x]\,dx \\ &= \frac{4}{\pi}\sum_{m=1}^{p}(2m-1)^2\int_0^{\pi}\operatorname{tr}Q(x)\,dx \end{aligned} \tag{2.5}$$

Now, we shall compute M_p^2. From (2.4), we write

$$M_p^2 = \frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\operatorname{tr}\left(QR_\lambda^0\right)^2 d\lambda = \frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\left(QR_\lambda^0\right)^2\psi_{mn},\ \psi_{mn}\right)_{H_1} d\lambda. \tag{2.6}$$

Moreover, we know that

$$QR_\lambda^0\psi_{mn} = \frac{Q\psi_{mn}}{\mu_m - \lambda}$$

and

$$\begin{aligned} \left(QR_\lambda^0\right)^2\psi_{mn} &= (\mu_m-\lambda)^{-1}QR_\lambda^0(Q\psi_{mn}) \\ &= (\mu_m-\lambda)^{-1}QR_\lambda^0\left\{\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}(\mu_r-\lambda)^{-1}(Q\psi_{mn},\psi_{rq})_{H_1}Q\psi_{rq}\right\} \end{aligned} \tag{2.7}$$

Substituting (2.7) in (2.6), we obtain

$$\begin{aligned} M_p^2 &= \frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{(Q\psi_{mn},\psi_{rq})_{H_1}(Q\psi_{rq},\psi_{mn})_{H_1}}{(\lambda-\mu_m)(\lambda-\mu_r)}\right]d\lambda \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}|(Q\psi_{mn},\psi_{rq})_{H_1}\|^2\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)}d\lambda \end{aligned}$$

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$$= \sum_{m=1}^{2p}\sum_{n=1}^{\infty}\sum_{r=1}^{2p}\sum_{q=1}^{\infty} |(Q\psi_{mn}, \psi_{rq})_{H_1}|^2 \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)}\lambda +$$

$$+ \sum_{m=1}^{2p}\sum_{n=1}^{\infty}\sum_{r=2p+1}^{\infty}\sum_{q=1}^{\infty} |(Q\psi_{mn}, \psi_{rq})_{H_1}|^2 \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)} d\lambda +$$

$$+ \sum_{m=2p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{2p}\sum_{q=1}^{\infty} |(Q\psi_{mn}, \psi_{rq})_{H_1}|^2 \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)} d\lambda$$

$$+ \sum_{m=2p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=2p+1}^{\infty}\sum_{q=1}^{\infty} |(Q\psi_{mn}, \psi_{rq})_{H_1}|^2 \frac{1}{2\pi i} \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)} d\lambda$$

$$= \sum_{m=1}^{2p}\sum_{n=1}^{\infty}\sum_{r=1}^{2p}\sum_{q=1}^{\infty} |(Q\psi_{mn}, \psi_{rq})_{H_1}|^2 +$$

$$+ 2\sum_{m=1}^{2p}\sum_{n=1}^{\infty}\sum_{r=2p+1}^{\infty}\sum_{q=1}^{\infty} |(Q\psi_{mn}, \psi_{rq})_{H_1}|^2 \frac{\mu_m}{\mu_m-\mu_r}$$

$$= \sum_{m=1}^{2p}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty} |(Q\psi_{mn}, \psi_{rq})_{H_1}|^2 -$$

$$- \sum_{m=1}^{2p}\sum_{n=1}^{\infty}\sum_{r=2p+1}^{\infty}\sum_{q=1}^{\infty} \left(1 + \frac{2\mu_m}{\mu_r-\mu_m}\right) |(Q\psi_{mn}, \psi_{rq})_{H_1}|^2$$

$$= \sum_{m=1}^{2p}\sum_{n=1}^{\infty} \|Q\psi_{mn}\|_{H_1}^2 - \sum_{m=1}^{2p}\sum_{n=1}^{\infty}\sum_{r=2p+1}^{\infty}\sum_{q=1}^{\infty} \frac{\mu_r+\mu_m}{\mu_r-\mu_m} |(Q\psi_{mn}, \psi_{rq})_{H_1}|^2$$

Here, if we take

$$\alpha_p = \alpha_p(n,q) = \sum_{m=1}^{2p}\sum_{r=2p+1}^{\infty} \frac{\mu_r+\mu_m}{\mu_r-\mu_m} |(Q\psi_{mn}, \psi_{rq})_{H_1}|^2$$

then we write

$$M_p^2 = \sum_{m=1}^{2p}\sum_{n=1}^{\infty} \|Q\psi_{mn}\|_{H_1}^2 - \sum_{n=1}^{\infty}\sum_{q=1}^{\infty} \alpha_p \tag{2.8}$$

$$\alpha_p = \alpha_{p1} + \alpha_{p2} + \alpha_{p3} \tag{2.9}$$

where

$$\alpha_{p1} = 2\pi^{-2} \sum_{m=1}^{p}\sum_{r=p+1}^{\infty} \frac{(2r-1)^2+(2m-1)^2}{(2r-1)^2-(2m-1)^2} \left\{ \left| \int_0^{\pi} (Q(x)\varphi_n, \varphi_q)_H \cos 2(m-r)x\, dx \right|^2 \right.$$

$$\left. + \left| \int_0^{\pi} (Q(x)\varphi_n, \varphi_q)_H \sin 2(m-r)x\, dx \right|^2 \right\}$$

$$\alpha_{p2} = 2\pi^{-2} \sum_{m=1}^{p}\sum_{r=p+1}^{\infty} \frac{(2r-1)^2+(2m-1)^2}{(2r-1)^2-(2m-1)^2} \left| \int_0^{\pi} (Q(x)\varphi_n, \varphi_q)_H \cos 2(m+r-1)x\, dx \right|^2$$

$$\alpha_{p3} = 2\pi^{-2} \sum_{m=1}^{p}\sum_{r=p+1}^{\infty} \frac{(2r-1)^2+(2m-1)^2}{(2r-1)^2-(2m-1)^2} \left| \int_0^{\pi} (Q(x)\varphi_n, \varphi_q)_H \sin 2(m+r-1)x\, dx \right|^2$$

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Now we shall find a formula for $\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p1}$. For the expression α_{p1}, we can write

$$\alpha_{p1}=2\pi^{-2}\sum_{i=1}^{\infty}\sum_{\substack{r-m=i\\ m\le p\\ r>p}}\left(1+\frac{2(2m-1)^2}{(2r-1)^2-(2m-1)^2}\right)\left\{\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\cos 2ix\,dx\right|^2\right.$$
$$\left.+\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\sin 2ix\,dx\right|^2\right\} \tag{2.10}$$

In case $i\le p$

$$\sum_{\substack{r-m=i\\ m\le p\\ r>p}}\frac{(2m-1)^2}{(2r-1)^2-(2m-1)^2}=\sum_{j=0}^{i-1}\frac{(2(p-j)-1)^2}{(2(p-j+i)-1)^2-(2(p-j)-1)^2}$$
$$=\sum_{j=0}^{i-1}\frac{(2(p-j)-1+i)^2-2(2(p-j)-1+i)i+i^2}{4i(2(p-j)-1+i)} \tag{2.11}$$
$$=\frac{2p-1}{4}+\frac{1-2i}{4}+\sum_{j=0}^{i-1}\frac{i}{4(2p-1-2j+i)}$$

For any integers p and i, we let

$$E=\{(r,m): r,m\in N;\ r-m=i;\ m\le p;\ r>p\}$$

Then using (2.11), we write

$$\sum_{m,r\in E}\left(1+\frac{2(2m-1)^2}{(2r-1)^2-(2m-1)^2}\right)=i+2\sum_{\substack{r-m=i\\ m\le p\\ r>p}}\frac{(2m-1)^2}{(2r-1)^2-(2m-1)^2}$$
$$=p+\frac{i}{2}\sum_{j=p-i+1}^{p}\frac{1}{2j-1+i},\quad (i\le p) \tag{2.12}$$

It is easy to see that

$$\frac{i}{2}\sum_{j=p-i+1}^{p}\frac{1}{2j-1+i}<\frac{i^2}{p}.$$

From here and (2.12), we obtain

$$\sum_{m,r\in E}\left(1+\frac{2(2m-1)^2}{(2r-1)^2-(2m-1)^2}\right)=p+i^2O(p^{-1}),\quad (i\le p) \tag{2.13}$$

Here, $O(p^{-1})$ depends on p and i, and it satisfies the inequality

$$|O(p^{-1})|<\text{const.}\,p^{-1}$$

In a similar form, we can show that

$$\sum_{m,r\in E}\left(1+\frac{2(2m-1)^2}{(2r-1)^2-(2m-1)^2}\right)=O(p) \tag{2.14}$$

for $i\ge p$. Here, $O(p)$ depends on p and i, and it satisfies the inequality

$$|O(p)|<\text{const.}\,p$$

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From (2.10), (2.13), and (2.14), we obtain

$$\begin{aligned}\alpha_{p1} &= 2\pi^{-2}p\sum_{i=1}^{\infty}\left\{\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\cos 2ix\,dx\right|^2\right.\\&\quad\left.+\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\sin 2ix\,dx\right|^2\right\}+\\&\quad+\sum_{i=1}^{p}i^2O(p^{-1})\left\{\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\cos 2ix\,dx\right|^2\right.\\&\quad\left.+\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\sin 2ix\,dx\right|^2\right\}+\\&\quad+\sum_{i=p+1}^{\infty}O(p)\left\{\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\cos 2ix\,dx\right|^2\right.\\&\quad\left.+\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\sin 2ix\,dx\right|^2\right\}\\&=\frac{p}{\pi}\int_0^{\pi}\left|(Q(x)\varphi_n,\ \varphi_q)_H\right|^2dx-\frac{p}{\pi^2}\int_0^{\pi}\left|(Q(x)\varphi_n,\ \varphi_q)_H\right|^2dx\\&\quad+\alpha_{p1}^{(1)}+\alpha_{p1}^{(2)}\end{aligned}\tag{2.15}$$

Here,

$$\begin{aligned}|\alpha_{p1}^{(1)}| &= \left|\sum_{i=1}^{p}i^2O(p^{-1})\left\{\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\sin 2ix\,dx\right|^2+\right.\right.\\&\quad\left.\left.+\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\sin 2ix\,dx\right|^2\right\}\right|\\&\le \text{const.}\,p^{-1}\int_0^{\pi}|(Q'(x)\varphi_n,\ \varphi_q)_H|^2dx\end{aligned}\tag{2.16}$$

and

$$\begin{aligned}|\alpha_{p1}^{(2)}| &= \left|\sum_{i=p+1}^{\infty}O(p)\left\{\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\sin 2ix\,dx\right|^2+\right.\right.\\&\quad\left.\left.+\left|\int_0^{\pi}(Q(x)\varphi_n,\ \varphi_q)_H\sin 2ix\,dx\right|^2\right\}\right|\\&\le \text{const.}\,p^{-1}\int_0^{\pi}|(Q'(x)\varphi_n,\ \varphi_q)_H|^2dx.\end{aligned}\tag{2.17}$$

To obtain these inequalities, we used conditions (1) and (4).
From (2.15)–(2.17), we obtain

$$\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p1}=\frac{p}{\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)dx-\frac{p}{\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)dx\right)^2+O(p^{-1})\tag{2.18}$$

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because $Q(x)$ satisfies the conditions (1) and (4), it can be shown that

$$\left|\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{pj}\right| < \text{const.}\, p^{-1}, \quad (j=2,3). \tag{2.19}$$

Now, let us compute the sum $\sum_{m=1}^{2p}\sum_{n=1}^{\infty}\|Q\psi_{mn}\|_{H_1}^2$:

$$\begin{aligned}\sum_{m=1}^{2p}\sum_{n=1}^{\infty}\|Q\psi_{mn}\|_{H_1}^2 &= \frac{2}{\pi}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\left[\int_0^{\pi}(Q^2(x)\varphi_n,\varphi_n)_H\cos^2(2m-1)x\,dx\right.\\ &\quad\left.+\int_0^{\pi}(Q^2(x)\varphi_n,\varphi_n)_H\sin^2(2m-1)x\,dx\right]\\ &= \frac{2p}{\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)dx.\end{aligned} \tag{2.20}$$

From (2.8), (2.9), and (2.18)–(2.20), we obtain

$$M_p^2 = \frac{p}{\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)dx + \frac{p}{\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)dx\right)^2 + O(p^{-1}). \tag{2.21}$$

3. The second regularized trace formula for the operator L

In this section, we will obtain the second regularized trace formula for the operator L. For this, we will first show that the formulas

$$\lim_{p\to\infty}M_p^j = 0, \quad j\geq 3, \tag{3.1}$$

$$\lim_{p\to\infty}M_{pN} = 0, \quad N\geq 6. \tag{3.2}$$

are satisfied.

The inequalities

$$\|QR_\lambda^0\|_{\sigma_1(H_1)} < C, \quad \|R_\lambda^0\|_{H_1} < Cp^{-1}, \quad \|R_\lambda\|_{H_1} < Cp^{-1}, \ (|\lambda| = b_p) \tag{3.3}$$

are true (see [2]). Here, $C>0$ is a constant. From (2.4), (3.3), and condition (2), we have

$$\begin{aligned}|M_p^j| &= \frac{1}{\pi j}\left|\int_{|\lambda|=b_p}\lambda\operatorname{tr}(QR_\lambda^0)^j d\lambda\right| \leq \frac{b_p}{2\pi j}\int_{|\lambda|=b_p}\|QR_\lambda^0\|_{\sigma_1(H_1)}^j|d\lambda|\\ &\leq \frac{b_p}{\pi j}\int_{|\lambda|=b_p}\|QR_\lambda^0\|_{\sigma_1(H_1)}\|(QR_\lambda^0)^{j-1}\|_{H_1}|d\lambda| \leq \frac{Cb_p}{\pi j}\int_{|\lambda|=b_p}\|Q\|_{H_1}^{j-1}\|R_\lambda^0\|_{H_1}^{j-1}|d\lambda|\\ &\leq \frac{C^j b_p}{\pi j}\int_{|\lambda|=b_p}4^{j-1}p^{1-j}|d\lambda| < C_1p^{5-j} \qquad (C_1>0)\end{aligned}$$

This gives

$$\lim_{p\to\infty}M_p^j = 0, \quad (j\geq 6),$$

but we claim that this is true also for $j=3,4,5$. Now, let us prove formula (3.1) for $j=3$.

$$\begin{aligned}M_p^3 = -\frac{1}{3\pi i}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{(\mu_m-\lambda)(\mu_r-\lambda)(\mu_k-\lambda)}\times\\ \times(Q\psi_{mn},\psi_{rq})_{H_1}\times(Q\psi_{rq},\psi_{kl})_{H_1}\times(Q\psi_{kl},\psi_{mn})_{H_1}\end{aligned} \tag{3.4}$$

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Let us take

$$F(m,r,k)=\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{l=1}^{\infty}\left(Q\sqrt{\frac{2}{\pi}}(\cos(2m-1)x+\sin(2m-1)x)\varphi_n,\ \sqrt{\frac{2}{\pi}}(\cos(2r-1)x+\sin(2r-1)x)\varphi_q\right)_{H_1}$$
$$\times\left(Q\sqrt{\frac{2}{\pi}}(\cos(2r-1)x+\sin(2r-1)x)\varphi_q,\sqrt{\frac{2}{\pi}}(\cos(2k-1)x+\sin(2k-1)x)\varphi_l\right)_{H_1}$$
$$\times\left(Q\sqrt{\frac{2}{\pi}}(\cos(2k-1)x+\sin(2k-1)x)\varphi_l,\sqrt{\frac{2}{\pi}}(\cos(2m-1)x+\sin(2m-1)x)\varphi_n\right)_{H_1}$$

Because $F(m,r,k)=F(r,m,k)=F(k,m,r)=F(k,r,m)=F(m,k,r)=F(r,k,m)$ and from (3.4), we obtain

$$\begin{aligned}M_p^3&=\frac{1}{\pi i}\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\sum_{k=p+1}^{\infty}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{(\lambda-(2m-1)^2)(\lambda-(2r-1)^2)(\lambda-(2k-1)^2)}F(m,r,k)\\&+\frac{1}{\pi i}\sum_{m=1}^{p}\sum_{\substack{r=1\\m\neq r}}^{p}\sum_{k=p+1}^{\infty}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{(\lambda-(2m-1)^2)(\lambda-(2r-1)^2)(\lambda-(2k-1)^2)}F(m,r,k)\\&+\frac{1}{\pi i}\sum_{m=1}^{p}\sum_{k=p+1}^{\infty}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{(\lambda-(2m-1)^2)(\lambda-(2k-1)^2)}F(m,m,k)\\&=2\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\sum_{k=p+1}^{\infty}\frac{(2m-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)}F(m,r,k)\\&+4\sum_{m=1}^{p}\sum_{\substack{r=1\\m\neq r}}^{p}\sum_{k=p+1}^{\infty}\frac{(2m-1)^2}{((2m-1)^2-(2r-1)^2)((2m-1)^2-(2k-1)^2)}F(m,r,k)\\&-2\sum_{m=1}^{p}\sum_{k=p+1}^{\infty}\frac{(2k-1)^2}{((2k-1)^2-(2m-1)^2)^2}F(m,m,k)\end{aligned}\tag{3.5}$$

Let

$$\begin{aligned}F_1(m,r,k)=8\pi^{-3}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{\pi}\left(Q(x)\varphi_n,\varphi_q\right)_H\cos 2(m-r)x\,dx\\\times\int_0^{\pi}\left(Q(x)\varphi_q,\varphi_l\right)_H\cos 2(r-k)x\,dx\\\times\int_0^{\pi}(Q(x)\varphi_l,\varphi_n)_H\cos 2(k-m)x\,dx,\end{aligned}\tag{3.6}$$

$$F_2(m,r,k)=F(m,r,k)-F_1(m,r,k),\tag{3.7}$$

$$A_{pi}=\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\sum_{k=p+1}^{\infty}\frac{(2m-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)}F_i(m,r,k),\tag{3.8}$$

$$B_{pi}=\sum_{m=1}^{p}\sum_{\substack{r=1\\m\neq r}}^{p}\sum_{k=p+1}^{\infty}\frac{(2m-1)^2}{((2m-1)^2-(2r-1)^2)((2m-1)^2-(2k-1)^2)}F_i(m,r,k),\tag{3.9}$$

$$C_{pi}=\sum_{m=1}^{p}\sum_{k=p+1}^{\infty}\frac{(2k-1)^2}{((2k-1)^2-(2m-1)^2)^2}F_i(m,m,k),\qquad(i=1,2)\tag{3.10}$$

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From (3.6), we obtain

$$F_1(m,r,k) = F_1(m,k,r), \qquad F_1(m,m,k) = F_1(m,k,k)$$

From here and (3.8), we have

$$A_{p1} = 2\sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>k}}^{\infty}\sum_{k=p+1}^{\infty}\frac{(2m-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)}F_1(m,r,k)+ \\ +\sum_{m=1}^{p}\sum_{k=p+1}^{\infty}\frac{(2m-1)^2}{((2k-1)^2-(2m-1)^2)}F_1(m,m,k), \tag{3.11}$$

In a similar form, we can show that

$$B_{pi} = -\sum_{m=1}^{p}\sum_{\substack{r=1\\ m<r}}^{p}\sum_{k=p+1}^{\infty}\frac{(2k-1)^2}{((2k-1)^2-(2m-1)^2)((2k-1)^2-(2r-1)^2)}F_i(m,r,k), \quad (i=1,2) \tag{3.12}$$

Let

$$A_p^1 = \sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>k}}^{\infty}\sum_{k=p+1}^{\infty}\frac{(2m-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)}F_1(m,r,k), \tag{3.13}$$

$$B_p^1 = -B_{p1} \tag{3.14}$$

$$C_p^1 = \sum_{m=1}^{p}\sum_{k=p+1}^{\infty}\frac{1}{(2k-1)^2-(2m-1)^2}F_1(m,m,k) \tag{3.15}$$

Hence, from (3.5)–(3.15), we obtain

$$M_p^3 = 4A_p^1 - 4B_p^1 - 2C_p^1 + 2A_{p2} + 4B_{p2} - 2C_{p2} \tag{3.16}$$

Let us find a formula for the A_p^1.

Let $E_1 = \{(m,r,k) : m,r,k \in N;\ r-m=i;\ k-m=j;\ m \le p;\ r,k > p\}$ where p, i and j are integers such that $p \ge j,\ i \ge j$. Then

$$A_p^1 = \sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>k\ ,}}^{\infty}\sum_{\substack{k=p+1\\ k-m\le p}}^{\infty}\frac{(2r-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)}F_1(m,r,k) \\ +\sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>k\ ,}}^{\infty}\sum_{\substack{k=p+1\\ k-m>p}}^{\infty}\frac{(2m-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)}F_1(m,r,k) \\ = 8\pi^{-3}\sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\Bigg[\sum_{m,r,k\in E_1}\frac{(2m-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)} \\ \times\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{\pi}\left(Q(x)\varphi_n,\varphi_q\right)_H\cos 2ix\,dx\int_0^{\pi}\left(Q(x)\varphi_q,\varphi_l\right)_H\cos 2(i-j)x\,dx \\ \times\int_0^{\pi}(Q(x)\varphi_l,\varphi_n)_H\cos 2jx\,dx\Bigg] \\ +\sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>k\ ,}}^{\infty}\sum_{\substack{k=p+1\\ k-m>p}}^{\infty}\frac{(2m-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)}F_1(m,r,k)$$

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If we let

$$\beta_{ij} = \frac{8}{\pi^3}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{\pi}\left(Q(x)\varphi_n,\varphi_q\right)_H\cos 2ix\,dx\int_0^{\pi}\left(Q(x)\varphi_q,\varphi_l\right)_H\cos 2(i-j)x\,dx \times\int_0^{\pi}(Q(x)\varphi_l,\varphi_n)_H\cos 2jx\,dx \tag{3.17}$$

$$A_p^{11} = \sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\left[\left(\sum_{m,r,k\in E_1}\frac{(2m-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)}\right)\beta_{ij}\right] \tag{3.18}$$

$$A_p^{12} = \sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>k}}^{\infty}\sum_{\substack{k=p+1\\ k-m>p}}^{\infty}\frac{(2m-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)}F_1(m,r,k)$$

then we write

$$A_p^1 = A_p^{11} + A_p^{12} \tag{3.19}$$

Similarly, to prove formula (2.13), we can show

$$\sum_{m,r,k\in E_1}\frac{(2m-1)^2}{((2r-1)^2-(2m-1)^2)((2k-1)^2-(2m-1)^2)} = \frac{1}{16i}+\frac{j}{p}O(1) \tag{3.20}$$

where $O(1)$ satisfies the condition

$$|O(1)| < \text{const.}$$

and it depends on p, i, and j. Moreover, if $Q(x)$ has a continuous derivative of second order at $[0,\pi]$ with respect to the norm in $\sigma_1(H)$, then we can show that

$$|\beta_{ij}| < \frac{\text{const.}}{i^2j^2} \tag{3.21}$$

From (3.18), (3.20), and (3.21), we obtain

$$A_p^{11} = \sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\left[\frac{\beta_{ij}}{16i}+\frac{O(1)}{pi^2j}\right]$$

Because

$$\left|\sum_{i=2}^{\infty}\sum_{j=1}^{p}\frac{O(1)}{pi^2j}\right| \le \text{const.}\,p^{-1}\left(\sum_{i=1}^{\infty}i^{-2}\right)\left(\sum_{j=1}^{p}j^{-1}\right) < \text{const.}\;p^{-1}\ln p,$$

we find

$$A_p^{11} = \sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\frac{\beta_{ij}}{16i}+o(1) \tag{3.22}$$

Here, $o(1)$ is an expression, which satisfies the condition

$$\lim_{p\to\infty}o(1) = 0$$

and it depends on p. From (3.19) and (3.22), we obtain

$$A_p^1 = \sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\frac{\beta_{ij}}{16i}+A_p^{12}+o(1) \tag{3.23}$$

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Now let us find a formula for the B_p^1:
Let

$$B_p^{11} = \sum_{\substack{m=1 \\ m<r}}^{p} \quad \sum_{\substack{r=1 \\ k-m\le p}}^{p} \quad \sum_{k=p+1}^{\infty} \frac{(2k-1)^2}{((2k-1)^2-(2m-1)^2)((2k-1)^2-(2r-1)^2)} F_1(m,r,k)$$

and

$$B_p^{12} = \sum_{\substack{m=1 \\ m<r}}^{p} \quad \sum_{\substack{r=1 \\ k-m>p}}^{p} \quad \sum_{k=p+1}^{\infty} \frac{(2k-1)^2}{((2k-1)^2-(2m-1)^2)((2k-1)^2-(2r-1)^2)} F_1(m,r,k)$$

Then from (3.12) and (3.14), we have

$$B_p^1 = B_p^{11} + B_p^{12} \tag{3.24}$$

By using (3.6) and (3.17), we write B_p^{11} as in the form

$$B_p^{11} = \sum_{\substack{j=2 \\ i>j}}^{p} \quad \sum_{i=1}^{p-1} \left(\sum_{m,r,k\in E_2} \frac{(2k-1)^2}{((2k-1)^2-(2m-1)^2)((2k-1)^2-(2r-1)^2)} \right) \beta_{ij} \tag{3.25}$$

where E_2 is a set defined by

$$E_2 = \{(m,r,k) : m,r,k \in N; r-m=i; k-m=j; m,r\le p; k>p\}$$

such that $i<j\le p$. Moreover, we can show that

$$\sum_{m,r,k\in E_2} \frac{(2k-1)^2}{((2k-1)^2-(2m-1)^2)((2k-1)^2-(2r-1)^2)} = \frac{1}{16j} + \frac{i}{p}O(1) \tag{3.26}$$

From (3.21), (3.25), and (3.26), we obtain

$$B_p^{11} = \sum_{\substack{j=2 \\ j>i}}^{p} \quad \sum_{i=1}^{p-1} \left[\frac{\beta_{ij}}{16j} + \frac{O(1)}{pj^2 i} \right]$$

and, because $\beta_{ij} = \beta_{ji}$, we write

$$B_p^{11} = \sum_{\substack{j=2 \\ i<j}}^{p} \quad \sum_{i=1}^{p-1} \frac{\beta_{ij}}{16j} + O(1)$$

From (3.24) and this last equation, we obtain

$$B_p^1 = \sum_{\substack{i=2 \\ i>j}}^{p} \quad \sum_{j=1}^{p-1} \frac{\beta_{ij}}{16i} + B_p^{12} + o(1) \tag{3.27}$$

By using (3.21), we have

$$\left| \sum_{i=2}^{p} \frac{B_{ip}}{i} \right| \le \sum_{i=2}^{p} \frac{1}{i^3 p^2} = o(1)$$

$$\left| \sum_{i=p+1}^{\infty} \sum_{j=1}^{p} \frac{\beta_{ij}}{i} \right| \le \sum_{i=p+1}^{\infty} \sum_{j=1}^{p} \frac{1}{i^3 j^2} = \left(\sum_{i=p+1}^{\infty} \frac{1}{i^3} \right) \left(\sum_{j=1}^{p} \frac{1}{j^2} \right) = o(1)$$

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From here and (3.27), we have

$$B_p^1 = \sum_{\substack{i=2 \\ i>j}}^{\infty} \sum_{j=1}^{p} \frac{\beta_{ij}}{16i} + B_p^{12} + o(1) \tag{3.28}$$

From (3.16), (3.23), and (3.28), we obtain

$$M_p^3 = 4A_p^{12} - 4B_p^{12} - 2C_p^1 + 2A_{p2} + 4B_{p2} - 2C_{p2} + o(1) \tag{3.29}$$

Here, it can be seen that

$$\lim_{p\to\infty} A_p^{12} = \lim_{p\to\infty} B_p^{12} = \lim_{p\to\infty} C_p^1 = \lim_{p\to\infty} A_{p2} = \lim_{p\to\infty} B_{p2} = \lim_{p\to\infty} C_{p2} = 0 \tag{3.30}$$

From (3.29) and (3.30), we obtain

$$\lim_{p\to\infty} M_p^3 = 0$$

In a similar form, it is proved that

$$\lim_{p\to\infty} M_p^4 = \lim_{p\to\infty} M_p^5 = 0$$

Now let us prove the formula (3.2); by employing (2.2) and (3.3), we obtain

$$\begin{aligned} |M_{pN}| &= \frac{1}{2\pi} \left| \int_{|\lambda|=b_p} \lambda^2 \operatorname{tr}\left[R_\lambda (QR_\lambda^0)^{N+1}\right] d\lambda \right| \\ &\le \frac{b_p^2}{2\pi} \int_{|\lambda|=b_p} \|R_\lambda (QR_\lambda^0)^{N+1}\|_{\sigma_1(H_1)} |d\lambda| \\ &\le b_p^2 \int_{|\lambda|=b_p} \|R_\lambda\|_{H_1} \|(QR_\lambda^0)^{N+1}\|_{\sigma_1(H_1)} |d\lambda| \\ &\le C b_p^2 p^{-1} \int_{|\lambda|=b_p} \|QR_\lambda^0\|_{H_1}^N \|(QR_\lambda^0)\|_{\sigma_1(H_1)} |d\lambda| \le \text{const.}\, p^{5-N} \end{aligned}$$

This shows that

$$\lim_{p\to\infty} M_{pN} = 0, \qquad N \ge 6.$$

By using the relations (2.2), (2.5), (2.21), (3.1), and (3.2), we find

$$\sum_{m=1}^{p} \sum_{n=1}^{\infty} (\lambda_{mn}^2 - 2(2m-1)^4) = \frac{4}{\pi} \sum_{m=1}^{p} (2m-1)^2 \int_0^\pi \operatorname{tr} Q(x)dx + \frac{p}{\pi} \int_0^\pi \operatorname{tr} Q^2(x)dx + \frac{p}{\pi^2} \operatorname{tr} \left(\int_0^\pi Q(x)dx \right)^2 + o(1)$$

From here, we get

$$\sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} (\lambda_{mn}^2 - 2(2m-1)^4) - \frac{4(2m-1)^2}{\pi} \int_0^\pi \operatorname{tr} Q(x)dx - \frac{1}{\pi} \int_0^\pi \operatorname{tr} Q^2(x)dx - \frac{1}{\pi^2} \operatorname{tr} \left(\int_0^\pi Q(x)dx \right)^2 \right] = 0 \tag{3.31}$$

The left-hand side of this equality is called the second regularized trace of the operator L. Thus, we have proven that the main result of this article is given by the following theorem.

Theorem 3.1

If the operator function $Q(x)$ satisfies the conditions (1)–(4), then the formula (3.31) holds for the second regularized trace of the operator L.

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