Q2) Evaluate $\int_0^{\infty} \frac{1}{(x+1)\sqrt{x}}dx$

SOLUTION:

To evaluate this integral, split it at a convenient point such that

$$\int_0^{\infty} \frac{1}{(x+1)\sqrt{x}}dx = \underbrace{\int_0^{c} \frac{1}{(x+1)\sqrt{x}}dx}_{\text{Type2}} + \underbrace{\int_c^{\infty} \frac{1}{(x+1)\sqrt{x}}dx}_{\text{Type1}} \text{ ,where } c>0.$$

Take $c=1$ for convenience:

$$\left(\int \frac{1}{(x+1)\sqrt{x}}\,dx = \underbrace{2\int \frac{du}{u^2+1}}_{\substack{u=\sqrt{x}\Rightarrow x=u^2 \\ dx=2udu}} = 2\arctan\sqrt{x} + C \right)$$

$$\int_0^{1} \frac{1}{(x+1)\sqrt{x}}dx = 2\lim_{k\to 0^+} \arctan\sqrt{x}\Big|_k^1 = 2\lim_{k\to 0^+}\left(\frac{\pi}{4} - \arctan\sqrt{k}\right) = \frac{\pi}{2}$$

$$\int_1^{\infty} \frac{1}{(x+1)\sqrt{x}}dx = 2\lim_{b\to \infty} \arctan\sqrt{x}\Big|_1^b = 2\lim_{b\to \infty}\left(\arctan\sqrt{b} - \frac{\pi}{4}\right) = \frac{\pi}{2}$$

So $\int_0^{\infty} \frac{1}{(x+1)\sqrt{x}}dx = \underbrace{\int_0^{1} \frac{1}{(x+1)\sqrt{x}}dx}_{\frac{\pi}{2}} + \underbrace{\int_1^{\infty} \frac{1}{(x+1)\sqrt{x}}dx}_{\frac{\pi}{2}} = \boxed{\pi}$