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On the stress field in a half-plane covered by the pre-stretched layer under the action of arbitrary linearly located time-harmonic forces

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Abstract

Within the framework of the piecewise homogeneous body model with the use of the Three-Dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies the stress field in a half-plane covered by the pre-stretched layer under the action of the inclined linearly located time-harmonic forces is investigated. The algorithm for obtaining numerical results is proposed. Concrete numerical results are presented for the pair of materials Aluminium (layer) + Steel (half-plane) as well as for Steel (layer) + Aluminium (half-plane). These numerical results are involved in the normal stress acting on the interface plane. The dependencies between this stress and the frequency of the external force and the influence of the pre-stretching of the covering layer on these dependencies is analysed.
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Keywords: Lamb's problem; Initial stress; Half-plane; Covered layer; Time-harmonic loading; Frequency

1. Introduction

An interesting and urgent elastodynamics problem, which cannot be solved within the framework of classical linear theory of elastic waves, is the elastodynamics problem for initially stressed bodies. These problems have a wide range of applications in almost all areas of natural sciences and engineering such as the mechanics of composites, seismology, biomechanics, etc. The intensive investigations in this field started approximately, in the second half of the 20th century and from that time to the present numerous theoretical and experimental investigations have been made. The review and their consideration of the early investigations are given by Biot [1], Eringen and Suhubi [2]. The systematic consideration and analyses of the results which were obtained before 1986 were made in monographs by Guz [3,4]. The review of the later researches is given in the papers

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by Guz [5], Akbarov and Guz [6], Akbarov and Ozisik [7,8], Ozisik and Akbarov [9], Chadwick [10], Hussain and Ogden [11], Rogerson and Sadiford [12] and many others. It follows from these reviews that almost all these investigations were made within the framework of the Three-dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies (TLTEWISB) and a considerable part of these refers to the wave propagation in the layered composite materials with homogeneous initial stresses.

It is evident that the investigations regarding the study of the influence of the initial strains (or stresses) on the dynamical stress field in the homogeneous and layered materials has also a great significance in the theoretical and practical sense. For investigation of such problems the solution to the Lamb's problem for the pre-strained homogeneous and layered (inhomogeneous) half-spaces has a fundamental significance. However, the number of the investigations regarding to this field is not enough. Here we consider briefly some of them which apply to the subject of the present paper.

In the paper by Kosman [13] the attempt was made for investigation of the Lamb's problem for the homogeneously pre-strained half-space. The Lamb's problem for the half-plane covered by the pre-stretched layer for the case where a harmonic lineal load applied perpendicularly to the free layer surface was studied by Akbarov and Ozaydin [14,15]. Particular numerical investigations were performed for cases where the material of the stratified layer was more rigid than that of the half-plane. In the paper by Guler and Akbarov [16] the investigations [14,15] are developed for the case where the elastic modulus of the pre-stretched cover layer is smaller than that of the half-plane. Note that the latter case is more complicated than the one considered in [14,15]. Because the case considered in [16] is suitable for considering the propagation of a generalized Rayleigh wave in an initially pre-strained covered half-plane which was studied in [7–9].

The study [14–16] is performed for the case where the time-harmonic lineal load is perpendicular to the face plane of the covered layer. Moreover, in [14–16] the numerical results are given for some selected discrete values of a frequency of the external forces.

In the present paper the investigations started in [14–16] are developed for the cases where on the free face plane of the covering layer the arbitrary inclined lineal located time-harmonic forces act. The algorithm for obtaining numerical results is also developed for the considered cases. The concrete numerical results are presented for the pair of materials Aluminium (layer) + Steel (half-plane) as well as for Steel (layer) + Aluminium (half-plane). These numerical results are involved in the normal stress acting on the interface plane. The analyses of these results are made in the view point of the new context, namely in the context of the analyses of the dependencies between this stress and the frequency of the external force and the influence of the pre-stretching of the covering layer on these dependencies. Note that such types of results and their analyses were not given in the above-mentioned papers by Akbarov and Ozaydin [14,15] and Guler and Akbarov [16]. In this sense, the present paper is the first attempt for the analyses of the mentioned type of dependencies.

Throughout the paper, repeated indices are summed over their ranges.

2. Formulation of the problem

Consider the half-plane covered by a pre-stretched layer (Fig. 1). With the covering layer we associate the Lagrangian coordinates $Ox_1x_2x_3$ which in the natural state coincide with the Cartesian coordinates. Note that the covering layer and half-plane occupy the regions $\{-\infty \leqslant x_1 \leqslant +\infty,\ -h \leqslant x_2 \leqslant 0,\ -\infty \leqslant x_3 \leqslant +\infty\}$ and $\{-\infty \leqslant x_1 \leqslant +\infty,\ -\infty \leqslant x_2 \leqslant -h,\ -\infty \leqslant x_3 \leqslant +\infty\}$ respectively. Assume that the free face plane of the layer is subjected to linearly located time-harmonic load extending to infinity in the x_3 direction, which is inclined to the $x_2 = 0$ plane.

Below we use the following notation: the values related to the layer and half-plane are denoted by upper indices (1) and (2) respectively and the values related to the initial stresses are denoted by upper indices (m), 0 where $m = 1, 2$.

We aim that the layer and half-plane before the compounding with each other be pre-strained separately along the Ox_1 axis and in each of them the following initial stress state appear.

$$\sigma_{11}^{(m),0} = \text{const}_m \neq 0, \quad m = 1, 2, \quad \sigma_{ij}^{(m),0} = 0 \quad \text{for } ij \neq 11. \tag{1}$$

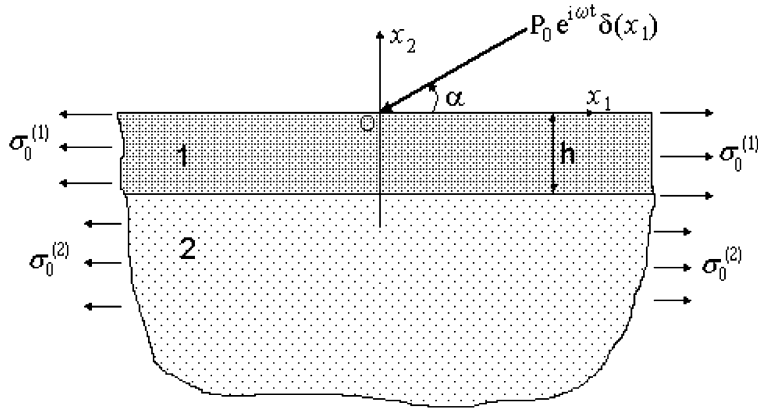


Fig. 1. The geometry of the structure of the half-plane covered by the layer.

The linear elastic material of the layer and the half-plane is assumed to be both homogeneous and isotropic and $\sigma_{11}^{(m),0}$ as the unique nonzero component of the initial stress tensor of both the layer and the half-plane; therefore, $\sigma_{11}^{(m),0}$ is denoted in short form as $\sigma_0^{(m)}$. Following on from the above, it can be assumed that the plane deformation state in the Ox_1x_2 plane prevails. According to Guz [3,4], for the considered case the equations TLTEWISB are

$$\frac{\partial \sigma_{ij}^{(m)}}{\partial x_j} + \sigma_0^{(m)} \frac{\partial^2 u_i^{(m)}}{\partial x_1^2} = \rho_0^{(m)} \frac{\partial^2 u_i^{(m)}}{\partial t^2}, \quad i;\ j = 1,2;\ \ m = 1,2. \tag{2}$$

In (2) by $\rho_0^{(m)}$ the density of the mth material in the natural state is denoted. The other notation in (2) is conventional.

For an isotropic compressible material we can write the following mechanical relations:

$$\sigma_{ij}^{(m)} = \lambda^{(m)} \theta^{(m)} \delta_{ij} + 2\mu^{(m)} \varepsilon_{ij}^{(m)}, \quad \theta^{(m)} = \varepsilon_{11}^{(m)} + \varepsilon_{22}^{(m)}, \tag{3}$$

where $\lambda^{(m)}$, $\mu^{(m)}$ are Lame's constants and

$$\varepsilon_{ij}^{(m)} = \frac{1}{2}\left(\frac{\partial u_i^{(m)}}{\partial x_j} + \frac{\partial u_j^{(m)}}{\partial x_i} \right). \tag{4}$$

We assume that the following complete contact conditions between the layer and half-plane are satisfied.

$$u_i^{(1)}\Big|_{x_2=-h} = u_i^{(2)}\Big|_{x_2=-h}, \quad \sigma_{i2}^{(1)}\Big|_{x_2=-h} = \sigma_{i2}^{(2)}\Big|_{x_2=-h}, \quad i = 1,2. \tag{5}$$

According to Fig. 1, in the free face plane of the covered layer we can write the following conditions:

$$\sigma_{12}^{(1)}\Big|_{\chi_2=0} = -P_0 \delta(x_1) e^{i\omega t} \cos\alpha, \quad \sigma_{22}^{(1)}\Big|_{\chi_2=0} = -P_0 \delta(x_1) e^{i\omega t} \sin\alpha, \tag{6}$$

where $\delta(x_1)$ is the Dirac delta function. In addition to the above-written conditions, we also assume that as $x_2 \to -\infty$ there is no reflection, which means that all waves travel in the negative x_2 direction. In other words, we assume that

$$|u_i^{(2)}|, |\sigma_{ij}^{(2)}| < M = \text{const}, \text{ as } x_2 \to -\infty. \tag{7}$$

Note that Eqs. (2) and (4) and the conditions (5) and (6) are obtained from the corresponding exact geometrically nonlinear equations by employing the linearization procedure. In this case it is assumed that the final deformed state of the material body can be reached by the composition of two deformations: the initial

static deformation characterized by expressions (1) and a superimposed infinitesimal deformation characterized by $\sigma_{ij}^{(m)}, \varepsilon_{ij}^{(m)}$ and $u_i^{(m)}$. Afterwards, the mentioned nonlinear equations are linearized with respect to the superimposed infinitesimal deformation field.

3. Method of solution

As a matter of fact, the formulated problem is a time-harmonic Lamb's problem for a pre-strained half-plane covered with a pre-strained layer. The Lamb's problem, a fundamental problem in the analysis of elastic waves generated by external forces in solid bodies bounded by plane surfaces, has been studied since Lamb [17]. The reference of these investigations are given in the papers by Tadeu et al. [18], Wang and Achenbach [19] and others. However, the mentioned investigations have been carried out within the framework of the classical linear theory of elastic waves for the homogeneous, isotropic or anisotropic half-spaces and main attention was addressed to the construction of the integral expressions for sought functions which are attractive for numerical applications and these expressions are examined for some numerical examples. Note that these investigations and the other ones listed therein have a great significance for the corresponding elastodynamics problems and are based on various types of integral transformations with respect to space and time variables. For time-harmonic excitations, the resulting multi-integrals are evaluated only in the far field by the methods of stationary phase.

For solution to the problem formulated in the present paper we will also use the integral (exponential Fourier) transformation with respect to space coordinate x_1 and will attempt to develop this method to study the elastodynamics problem for initially strained piecewice homogeneous body. In this case we will obtain not only the integral expressions for sought values, but also will attempt to develop the computer-aided algorithm for numerical investigations. Thus, now we return to the consideration of this method.

Substituting (3) and (4) in (2) we derive the following linearized equations of motion in terms of the displacement perturbations:

$$\begin{aligned}
&(\lambda^{(m)}+2\mu^{(m)}+\sigma_0^{(m)})\frac{\partial^2 u_1^{(m)}}{\partial x_1^2}+\mu^{(m)}\frac{\partial^2 u_1^{(m)}}{\partial x_2^2}+(\lambda^{(m)}+\mu^{(m)})\frac{\partial^2 u_2^{(m)}}{\partial x_1\partial x_2}=\rho_0^{(m)}\frac{\partial^2 u_1^{(m)}}{\partial t^2},\\
&(\lambda^{(m)}+\mu^{(m)})\frac{\partial^2 u_1^{(m)}}{\partial x_1\partial x_2}+(\mu^{(m)}+\sigma_0^{(m)})\frac{\partial^2 u_2^{(m)}}{\partial x_1^2}+(\lambda^{(m)}+2\mu^{(m)})\frac{\partial^2 u_2^{(m)}}{\partial x_2^2}=\rho_0^{(m)}\frac{\partial^2 u_2^{(m)}}{\partial t^2}.
\end{aligned} \tag{8}$$

According to Akbarov and Ozaydin [14,15], we attempt to present the displacement in terms of potentials $\varphi^{(m)}$ and $\psi^{(m)}$

$$u_1^{(m)}=\frac{\partial\varphi^{(m)}}{\partial x_1}+\frac{\partial\psi^{(m)}}{\partial x_2},\quad u_2^{(m)}=\frac{\partial\varphi^{(m)}}{\partial x_2}-\frac{\partial\psi^{(m)}}{\partial x_1}. \tag{9}$$

By direct verification we prove that the potentials must satisfy the following equations:

$$\begin{aligned}
&\nabla^2\varphi^{(m)}+\frac{\sigma_0^{(m)}}{\lambda^{(m)}+2\mu^{(m)}}\frac{\partial^2\varphi^{(m)}}{\partial x_1^2}=\frac{1}{(c_1^{(m)})^2}\frac{\partial^2\varphi^{(m)}}{\partial t^2},\\
&\nabla^2\psi^{(m)}+\frac{\sigma_0^{(m)}}{\mu^{(m)}}\frac{\partial^2\psi^{(m)}}{\partial x_1^2}=\frac{1}{(c_2^{(m)})^2}\frac{\partial^2\psi^{(m)}}{\partial t^2},
\end{aligned} \tag{10}$$

where

$$c_1^{(m)}=\sqrt{\frac{\lambda^{(m)}+2\mu^{(m)}}{\rho_0^{(m)}}},\quad c_2^{(m)}=\sqrt{\frac{\mu^{(m)}}{\rho_0^{(m)}}} \tag{11}$$

are the speeds of the dilatation and distortion waves, respectively in the absence of the initial stresses. It follows from (10) that under the condition $\sigma_0^{(m)}=0$, Eq. (10) coincide with the corresponding ones derived in the classical linear theory of elastodynamics [20].

Now we consider the solution to Eq. (10). As the external load is harmonic in time, only the stationary case is considered; all the dependent variables become harmonic and can be represented as:

$$\left\{u_n^{(m)}, \sigma_{nj}^{(m)}, \varepsilon_{nj}^{(m)}\right\}(x_1, x_2, t) = \left\{\bar{u}_n^{(m)}, \bar{\sigma}_{nj}^{(m)}, \bar{\varepsilon}_{nj}^{(m)}\right\}(x_1, x_2)\mathrm{e}^{\mathrm{i}\omega t}, \tag{12}$$

where a superimposed dash denotes the amplitude of the relevant quantity. Below we will omit this superimposed dash.

If (12) is employed in (2)–(10), by replacing $\frac{\partial^2 u_i^{(m)}}{\partial t^2}$ with $-\omega^2 u_i^{(m)}$ we obtain the same equations and conditions for the amplitude of the sought quantities. Consequently, introducing the dimensionless coordinates $x_i \to x_i/h$ and dimensionless frequency

$$\Omega = \frac{\omega h}{c_2^{(1)}} \tag{13}$$

we obtain the following equations for potentials $\varphi^{(m)}$ and $\psi^{(m)}$

$$\begin{aligned}
&\nabla^2\varphi^{(m)} + \frac{\sigma_0^{(m)}}{\lambda^{(m)} + 2\mu^{(m)}}\frac{\partial^2\varphi^{(m)}}{\partial x_1^2} + \frac{\left(c_2^{(1)}\right)^2}{\left(c_1^{(m)}\right)^2}\Omega^2\varphi^{(m)} = 0,\\
&\nabla^2\psi^{(m)} + \frac{\sigma_0^{(m)}}{\mu^{(m)}}\frac{\partial^2\psi^{(m)}}{\partial x_1^2} + \frac{\left(c_2^{(1)}\right)^2}{\left(c_2^{(m)}\right)^2}\Omega^2\psi^{(m)} = 0.
\end{aligned} \tag{14}$$

For the solution to Eq. (14) we employ the exponential Fourier transformation

$$f_{\mathrm{F}}(sx_2) = \int_{-\infty}^{+\infty} f(x_1, x_2)\mathrm{e}^{\mathrm{i}sx_1}\mathrm{d}x_1. \tag{15}$$

After applying (15) to Eq. (14) the functions $\varphi_{\mathrm{F}}^{(m)}$ and $\psi_{\mathrm{F}}^{(m)}$ are determined as follows:

$$\begin{aligned}
&\bar{\varphi}_{\mathrm{F}}^{(1)} = A_1^{(1)}(s)\mathrm{e}^{\gamma_1^{(1)}x_2} + A_2^{(1)}(s)\mathrm{e}^{-\gamma_1^{(1)}x_2}, \quad \bar{\varphi}_{\mathrm{F}}^{(2)} = A_1^{(2)}(s)\mathrm{e}^{\gamma_1^{(2)}x_2},\\
&\bar{\psi}_{\mathrm{F}}^{(1)} = B_1^{(1)}(s)\mathrm{e}^{\gamma_2^{(1)}x_2} + B_2^{(1)}(s)\mathrm{e}^{-\gamma_2^{(1)}x_2}, \quad \bar{\psi}_{\mathrm{F}}^{(2)} = B_1^{(2)}(s)\mathrm{e}^{\gamma_2^{(2)}x_2},
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
&\left(\gamma_1^{(m)}\right)^2 = s^2\left(1 + \frac{\sigma_0^{(m)}}{\lambda^{(m)} + 2\mu^{(m)}}\right) - \frac{\left(c_2^{(1)}\right)^2}{\left(c_1^{(m)}\right)^2}\Omega^2,\\
&\left(\gamma_2^{(m)}\right)^2 = s^2\left(1 + \frac{\sigma_0^{(m)}}{\mu^{(m)}}\right) - \frac{\left(c_2^{(1)}\right)^2}{\left(c_2^{(m)}\right)^2}\Omega^2.
\end{aligned} \tag{17}$$

The unknowns $A_1^{(1)}, A_2^{(1)}, A_1^{(2)}, B_1^{(1)}, B_2^{(1)}$ and $B_1^{(2)}$ which enter the expressions (16) are determined from the contact conditions (5) and boundary conditions (6). By the usual mathematical procedure we also determine the expressions for $u_{n\mathrm{F}}^{(m)}, \varepsilon_{nj\mathrm{F}}^{(m)}$ and $\sigma_{nj\mathrm{F}}^{(m)}$ from which the original of the sought values are presented as

$$\left\{u_n^{(m)}, \sigma_{nj}^{(m)}, \varepsilon_{nj}^{(m)}\right\}(x_1, x_2) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{u_{n\mathrm{F}}^{(m)}, \sigma_{nj\mathrm{F}}^{(m)}, \varepsilon_{nj\mathrm{F}}^{(m)}\right\}\mathrm{e}^{-\mathrm{i}sx_1}\mathrm{d}s. \tag{18}$$

The expressions of $u_{n\mathrm{F}}^{(m)}$, $\varepsilon_{nj\mathrm{F}}^{(m)}$ and $\sigma_{nj\mathrm{F}}^{(m)}$ are cumbersome, therefore we do not give them here.

For the determination of the stresses acting on the interface we calculate the integrals

$$\sigma_{n2}^{(m)}(x_1, -1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sigma_{n2\mathrm{F}}^{(m)}(s)|_{x_2=-1}\mathrm{e}^{-\mathrm{i}sx_1}\mathrm{d}s \tag{19}$$

numerically.

Using the notation

$$I^{(s)}_{n1\mathrm{F}} = \sigma_{n2\mathrm{F}}(s)|_{x_2=-1} + \sigma_{n2\mathrm{F}}(-s)|_{x_2=-1},$$
$$I^{(s)}_{n2\mathrm{F}} = \sigma_{n2\mathrm{F}}(s)|_{x_2=-1} - \sigma_{n2\mathrm{F}}(-s)|_{x_2=-1}, \tag{20}$$

the calculation of the integrals (19) is reduced to the calculation of the following ones

$$I_{n1}(x_1) = \frac{1}{2\pi}\int_0^\infty I_{n1\mathrm{F}}(s)\cos sx_1 \mathrm{d}s, \quad I_{n2}(x_1) = \frac{1}{2\pi}\int_0^\infty I_{n2\mathrm{F}}(s)\sin sx_1 \mathrm{d}s. \tag{21}$$

4. Some remarks on the calculation of the integrals (21)

The integrals (21) are the improved ones and under the calculation procedure they are replaced by the corresponding definite integrals, i.e. the relation used is $\int_0^\infty (.)\mathrm{d}s = \int_0^{S_*}(.)\mathrm{d}s$. The values of S_* are determined from the convergence criterion of the improved integrals.

Consider the determination of the singular points of the integrated expressions in (21). As the unknowns in (16) can be expressed as

$$\left(A_1^{(1)}(s); A_2^{(1)}(s); \ldots; B_2^{(2)}(s)\right) = \frac{1}{\det\left\|\alpha_{ij}(s)\right\|}\left(\det\left\|\beta_{ij}^{A_1^{(1)}(s)}\right\|; \det\left\|\beta_{ij}^{A_2^{(1)}(s)}\right\|; \ldots; \det\left\|\beta_{ij}^{B_2^{(2)}(s)}\right\|\right), \tag{22}$$

these singular points coincide with the roots of the equations

$$\det\|\alpha_{ij}(s)\| = 0, \quad (i,j = 1,2,3,4,5,6) \tag{23}$$

with respect to s, where α_{ij} are the coefficients of the unknowns in the algebraic equation system obtained from contact (5) and boundary (6) conditions for those. Note that, the expressions for $\left\|\beta_{ij}^{A_1^{(1)}(s)}\right\|$, $\left\|\beta_{ij}^{A_2^{(1)}(s)}\right\|, \ldots,$ $\left\|\beta_{ij}^{B_2^{(2)}(s)}\right\|$ are obtained from $\|\alpha_{ij}(s)\|$ by replacing the corresponding column of $\|\alpha_{ij}(s)\|$ with the right side of the above mentioned algebraic equation system.

Numerical analysis shows that the order of the roots of Eq. (23) with respect to s is one. Therefore, the order of all singular points of the integrated expressions in (21) is also one. Taking this situation into account for the solution to Eq. (23) with respect to s we employ the well-known bisection method.

Let us denote the roots of Eq. (23) through

$$s_1 < s_2 < \cdots < s_k < \cdots < s_N. \tag{24}$$

The number N in (24) depends on the values of the dimensionless frequency Ω, mechanical and geometrical parameters of the layer and half-plane.

Thus, after the determination of the roots (24) the integrating interval $[0,S_*]$ in (21) is separated into the following parts:

$$\int_0^{S_*}(\cdot)\mathrm{d}s = \left[\int_0^{s_1-\varepsilon}(.)\mathrm{d}s + \int_{s_1+\varepsilon}^{s_2-\varepsilon}(.)\mathrm{d}s + \cdots + \int_{s_N+\varepsilon}^{S_*}(\cdot)\mathrm{d}s\right]. \tag{25}$$

Consequently, the calculation of the integrals (21) is made in the Cauchy's principal value sense and in (25) ε is a very small value which is numerically determined from the convergence requirement of the integrals (21). Further, each of the integration intervals $[s_k+\varepsilon,\ s_{k+1}-\varepsilon]$ is also divided into a certain number of shorter intervals and for each of them the Gauss integration algorithm is employed. All these procedures are made automatically in PC by using the programmes written in MATLAB.

5. Numerical results and discussion

The following materials are selected for the considerations: Aluminium (shortly Al) with properties $\rho_0 = 2.7\ \mathrm{kg/m^3}$, $v = 0.35$, $c_1 = 6420\ \mathrm{m/s}$, $c_2 = 3110\ \mathrm{m/s}$; Steel (shortly St) with properties $\rho_0 = 7.86\ \mathrm{kg/m^3}$,

$v = 0.29$, $c_1 = 5890$ m/s, $c_2 = 3210$ m/s, where the density, Poisson's ratio, the speed of dilatation and distortion waves are denoted by ρ_0, v, c_1, and c_2 respectively. The numerical investigations will be carried out for the following two cases: in case I as the materials of the layer and half-plane the Al and St (denoted as Al + St) and in case II as the materials of the layer and half-plane the St and Al (denoted as St + Al) are selected.

Introduce the notation

$$\sigma_{22}^{(1)}(x_1, -h) = \sigma_{22}^{(2)}(x_1, -h) = \sigma_{22}(x_1) \tag{26}$$

and consider the numerical results obtained for the stress $\sigma_{22}(x_1)$.

The validity of the algorithm and programmes by the use of which the numerical investigations are carried out, is proven by the following statements:

- for the cases where the materials of the layer and half-plane are the same the values of σ_{22} approach the corresponding ones obtained for the same static loading (see [20]) under $\alpha = 0°$; 90° (6) as $\Omega \to 0$;
- in the case where $\alpha = 90°$ the results for the σ_{22} coincide with the corresponding ones obtained in the papers [14–16].

Assume that $\sigma_0^{(1)} > 0$, $\sigma_0^{(2)} = 0$ and introduce the parameter $\eta_1 = \sigma_0^{(1)}/\mu^{(1)}$ for the estimation of the initial stretching of the covered layer.

First we analyse the influence of the dimensionless frequency Ω on the values of $\sigma_{22}h/P_0$ in the characteristic points of the interface. Figs. 2 and 3 show the graphs of this dependence for the pairs Al + St and St + Al respectively. Note that in these figures the graphs separated by the letters a, b and c correspond to the cases where $\alpha = 90°$, 45° and 0° respectively. Moreover, the graphs separated by the letters a and b are constructed for the values of $\sigma_{22}h/P_0$ calculated at the point $x_1/h = 0$, but under construction of the graphs separated by the letter c the values of $\sigma_{22}h/P_0$ are calculated at the point $x_1/h = 0.7$.

It follows from these graphs that for both pairs the dependence between $\sigma_{22}h/P_0$ and Ω is nonmonotonic. There is such a value of Ω for which $\sigma_{22}h/P_0$ has its absolute maximum for the considered range of the change of Ω. We call these values of Ω as ''resonance'' values. The numerical results show that the ''resonance'' values of Ω depend on the selected pair, (i.e. on the mechanical properties of the layer and half-space materials) on the inclination angle (i.e. on α) and on the initial tension of the covered layer.

The nonmonotonic character of the dependence between σ_{22} and Ω agrees with the well-known results by Lamb [17], Robertson [21], Gladwell [22] and others, according to which, the behaviour of the half-space under forced vibration is similar to that of the forced vibration of the system which comprises a mass, a parallel connected spring and a dashpot. It follows from the graphs given in Figs. 2 and 3 that the behaviour of the forced vibration of the half-plane covered by the layer is also similar to that of the mentioned system. These results confirm again the validity and trustiness of the used algorithm and programmes constructed for PC. Moreover, the results show that by selection of the material of the covered layer the ''resonance'' values of Ω and the absolute maximum value of $\sigma_{22}h/P_0$ can be changed. As for the half-plane without the covered layer, the ''resonance'' value of $\Omega \approx 1.8$ and the absolute maximum of $\sigma_{22}h/P_0 \approx 0.6$. for $\alpha = 90°$, $x_2/h = 1$, $x_1/h = 0$ are obtained. Consequently, the covering of the half-plane by the layer the material stiffness of which is smaller (greater), such as Al + St (St + Al), than that for the half-plane material causes to increase (to decrease) significantly the ''resonance'' values of Ω and the absolute maximum values of $\sigma_{22}h/P_0$. In this case, according to the graphs in Figs. 2 and 3, the values of $\sigma_{22}h/P_0$ decrease monotonically with pre-stretching (i.e. with η) of the covering layer. Moreover, the noted decreasing becomes more significant for the values of $\sigma_{22}h/P_0$ attained for the ''resonance'' values of Ω.

As it can be foreseen for both pairs of the materials the values of $\sigma_{22}h/P_0$ decrease as $\alpha \to 0$.

Although the above discussed results are obtained for the concrete selected pairs of materials, They have also a meaning in a general sense. In the viewpoint of the authors, these results have not only the theoretical, but also the practical importance.

The results which are not given here show that the observed character of the dependence between $\sigma_{22}h/P_0$ and Ω holds for all points of the interface and for the other stresses and displacements which are not considered here.

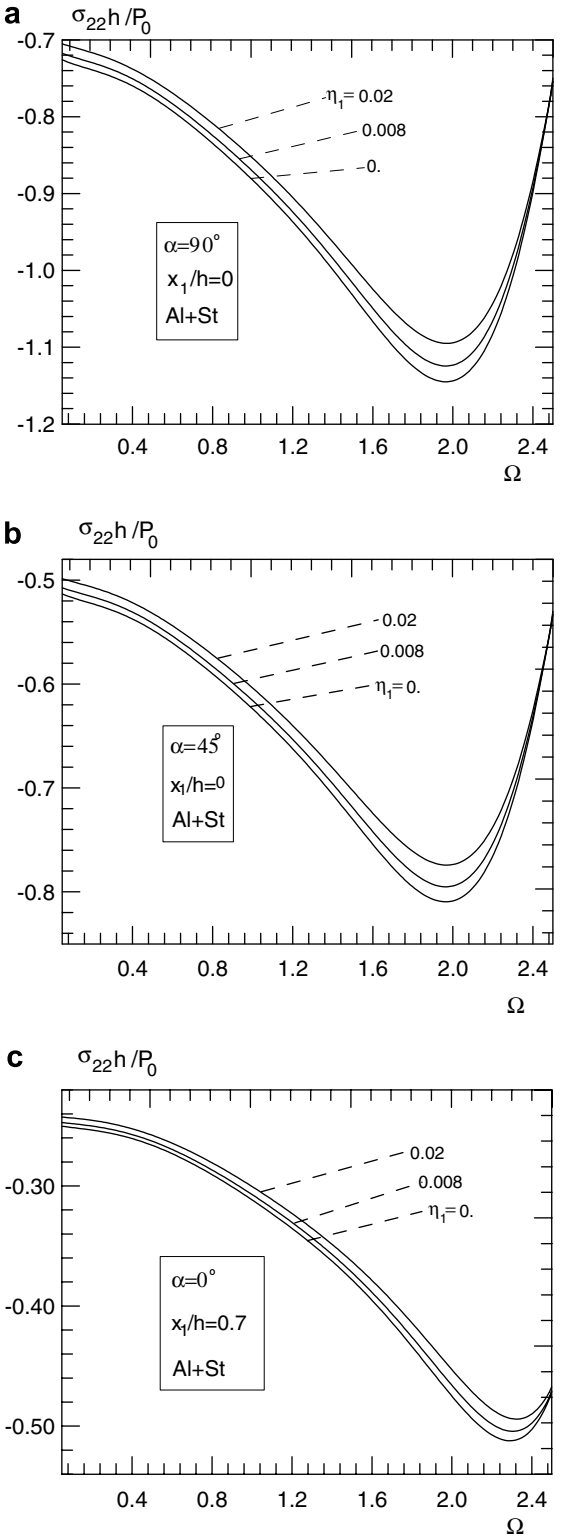


Fig. 2. The graphs of the dependence between $\sigma_{22}h/P_0$ and Ω for: (a) $\alpha = 90°$, $x_1/h = 0$; (b) $\alpha = 45°$, $x_1/h = 0$ and (c) $\alpha = 0°$, $x_1/h = 0.7$. The pair of material is (Al + St).

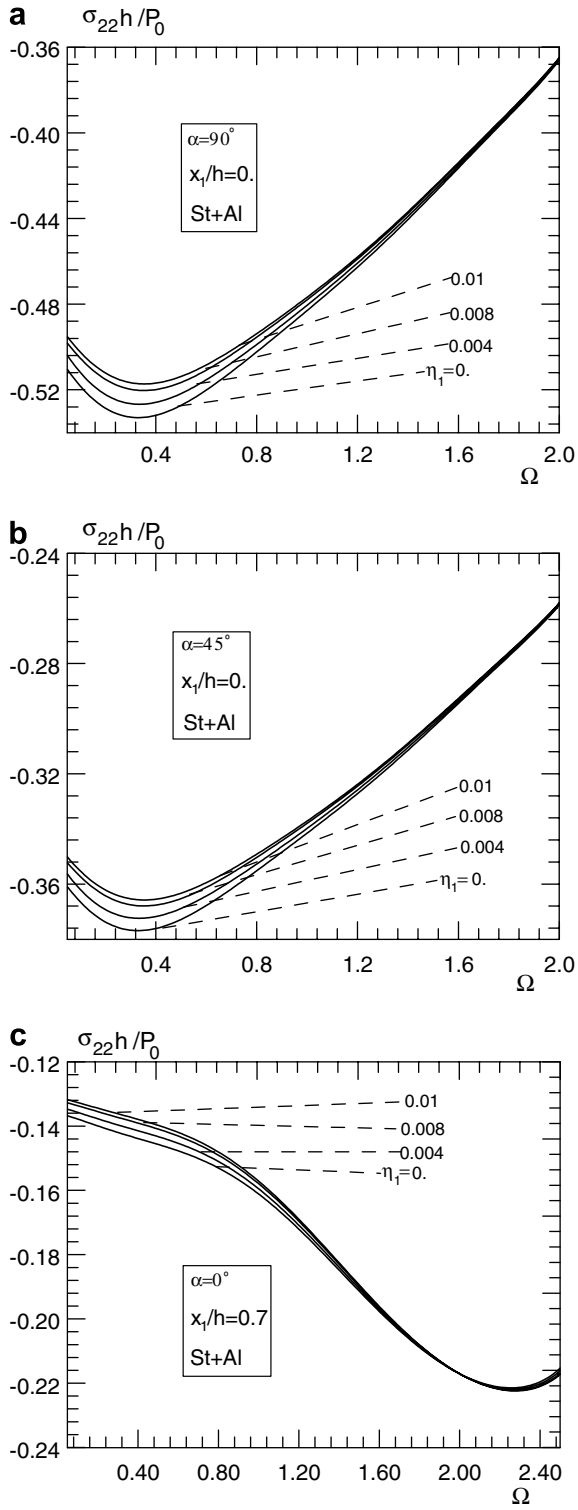


Fig. 3. The graphs of the dependence between $\sigma_{22}h/P_0$ and Ω for: (a) $\alpha = 90°$, $x_1/h = 0$; (b) $\alpha = 45°$, $x_1/h = 0$ and (c) $\alpha = 0°$, $x_1/h = 0.7$. The pair of materials is (St + Al).

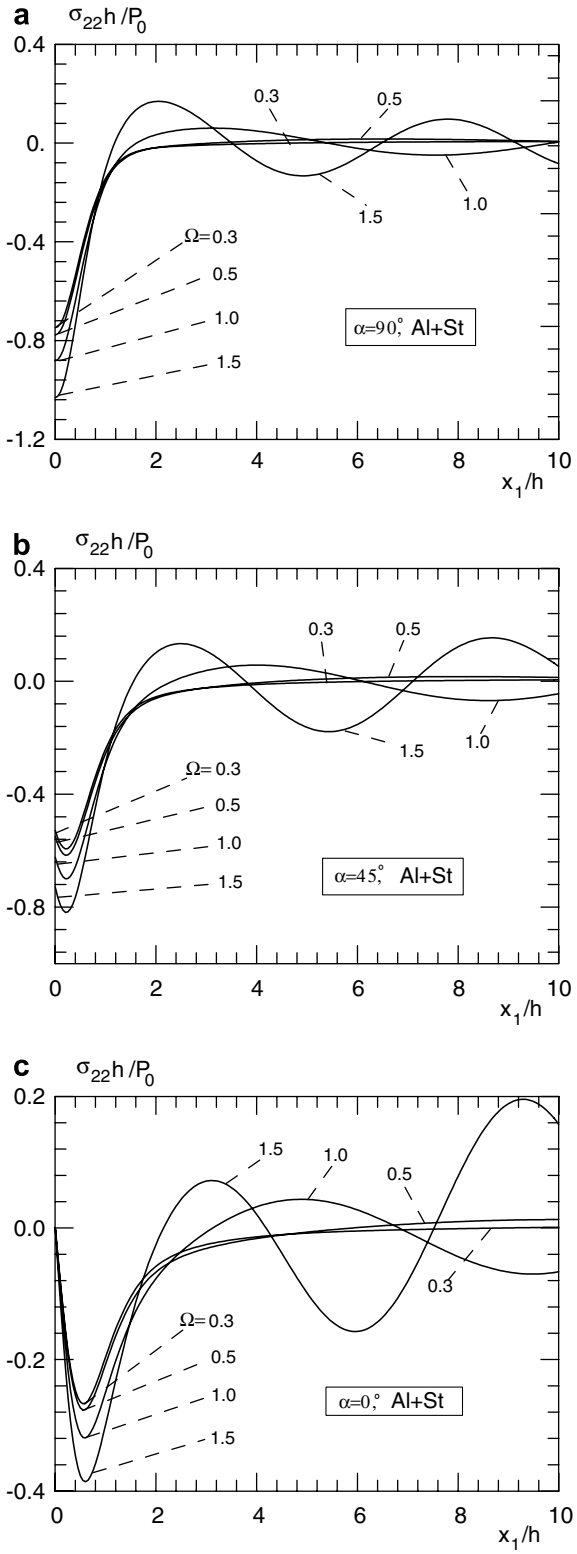


Fig. 4. The distribution of $\sigma_{22}h/P_0$ with respect to x_1/h for various Ω in the case where $\eta_1 = 0.0$: (a) $\alpha = 90°$; (b) $\alpha = 45°$ and (c) $\alpha = 0°$. The pair of materials is (Al + St).

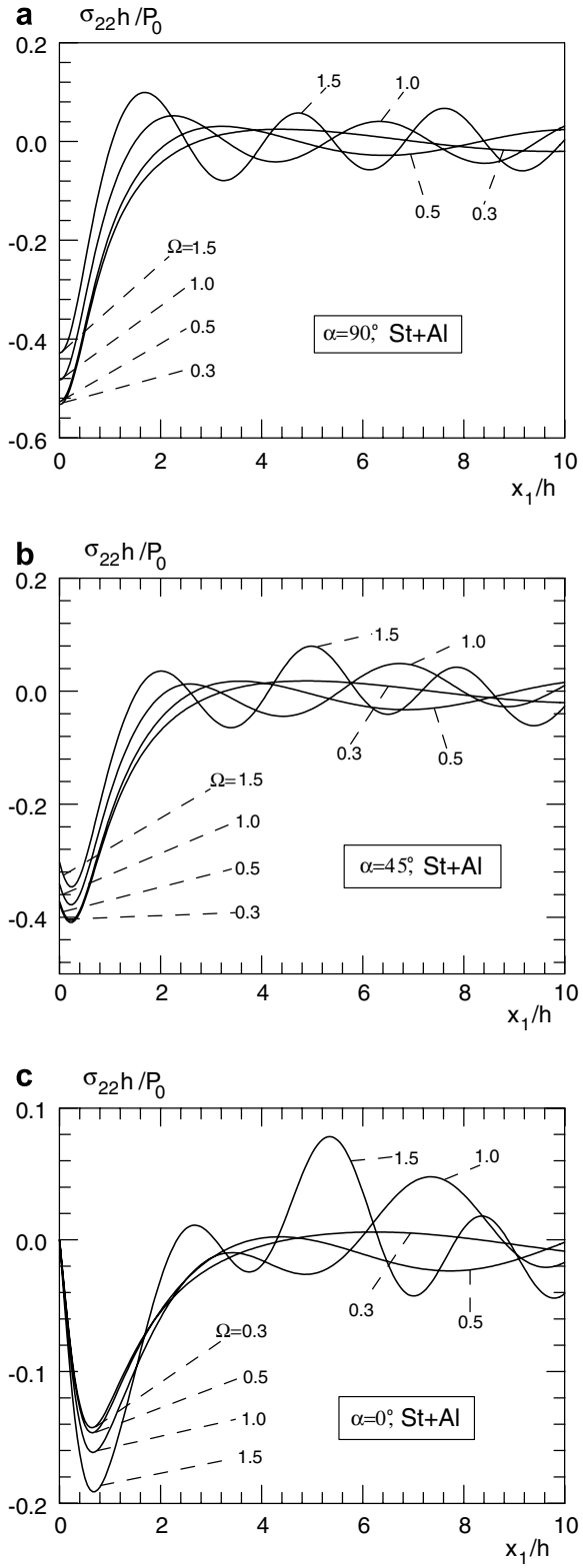


Fig. 5. The distribution of $\sigma_{22}h/P_0$ with respect to x_1/h for various Ω in the case where $\eta_1 = 0.0$: (a) $\alpha = 90°$; (b) $\alpha = 45°$ and (c) $\alpha = 0°$. The pair of materials is (St + Al).

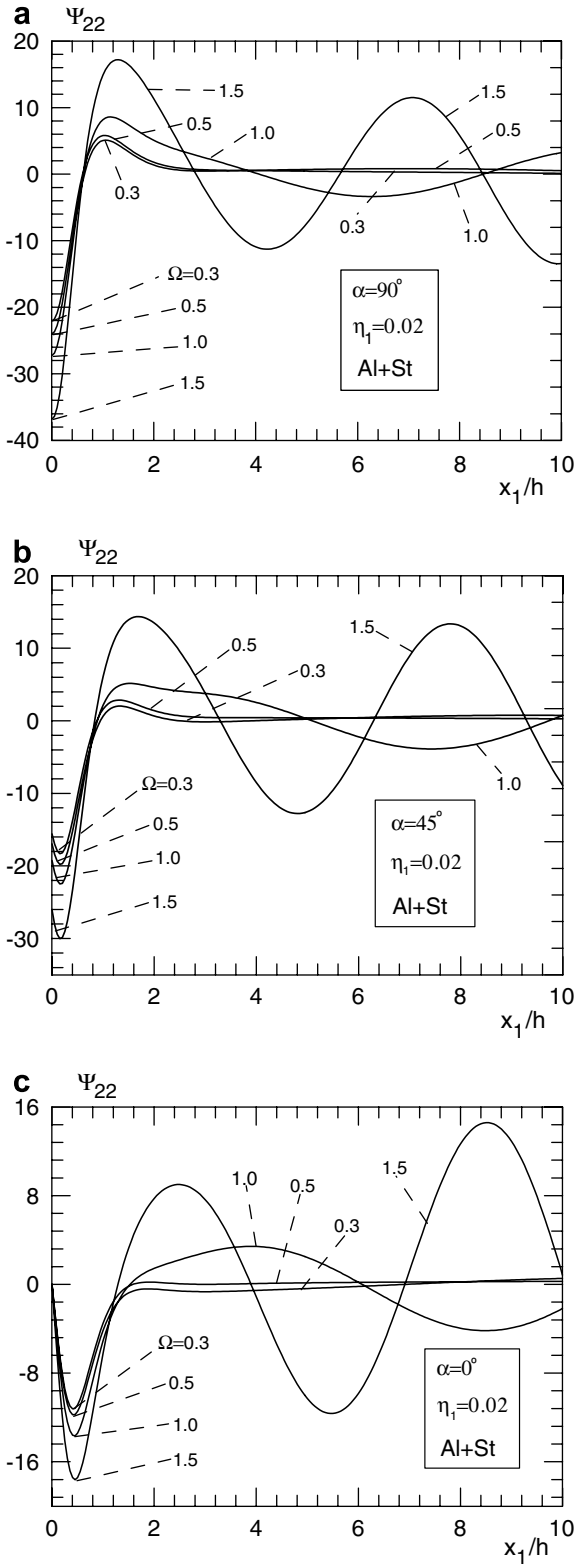


Fig. 6. The distribution of Ψ_{22}(27) with respect to x_1/h for various Ω in the case where $\eta_1 = 0.02$: (a) $\alpha = 90°$; (b) $\alpha = 45°$ and (c) $\alpha = 0°$. The pair of materials is (Al + St).

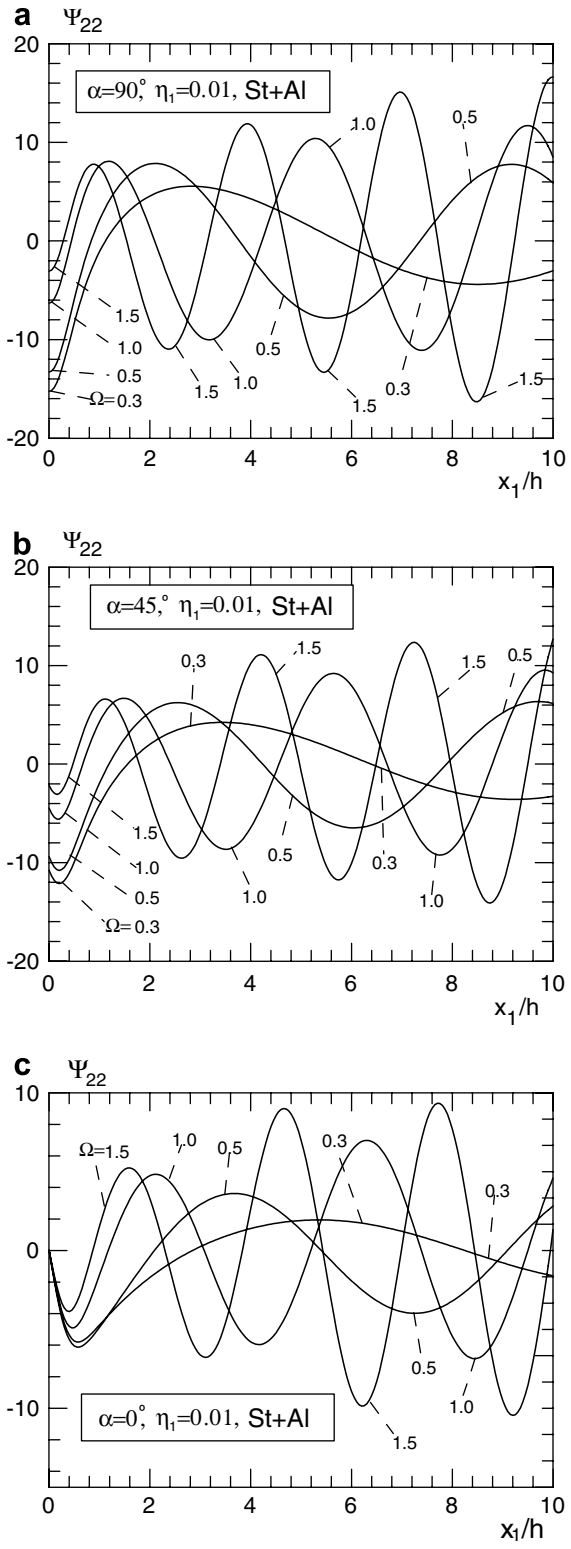


Fig. 7. The distribution of Ψ_{22}(27) with respect to x_1/h for various Ω in the case where $\eta_1 = 0.01$: (a) $\alpha = 90°$; (b) $\alpha = 45°$ and (c) $\alpha = 0°$. The pair of materials is (St + Al).

Now we consider the distribution of the stress σ_{22} along the Ox_1 axis in the case where $\eta_1 = 0.0$. The graphs of this distribution are given in Fig. 4 (for the pair Al + St) and in Fig. 5 (for the pair St + Al) with various values of Ω. In these figures the graphs separated by the letters a, b and c correspond to the cases where $\alpha = 90°$, 45° and $\alpha = 0°$ respectively.

The comparison of the graphs given in Fig. 4 with the corresponding ones given in Fig. 5 reveals that for all the considered α the absolute maximum values of σ_{22} obtained for the pair Al + St are greater than those obtained for the pair St + Al. This result is explained by the fact that Young's modulus of Al is smaller than that of St. With this also the difference between the vibrational character of the distribution σ_{22} obtained for the considered pairs of the materials is explained. It follows from these distributions that there are almost periodically located interface points at which $\sigma_{22} = 0$. The "frequency" of these points depends on Ω. Furthermore, the frequency of these points obtained for the pair (St + Al) is significantly greater than those obtained for the pair (Al + St).

Note that the above discussed results are observed daily in the engineering practice under an impact treatment of metals which lie on the others.

According to the obtained numerical results, the σ_{22} has its absolute maximum at $x_1/h = 0$, but at this point $\sigma_{22} \to 0$ as $\alpha \to 0°$. In the case where $0° < \alpha < 90°$ the absolute maximum of the values of σ_{22} arises in the region $0 < x_1/h \leqslant 0.7$. These results also agree well with the well known mechanical consideration.

Now introduce the parameter

$$\Psi_{22} = \left(\sigma_{22}|_{\eta_1=0} - \sigma_{22}|_{\eta_1>0}\right) x 10^3 h/P_0 \tag{27}$$

and through this parameter we analyse the influence of the pre-stretching of the covering layer on the distribution of the stress σ_{22}.

Figs. 6 and 7 show the distribution of Ψ_{22} with respect to x_1/h for various values of Ω under $\eta_1 = 0.02$ and $\eta_1 = 0.01$ for the pairs Al + St and St + Al respectively. As in the previous figures, here also the cases where $\alpha = 90°$, 45° and $\alpha = 0°$ are indicated respectively by the letters a, b and c.

Fig. 6 shows that for the pair Al + St for the considered range of the change of Ω (i.e. for $0.3 \leqslant \Omega \leqslant 1.5$) the influence of the pre-stretching of the covering layer (i.e. the absolute maximum values of Ψ_{22}) increase with Ω. Because in the considered range of the change of Ω the selected values of that is smaller than the "resonance" values of Ω. According to the graphs in Fig. 2, the "resonance" values of Ω are $\Omega \approx 1.92$; 2.00 and 2.28 for $\alpha = 90°$; 45°; and 0° respectively. Moreover, Fig. 6 shows that the point at which the Ψ_{22} has its absolute maximum moves from $x_1/h = 0.0$ to $x_1/h \approx 0.7$ by decreasing the inclination angle from $\alpha = 90°$ to $\alpha = 0°$.

Fig. 7 shows that for the pair St + Al the absolute maximum values of Ψ_{22} decrease with $\Omega \in [0.3,1.5]$ in the region $0 \leqslant x_1/h < 1.0$. However, getting away from $x_1/h = 0$ the absolute local maxima of the Ψ_{22} increase with Ω. These results are also explained by the character of the dependencies between $\sigma_{22}h/P_0$ and Ω (Fig. 3) and by the character of the distribution of $\sigma_{22}h/P_0$ with respect to x_1/h for selected values of Ω (Fig. 5).

As the distribution of the σ_{22}, the distribution of Ψ_{22} has also a vibrational character. The "frequency" of this distribution increases with Ω and the number of those obtained for the pair St + Al is significantly greater than that obtained for the pair Al + St. Moreover it follows from the graphs given in Figs. 6 and 7 that there are such points on the interface plane at which the initial strains of the covering layer does not influence the values of the considered stress.

6. Conclusions

In the present paper within the framework of the piecewise homogeneous body model with the use of Three-Dimensional Linearized Theory of Elastic Waves in Initially Stressed Bodies the time-harmonic interface stress field in the half-plane covered with the pre-stretched layer was studied. The investigations are carried out for the plane-strain state in the plane Ox_1x_2 (Fig. 1). It was assumed that on the free face plane of the covered layer the arbitrary inclined lineal located time-harmonic force acts.

The concrete numerical results are presented for the pair of materials Aluminium (layer) + Steel (half-plane) and for the pair Steel (layer) + Aluminium (half-plane). These pairs are denoted shortly as Al + St and St + Al respectively. These results are obtained for the case where the covering layer is stretched initially

and in the half-plane there is no initial stress. According to the carried out investigations which are involved in the normal stress σ_{22} (26) acting on the interface plane, the following conclusions can be made:

- the mechanical behaviour of the forced vibration of the half-plane covered with the layer is similar to that of the system comprising a mass, a spring and a dashpot;
- the ''resonance'' values of the frequency of the external force and the absolute maximum values of the stress σ_{22} (26) attained for the homogeneous half-plane can be decreased (increased) significantly by the covering of the half-plane with the layer the material stiffness of which is greater (smaller) than that of the half-plane material, such as in the pair St + Al (Al + St);
- the absolute values of σ_{22} decrease with pre-stretching of the covering layer;
- the absolute maximum values of σ_{22} decrease with decreasing α (Fig. 1);
- the influence of the pre-stretching of the covering layer on the absolute values of the stress σ_{22} decreases with decreasing inclination angle α from $\alpha = 90°$ to $\alpha = 0°$;
- the distribution of σ_{22} and Ψ_{22} (27) with respect to the x_1/h (Fig. 1) have a vibrational character. Besides, the ''frequency'' obtained for the pair St + Al is significantly greater than that obtained for the pair Al + St;
- there are such points on the interface plane at which the initial strains of the covering layer does not influence the values of the stress σ_{22}.

Although the above-written conclusions were attained for the concrete selected pairs of materials, in the qualitative sense they have also a general meaning.

The numerical results obtained in the case where the initial stress is absent (i.e. $\eta_1 = 0$) correspond to those attained within the framework of the classical linear theory of elasticity. Consequently, the numerical results presented in the paper are also significant from this viewpoint. Moreover, the obtained numerical result can be taken as standard obtained for the vibration of the system containing the covering layer and ground which is modelled by a system composed of a mass, spring and a dashpot.

In recent years a new scientific area called acousto-elasticity has been introduced in theory of waves. It addresses the propagation of elastic waves and vibrations in initially stressed bodies and includes practical issues such as determination of the physico-mechanical properties and nondestructive analyses of the residual stress-strain state in structural members of engineering design. A review of the relevant investigations has been given in the paper by Guz and Makhort [23]. Consequently, within the framework of certain conditions the results of the present investigations can be taken as theoretical ones for the study of nondestructive residual stress analyses in structural members of engineering.

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