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Advanced semi-analytical techniques for fractional shallow water wave models through the analogical structure of generalized ϕ-Caputo derivative operators

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Abstract

This paper presents advanced semi-analytical solution techniques for fractional shallow water wave systems governed by the Whitham–Broer–Kaup equations using the generalized ϕ-Caputo fractional derivative. By exploiting the analogical structure of the ϕ-Caputo operator, two extended solution procedures are developed, namely the ϕ-Caputo residual power series method (ϕ-CRPSM) and the ϕ-Caputo new iterative method (ϕ-CNIM). These formulations generalize classical RPSM and NIM approaches to a wider class of nonlocal fractional operators, providing enhanced flexibility for modeling memory-dependent nonlinear dispersive wave dynamics. The proposed methods generate rapidly convergent fractional power-series solutions and can be systematically implemented for nonlinear fractional systems without linearization or discretization. Applications to fractional Whitham–Broer–Kaup models demonstrate that the obtained solutions exhibit high accuracy, strong convergence behavior, and close agreement with exact solutions in the integer-order limit. Numerical illustrations confirm the effectiveness, stability, and computational efficiency of the proposed approaches in capturing the nonlinear, dispersive, and memory-influenced characteristics of shallow-water wave propagation. These results establish the proposed ϕ-CRPSM and ϕ-CNIM techniques as efficient and robust analytical–numerical tools for a broad class of nonlinear fractional partial differential equations.

Keywords: Fractional WBKEs; Shallow water waves; ϕ-Caputo derivative; RPSM; NIM

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1 Introduction

Fractional-order operators extend the traditional concepts of differentiation and integration beyond integer orders, providing a flexible mathematical framework for describing processes with memory and hereditary characteristics. Fractional calculus (FC) generalizes classical differentiation and integration by allowing operators of arbitrary, including non-integer, order. Its origins trace back to the seventeenth century, when Leibniz and L'Hôpital first raised the question of defining a fractional derivative. Significant progress was later achieved by Riemann, Liouville, and Grünwald, whose contributions established rigorous formulations for fractional integrals and derivatives [1, 4]. Since then, FC has developed into a comprehensive theoretical structure that unifies many generalized operators and offers powerful tools for modeling memory and nonlocal effects in complex systems. Recent advances in numerical

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methods have further broadened its applicability, enabling efficient approximation of fractional differential equations (FDEs), which are often analytically intractable [5, 6]. Widely used fractional operators, including the Caputo, Riemann–Liouville, Atangana–Baleanu, and Caputo–Fabrizio definitions, incorporate different kernel structures to represent experimentally observed nonlocal behaviors. When combined with modern computational techniques, these operators facilitate accurate modeling of viscoelastic materials, anomalous diffusion, biological systems with memory, and various nonlinear physical phenomena [7]. Consequently, fractional partial differential equations (FPDEs) have become increasingly significant in the analysis of complex systems across fluid dynamics, quantum mechanics, plasma physics, and electromagnetism. Their capability to capture long-range temporal memory and spatial heterogeneity provides a more versatile modeling framework than classical integer-order partial differential equations [8, 9]. Applications of FPDEs extend to numerous physical, biological, financial, and engineering contexts, particularly in systems governed by hereditary mechanisms such as charge transport, heat conduction in heterogeneous media, and groundwater flow [10]. This broad applicability has stimulated extensive research into the theoretical aspects of FPDEs, including existence, uniqueness, stability, and convergence analyses [11]. At the same time, a variety of analytical and approximate solution techniques for nonlinear FPDEs have been developed, including perturbation approaches [12, 9], integral and hyperbolic-type formulations [8, 13], decomposition methods [14], homotopy-based schemes [15], convection–diffusion modeling frameworks [16], and perturbation techniques for nonsingular fractional derivatives [17]. Collectively, these efforts reflect the continuing pursuit of reliable and efficient methodologies for solving nonlinear fractional partial differential equations.

The classical WBKEs, which describe nonlinear shallow water wave interactions, are given by [18]

$$\begin{cases} \omega_t + \omega\,\omega_x + \psi_x + \beta\,\omega_{xx} = 0, \\ \psi_t + (\omega\psi)_x - \beta\,\psi_{xx} + \beta'\,\omega_{xxx} = 0, \end{cases} \tag{1}$$

where ω and ψ denote horizontal and height velocities, respectively, and the parameters β and β' correspond to dispersive effects. These equations describe nonlinear wave propagation and have been applied to shallow water flows, plasma oscillations, and nonlinear acoustic waves. Significant research efforts have also been directed toward fractional extensions of the WBKEs. The introduction of various fractional operators has motivated the use of numerous analytical and numerical strategies for approximating solutions to FPDEs. Examples, Akgul et al. [19] introduced a fractional extension of Newton's method that achieves a 2α-order convergence rate. Their analysis shows that incorporating fractional operators can significantly improve the performance and stability of classical iterative numerical schemes. Seadawy et al. [20] examined kink and anti-kink soliton solutions of a nonlinear damped modified KdV equation arising in ion-acoustic wave propagation within collisional dusty plasma. Their results illustrate how nonlinear, dispersive, and damping effects interact to shape complex wave structures. He [21] proposed the homotopy perturbation method, a pioneering semi-analytical technique that combines homotopy ideas with perturbation theory to produce rapidly convergent series solutions for nonlinear differential equations. Yazgan et al. [22] developed new hyperbolic type exact solutions for the Wu-Zhang system, thereby expanding the analytical solution set available for nonlinear wave propagation models. Damag and Saif

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[23] introduced the Laplace residual power series method for solving modified time fractional Kawahara equations within Hilbert algebra frameworks. Their method yields highly accurate analytical numerical series approximations for fractional dispersive systems. Boutarfa et al. [24] proposed a new analytical numerical approach to the Fornberg-Whitham type equations, demonstrating improved stability and efficiency in approximating nonlinear dispersive wave solutions. Akgul [25] presented a novel fractional derivative operator characterized by a nonlocal and nonsingular kernel. This operator avoids singular behaviors typical of classical fractional derivatives and provides enhanced modeling flexibility for systems with memory. Recent studies have introduced generalized fractional operators that provide greater flexibility for modeling complex physical systems and nonlocal phenomena [26]. In addition, generalized Laplace transform frameworks combined with fractional operators have been successfully applied to computational modeling of electromagnetic wave propagation and related physical applications [27]. Sontakke et al. [28] applied the variational iteration method to nonlinear fractional differential equations, showing its precision and adaptability for physical problems governed by hereditary effects. Inc et al. [29] used the reproducing kernel Hilbert space method to derive explicit analytical solutions for the telegraph equation, offering a reliable convergent analytical framework for dissipative wave models. Dhaigude et al. [30] formulated a monotone iterative scheme for weakly coupled finite difference reaction-diffusion systems. Their method guarantees convergence under mild monotonicity assumptions and provides an effective means of approximating nonlinear coupled PDE solutions. Kilic and Celik [31] obtained complex analytical solutions for higher order nonlinear Boussinesq type wave equations, contributing important analytical results to the study of nonlinear dispersive wave dynamics. Rahman et al. [32] developed a fractional mathematical model for drinking behavior using the Atangana-Baleanu Caputo derivative, highlighting the capability of fractional operators to describe memory driven biological processes more accurately than classical models. Nadeem et al. [33] investigated a two-dimensional fuzzy fractional heat equation with an external source term. Their numerical results combine fractional diffusion with fuzzy uncertainty, providing a more general and realistic representation of heat transfer phenomena involving imprecision and memory effects. A wide range of analytical and hybrid numerical techniques has been introduced specifically for fractional WBKEs. These include the residual power series method [35], quintic B-spline approximations with Von Neumann analysis [36], the Adomian decomposition method [37], the Yang-Adomian technique [38], the q-homotopy analysis transform [42], the YRPST approach [39], the Sumudu decomposition method [40], Laplace-Adomian strategies [41], the natural decomposition method [42], the scaling transform method [43], and the optimal homotopy asymptotic method [44]. Collectively, these works underline the versatility and effectiveness of modern analytical numerical schemes for tackling nonlinear fractional PDEs.

Although several analytical and semi-analytical approaches have been proposed for solving fractional nonlinear partial differential equations, most existing formulations are primarily developed for classical Caputo-type operators and do not directly extend to generalized ϕ-Caputo fractional derivatives. This limitation reduces their applicability to complex physical models where memory effects are governed by more general kernel structures. Furthermore, efficient semi-analytical frameworks capable of producing rapidly convergent approximate solutions for fractional Whitham–Broer–Kaup shallow water wave systems remain relatively limited. Motivated by these challenges, this work develops generalized extensions of two pow-

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erful analytical–numerical techniques, namely the residual power series method and the new iterative method, within the generalized ϕ-Caputo fractional operator setting. The study introduces the ϕ-CRPSM for constructing fractional power-series solutions of nonlinear fractional systems and formulates the ϕ-CNIM to generate rapidly convergent iterative solution sequences without the need for linearization. The effectiveness of the proposed approaches is demonstrated through their application to fractional Whitham–Broer–Kaup shallow water wave equations, where the obtained results confirm strong convergence behavior, high computational efficiency, and accurate agreement with exact solutions in the integer-order limit. These developments provide a generalized and flexible analytical–numerical framework suitable for a wide class of nonlinear fractional partial differential equations arising in fluid dynamics and related physical applications. In this work we develop generalized semi-analytical techniques based on the ϕ-Caputo fractional operator and apply them to fractional shallow water WBKEs. Section 2 defines the ϕ-fractional integral and ϕ-Caputo derivative and establishes the mathematical tools required for constructing our solution approaches. Section 3 presents the two new methods: ϕ-CRPSM, which builds fractional power series solutions using residual conditions, and ϕ-CNIM, which generates iterative corrections through ϕ-fractional integral formulations. Section 4 applies these methods to fractional WBKEs and shows through examples that they produce accurate, fast convergent approximations capturing nonlinear, dispersive, and memory dependent shallow water wave behavior.

2 On fractional operators

Definition 2.1. [45] Let Ω be any subset of $\mathbb{R}$ and ω be any map on $\Omega \times [a, t_a]$, $0 < a < t_a < \infty$, in relation with a map ϕ of order $\alpha \geq 0$, such that $\phi \in C^1([a, t_a])$ is a growing map, $\phi'(t) \neq 0$ for each $t \in [a, t_a]$. The ϕ-fractional integral of ω with respect to t is given by

$$J_{a\,t}^{\alpha;\phi}[\omega(x,t)] = \frac{1}{\Gamma(\alpha)} \int_a^t \phi'(\mathcal{T})\big(\phi(t) - \phi(\mathcal{T})\big)^{\alpha-1} \omega(x,\mathcal{T})\, d\mathcal{T}, \quad \alpha > 0 \tag{2}$$

We remark that the results in this section, originally proved for the single variable case, are extended here to two variables. For simplicity, we cite only the one variable sources, as the generalization is direct.

Theorem 2.2. [46] Let ω be any map on $\Omega \times [a, t_a]$, in relation to a map ϕ of order α, such that $\phi \in C^n([a, t_a])$ is a growing map, $\phi'(t) \neq 0$ for each $t \in [a, t_a]$, and $\alpha > 0$. Let $\omega(x, \cdot) \in C^n([a, t_a])$ for all $x \in \Omega$. The ϕ-Caputo derivative operator of ω of order α with respect to t is given by ${}^{c}D_{a\,t}^{\alpha;\phi}\omega(x,t) = J_{a\,t}^{n-\alpha;\phi}\omega^{[n\phi]}(x,t)$, where $n-1 \leq \alpha < n$ for $n \in \mathbb{N}$ and

$$\omega^{[n\phi]}(x,t) := \left(\frac{1}{\phi'(t)}\frac{\partial}{\partial t}\right)^n \omega(x,t). \tag{3}$$

Recall from [47, 48] that choosing $\phi(t) = t$ leads to the classical Caputo derivative operator, while taking $\phi(t) = \ln(t)$ results in the Caputo-Hadamard derivative operator ${}^{CH}D_{a,t}^{\alpha;\phi}\omega(x,t)$.

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Theorem 2.3. [48] Let $\Re(\alpha) \geq 0$, $n = [\Re(\alpha)] + 1$. If $\omega(x,\cdot) \in AC_{\delta}^{n}([a,t_a])$ for all $x \in \Omega$ and

$$AC_{\delta}^{n}([a,t_a]) := \left\{ g : [a,t_a] \to \mathbb{C} : \delta^{n-1} g \in AC[a,t_a],\ \delta = t\frac{d}{dt} \right\},$$

then ${}^{c}D_{a\,t}^{\alpha;\ln t}\omega(x,\cdot)$ exists on $[a,t_a]$ for all $x \in \Omega$ and

$${}^{CH}D_{a\,t}^{\alpha;\ln t}\omega(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln\frac{t}{\mathcal{T}}\right)^{n-\alpha-1} \left(\mathcal{T}\frac{\partial}{\partial \mathcal{T}}\right)^{n} \omega(x,\mathcal{T})\,\frac{d\mathcal{T}}{\mathcal{T}}. \tag{4}$$

Theorem 2.4. [48] Let $\omega(x,\cdot) \in AC_{\phi}^{n}([a,t_a])$ for all $x \in \Omega$, and $\alpha \in \mathbb{C}$, then

$$J_{a\,t}^{\alpha;\ln t}\left({}^{CH}D_{a\,t}^{\alpha;\phi}\omega(x,t)\right) = \omega(x,t) - \sum_{k=0}^{n-1} \frac{\delta^k \omega(x,a)}{k!} \left(\ln\frac{t}{a}\right)^{k}. \tag{5}$$

Lemma 2.5. [48] Let $\gamma > -1$. Then

$$J_{a\,t}^{\alpha;\phi}\left[(\phi(t)-\phi(a))^{\gamma}\right] = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\big(\phi(t)-\phi(a)\big)^{\alpha+\gamma}. \tag{6}$$

and

$${}^{c}D_{a\,t}^{\alpha;\phi}\left[(\phi(t)-\phi(a))^{\gamma}\right] = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\big(\phi(t)-\phi(a)\big)^{\gamma-\alpha}. \tag{7}$$

Theorem 2.6. [48] Let ω be any map on $\Omega \times [a,t_a]$ and $\omega(x,\cdot) \in C^{n}([a,t_a])$ for all $x \in \Omega$, and $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\gamma > 0$, then

$$J_{a\,t}^{\alpha;\phi}\left({}^{c}D_{a\,t}^{\alpha;\phi}[\omega(x,t)]\right) = \omega(x,t) - \sum_{k=0}^{n-1} \frac{\omega^{[k\phi]}(x,a)}{k!}\left(\phi(t)-\phi(a)\right)^{k}. \tag{8}$$

3 Methodologies of some semi-analytical techniques

In this sections, we will develop the RPSM and NIM in new techniques in solving fractional WBKE type systems based on ϕ-Caputo derivative operators. A power series

$$\sum_{n=0}^{\infty} c_n\big(\phi(t)-\phi(a)\big)^{n\alpha} = C_0 + C_1\big(\phi(t)-\phi(a)\big)^{\alpha} + C_2\big(\phi(t)-\phi(a)\big)^{2\alpha} + \cdots, \tag{9}$$

is called a fractional power series (FPS) about $\phi(a)$, such that t is a variable, c_n are called the coefficients of the series, where $0 \leq n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $\phi(t) \geq \phi(a)$.

Lemma 3.1. Consider the FPS notation of $\mathcal{Q}$ about $\phi(a)$ has the form

$$\mathcal{Q}(t) = \sum_{n=0}^{\infty} c_n\big(\phi(t)-\phi(a)\big)^{n\alpha}, \tag{10}$$

where $0 \leq m-1 < \alpha \leq m$, $\phi(a) \leq \phi(t) \leq \phi(a)+R$. If $\mathcal{Q}(t)$, ${}^{c}D_{at}^{n\alpha;\phi}\mathcal{Q}(t) \in C[\phi(a),\phi(a)+R]$ for $m = 1,2,3,\ldots$, then the terms c_n are given by

$$c_n = \frac{{}^{c}D_{at}^{n\alpha;\phi}\mathcal{Q}(t)}{\Gamma(n\alpha+1)}\Big|_{t=a}, \tag{11}$$

where ${}^{c}D_{at}^{n\alpha;\phi} = {}^{c}D_{at}^{\alpha;\phi} \circ {}^{c}D_{at}^{\alpha;\phi} \circ \cdots \circ {}^{c}D_{at}^{\alpha;\phi}$ (n times) and R is the convergence radius.

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Proof. By Lemma(2.5) we have

$$ {}^{c}D_{at}^{\alpha;\phi}(\phi(t)-\phi(a))^{n\alpha} = \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}(\phi(t)-\phi(a))^{(n-1)\alpha}. \tag{12} $$

Hence

$$ {}^{c}D_{at}^{\alpha;\phi}\mathcal{Q}(t) = \sum_{n=1}^{\infty} c_n\, {}^{c}D_{at}^{\alpha;\phi}(\phi(t)-\phi(a))^{n\alpha} = \sum_{n=1}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}(\phi(t)-\phi(a))^{(n-1)\alpha}. $$

Reindex with $k = n-1$:

$$ {}^{c}D_{at}^{\alpha;\phi}\mathcal{Q}(t) = \sum_{k=0}^{\infty} c_{k+1} \frac{\Gamma((k+1)\alpha+1)}{\Gamma(k\alpha+1)}(\phi(t)-\phi(a))^{k\alpha}. $$

Evaluating at $t=a$ kills all $k \geq 1$, since $(\phi(a)-\phi(a))^{k\alpha} = 0$, giving

$$ {}^{c}D_{at}^{\alpha;\phi}\mathcal{Q}(t)\Big|_{t=a} = c_1\,\Gamma(\alpha+1). $$

Define recursively $\mathcal{Q}_0(t) = \mathcal{Q}(t)$,

$$ \mathcal{Q}_k(t) = {}^{c}D_{at}^{\alpha;\phi}\mathcal{Q}_{k-1}(t) = {}^{c}D_{at}^{k\alpha;\phi}\mathcal{Q}(t). $$

Using (12) repeatedly, by induction we get

$$ {}^{c}D_{at}^{k\alpha;\phi}\mathcal{Q}(t) = \mathcal{Q}_k(t) = \sum_{n=k}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma((n-k)\alpha+1)}(\phi(t)-\phi(a))^{(n-k)\alpha}. \tag{13} $$

Set $k=n$ in (13) to get ${}^{c}D_{at}^{n\alpha;\phi}\mathcal{Q}(t)\Big|_{t=a} = c_n\Gamma(n\alpha+1)$. Hence $c_n = \frac{{}^{c}D_{at}^{n\alpha;\phi}\mathcal{Q}(t)}{\Gamma(n\alpha+1)}\Big|_{t=a}$. □

A power series

$$ \omega(x,t) = \sum_{n=0}^{\infty} \mathcal{Q}_n(x)\left(\phi(t)-\phi(a)\right)^{n\alpha}, x \in \Omega,\ \phi(a) \leq \phi(t) \leq \phi(a)+R. \tag{14} $$

is called a multiple FPS of ω about $\phi(a)$.

Theorem 3.2. If ${}^{c}D_{at}^{n\alpha;\phi}\omega(x,t), n = 0,1,2,3,\ldots$, are continuous on $\Omega \times (\phi(a), \phi(a)+R)$, in the series (14) then

$$ \mathcal{Q}_n(x) = \frac{{}^{c}D_{at}^{n\alpha;\phi}\omega(x,\phi^{-1}(a))}{\Gamma(n\alpha+1)}, \tag{15} $$

where $R = \min_{x\in\Omega} R_x$ with R_x the convergence radius of the FPS

$$ \sum_{n=0}^{\infty} \mathcal{Q}_n(x)\big(\phi(t)-\phi(a)\big)^{n\alpha}. \tag{16} $$

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Proof. Fix an arbitrary x and define the function $\mathcal{Q}_x$ by $\mathcal{Q}_x(t) = \omega(x,t)$ Then, for this fixed x, the FPS (14) becomes

$$\mathcal{Q}_x(t) = \sum_{n=0}^{\infty} \mathcal{Q}_n(x)\big(\phi(t) - \phi(a)\big)^{n\alpha}.$$

By the continuity assumptions, $\mathcal{Q}_x(t)$ and ${}^cD_{a\,t}^{n\alpha;\phi}\mathcal{Q}_x(t) = {}^cD_{a\,t}^{n\alpha;\phi}\omega(x,t)$ satisfy the hypotheses of Lemma(3.1), we get $\mathcal{Q}_n(x) = \frac{{}^cD_{a\,t}^{n\alpha;\phi}\mathcal{Q}_x(t)}{\Gamma(n\alpha+1)}\Big|_{t=a} = \frac{{}^cD_{a\,t}^{n\alpha;\phi}\omega(x,t)}{\Gamma(n\alpha+1)}\Big|_{t=a}$. □

Theorem 3.3. Consider the multiple FPS notation of ω about $\phi(a)$ has the form

$$\omega(x,t) = \sum_{n=0}^{\infty} \mathcal{Q}_n(x)\left(\phi(t) - \phi(a)\right)^{n\alpha}, \tag{17}$$

where $x \in \Omega_1 \times \Omega_2 \times \cdots \Omega_n,\ \phi(a) \le \phi(t) \le \phi(a) + R$. If ${}^cD_{at}^{n\alpha;\phi}\omega(x,t), n = 0,1,2,3,\ldots,$ are continuous on $\Omega_1 \times \Omega_2 \times \cdots \Omega_m \times (\phi(a), \phi(a)+R)$, then

$$\mathcal{Q}_n(x) = \frac{{}^cD_{at}^{n\alpha;\phi}\omega(x,t)}{\Gamma(n\alpha+1)}\Big|_{t=\phi^{-1}(a)}, \tag{18}$$

where $R = \min_{x\in\Omega_1\times\Omega_2\times\cdots\Omega_m} R_x$ with R_x the convergence radius of the FPS

$$\sum_{n=0}^{\infty} \mathcal{Q}_n(x)\big(\phi(t) - \phi(a)\big)^{n\alpha}. \tag{19}$$

Proof. The proof is directly from Theorem(3.2) and the Cartesian product properties. □

3.1 Framework of ϕ-CRPSM

Consider the following $\phi-$Caputo fractional system

$$\begin{cases} {}^c\mathcal{D}_{0t}^{\alpha;\phi}\omega(x,t) = \mathcal{N}_\omega(x,t,\omega,\psi) + \mathcal{R}_\omega(x,t,\omega,\psi) \\ {}^c\mathcal{D}_{0t}^{\alpha;\phi}\psi(x,t) = \mathcal{N}_\psi(x,t,\omega,\psi) + \mathcal{R}_\psi(x,t,\omega,\psi) \end{cases} \tag{20}$$

where $0 < \alpha \le 1$, $\mathcal{N}_\omega, \mathcal{N}_\psi$ denote the nonlinear parts, and $\mathcal{R}_\omega, \mathcal{R}_\psi$ denote any known functions. The initial conditions are met by ω and ψ, allowing them to be rewritten as:

$$\omega(x,0) = h_0(x) \text{ and } \psi(x,0) = \hat{h}_0(x). \tag{21}$$

The approximate series solution of (20) can be expressed as:

$$\omega(x,t) = \sum_{n=0}^{\infty} h_n(x)\frac{\big(\phi(t)-\phi(a)\big)^{n\alpha}}{\Gamma(n\alpha+1)} \text{ and } \psi(x,t) = \sum_{n=0}^{\infty} \hat{h}_n(x)\frac{\big(\phi(t)-\phi(a)\big)^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{22}$$

Let $\langle\omega_m\rangle$ and $\langle\psi_m\rangle$ be the partial sum sequences of (22) where expressed as:

$$\omega_m(x,t) = \sum_{n=0}^{m} h_n(x)\frac{\big(\phi(t)-\phi(a)\big)^{n\alpha}}{\Gamma(n\alpha+1)} \text{ and } \psi_m(x,t) = \sum_{n=0}^{m} \hat{h}_n(x)\frac{\big(\phi(t)-\phi(a)\big)^{n\alpha}}{\Gamma(n\alpha+1)} \tag{23}$$

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Let Res_ω and Res_ψ be the residual maps of (20) where expressed as:

$$\begin{cases} Res_\omega(x,t) = {}^c\mathcal{D}_{0t}^{\alpha;\phi}\omega(x,t) - \mathcal{N}_\omega(x,t,\omega,\psi) - \mathcal{R}_\omega(x,t,\omega,\psi) \\ Res_\psi(x,t) = {}^c\mathcal{D}_{0t}^{\alpha;\phi}\psi(x,t) - \mathcal{N}_\psi(x,t,\omega,\psi) - \mathcal{R}_\psi(x,t,\omega,\psi). \end{cases} \tag{24}$$

Thus, the residual functions truncated at index m are given by

$$\begin{cases} Res_{\omega_m}(x,t) = {}^c\mathcal{D}_{0t}^{\alpha;\phi}\omega_m(x,t) - \mathcal{N}_{\omega_m}(x,t,\omega_m,\psi_m) - \mathcal{R}_{\omega_m}(x,t,\omega_m,\psi_m) \\ Res_{\psi_m}(x,t) = {}^c\mathcal{D}_{0t}^{\alpha;\phi}\psi_m(x,t) - \mathcal{N}_{\psi_m}(x,t,\omega_m,\psi_m) - \mathcal{R}_{\psi_m}(x,t,\omega_m,\psi_m). \end{cases} \tag{25}$$

We have

$$\lim_{m\to\infty} Res_{\omega_m}(x,t) = Res_\omega(x,t), \quad \lim_{m\to\infty} Res_{\psi_m}(x,t) = Res_\psi(x,t),$$

$Res_\omega(x,t) = 0$ and $Res_\psi(x,t) = 0$ for all $x) \in \Omega$ and $\phi(t) \in [\phi(0), \phi(0)+R]$, where R is convergence radius. Consequently, ${}^cD_{0t}^{m\alpha;\phi}\, Res_\omega(x,t) = 0$ and ${}^cD_{0t}^{m\alpha;\phi}\, Res_\psi(x,t) = 0$. Since the Caputo fractional derivative of a constant function is zero, the operators ${}^cD_{0t}^{m\alpha;\phi}$ applied to $Res_\omega(x,t)$, $Res_\psi(x,t)$, $Res_{\omega_m}(x,t)$ and $Res_{\psi_m}(x,t)$ correspond to taking $\phi(0)$ for all $m \in \mathbb{N}$. If $\phi(0) = 0$ then ${}^cD_{0t}^{(m-1)\alpha;\phi}\, Res_{\omega_m}(x,0) = 0$ and ${}^cD_{0t}^{(m-1)\alpha;\phi}\, Res_\psi(x,0) = 0$. We then apply the RPS technique to determine the terms of $\langle h_m \rangle$ and $\langle \hat{h}_m \rangle$ to get the analytical solution of (20).

3.2 Framework of ϕ-CNIM

Consider the following Caputo fractional system

$$\begin{cases} {}^c\mathcal{D}_{0t}^{\alpha;\phi}\omega(x,t) = \mathcal{N}_\omega(x,t,\omega,\psi) + \mathcal{R}_\omega(x,t,\omega,\psi) \\ {}^c\mathcal{D}_{0t}^{\alpha;\phi}\psi(x,t) = \mathcal{N}_\psi(x,t,\omega,\psi) + \mathcal{R}_\psi(x,t,\omega,\psi) \end{cases} \tag{26}$$

where $0 < \alpha \leq 1$, $\mathcal{N}_\omega, \mathcal{N}_\psi$ denote the nonlinear parts, and $\mathcal{R}_\omega, \mathcal{R}_\psi$ denote any known functions. The initial conditions are met by ω and ψ, allowing them to be rewritten as:

$$\omega(x,0) = h_0(x) \text{ and } \psi(x,0) = \hat{h}_0(x). \tag{27}$$

To apply the NIM to system (26), we first rewrite the fractional differential equations in an equivalent integral form by using the inverse operator of the $\phi-$Caputo fractional derivative with respect to ϕ. Then (26) can be written schematically as

$$\begin{cases} \omega(x,t) = h_0(x) + J_{0t}^{\alpha;\phi}\big\{\mathcal{N}_\omega(x,t,\omega,\psi) + \mathcal{R}_\omega(x,t,\omega,\psi)\big\} \\ \psi(x,t) = \hat{h}_0(x) + J_{0t}^{\alpha;\phi}\big\{\mathcal{N}_\psi(x,t,\omega,\psi) + \mathcal{R}_\psi(x,t,\omega,\psi)\big\}. \end{cases} \tag{28}$$

Next, we decompose the unknown functions as infinite series of correction terms,

$$\omega(x,t) = \sum_{n=0}^{\infty} \omega_n(x,t) \text{ and } \psi(x,t) = \sum_{n=0}^{\infty} \psi_n(x,t). \tag{29}$$

We also separate the known (linear or source) parts from the nonlinear operators by introducing

$$F_\omega(x,t) = h_0(x) + J_{0t}^{\alpha;\phi}\big\{\mathcal{R}_\omega(x,t,\omega,\psi)\big\} \text{ and } F_\psi(x,t) = \hat{h}_0(x) + J_{0t}^{\alpha;\phi}\big\{\mathcal{R}_\psi(x,t,\omega,\psi)\big\}. \tag{30}$$

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and the nonlinear operators

$$[G_\omega(\omega,\psi)](x,t) = J_{0t}^{\alpha;\phi}\{\mathcal{N}_\omega(x,t,\omega,\psi)\} \text{ and } [G_\psi(\omega,\psi)](x,t) = J_{0t}^{\alpha;\phi}\{\mathcal{N}_\psi(x,t,\omega,\psi)\}. \quad (31)$$

Thus (26) can be written as

$$\omega = F_\omega + G_\omega(\omega,\psi) \text{ and } \psi = F_\psi + G_\psi(\omega,\psi). \quad (32)$$

Let $\langle S_\omega^m\rangle$ and $\langle S_\psi^m\rangle$ be the partial sum sequences of the series (29), that is,

$$S_\omega^m(x,t) = \sum_{n=0}^{m}\omega_n(x,t) \text{ and } S_\psi^m(x,t) = \sum_{n=0}^{m}\psi_n(x,t).$$

Hence the terms of the series (29) are given by

$$\omega_{m+1}(x,t) = G_\omega\big(S_\omega^m(x,t), S_\psi^m(x,t)\big) - G_\omega\big(S_\omega^{m-1}(x,t), S_\psi^{m-1}(x,t)\big) \quad (33)$$

and

$$\psi_{m+1}(x,t) = G_\psi\big(S_\omega^m(x,t), S_\psi^m(x,t)\big) - G_\psi\big(S_\omega^{m-1}(x,t), S_\psi^{m-1}(x,t)\big) \quad (34)$$

for $m = 0, 1, 2, 3, \cdots$, where

$$G_\omega\big(S_\omega^{-1}(x,t), S_\psi^{-1}(x,t)\big) = 0 \text{ and } G_\psi\big(S_\omega^{-1}(x,t), S_\psi^{-1}(x,t)\big) = 0.$$

This construction yields the above successive approximations which provide the m-th order approximate solutions of system (26). Under suitable conditions on the nonlinear operators, the series $\sum_{n=0}^{\infty}\omega_n$ and $\sum_{n=0}^{\infty}\psi_n$ converge to the exact solutions ω and ψ, respectively.

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4 Some applications

Consider the $\phi-$Caputo fractional WBKEs in shallow water wave phenomena:

$$\begin{cases} {}^c\mathcal{D}_{0,t}^{\alpha;\phi}\omega(x,t) + \omega(x,t)\omega_x(x,t) + \psi_x(x,t) + \beta\omega_{xx}(x,t) = 0, \\ {}^c\mathcal{D}_{0,t}^{\alpha;\phi}\psi(x,t) + [\omega(x,t)\psi(x,t)]_x - \beta\psi_{xx}(x,t) + \beta'\omega_{xxx}(x,t) = 0, \\ \omega(x,0) = h_0(x), \quad \psi(x,0) = \hat{h}_0(x) \end{cases} \quad (35)$$

where $0 < \alpha \leq 1$, x is spatial dimension of wave, t denotes the time, ω represents the depth averaged horizontal velocity of wave, ψ denotes free surface elevation (surface wave height), and the parameters β, β' are coefficients controlling dissipation and dispersion. This fractional model captures the influence of memory and hereditary properties on nonlinear wave propagation and extends the applicability of the WBKEs to systems where dissipative and dispersive effects coexist with long-term temporal dependence. In the following example, we will consider that wave maintains dispersive character and doesn't have dissipative smoothing, that is, $\beta = 0$ and $\beta' = 1$.

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Example 4.1. Consider the $\phi-$Caputo fractional WBKEs in shallow water wave phenomena:

$$\begin{cases} {}^{c}\mathcal{D}_{0,t}^{\alpha;\phi}\omega(x,t)+\omega(x,t)\omega_x(x,t)+\psi_x(x,t)=0, \\ {}^{c}\mathcal{D}_{0,t}^{\alpha;\phi}\psi(x,t)+[\omega(x,t)\psi(x,t)]_x-\omega_{xxx}(x,t)=0, \\ \omega(x,0)=5-0.2\coth(0.1x+1) \\ \psi(x,0)=-0.02\ \mathrm{csch}(0.1x+1). \end{cases} \tag{36}$$

If $\alpha=1$ then the exact solution of (36) is

$$\omega(x,t)=5-0.2\coth(0.1x-5t+1) \text{ and } \psi(x,t)=-0.02\ \mathrm{csch}(0.1x-5t+1). \tag{37}$$

Example 4.2. Consider the $\phi-$Caputo fractional WBKEs in shallow water wave phenomena:

$$\begin{cases} {}^{c}\mathcal{D}_{0,t}^{\alpha;\phi}\omega(x,t)+\omega(x,t)\omega_x(x,t)+\psi_x(x,t)=0, \\ {}^{c}\mathcal{D}_{0,t}^{\alpha;\phi}\psi(x,t)+[\omega(x,t)\psi(x,t)]_x-\omega_{xxx}(x,t)=0, \\ \omega(x,0)=4.74+0.0145x \\ \psi(x,0)=-0.017+0.00223x. \end{cases} \tag{38}$$

4.1 Using ϕ-CRPSM

In this subsection we will use ϕ-CRPSM in solving (36) in Example(4.1) and (38) in Example(4.2). In Example(4.1) by using the RPS technique, we assume the following series representations for the solution of (36):

$$\begin{cases} \omega(x,t)=h_0(x)+\sum_{n=1}^{\infty}h_n(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)} \\ \psi(x,t)=\hat{h}_0(x)+\sum_{n=1}^{\infty}\hat{h}_n(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}. \end{cases} \tag{39}$$

That is,

$$\begin{cases} \omega(x,t)=5-0.2\coth(0.1x+1)+\sum_{n=1}^{\infty}h_n(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)} \\ \psi(x,t)=-0.02\ \mathrm{csch}(0.1x+1)+\sum_{n=1}^{\infty}\hat{h}_n(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}. \end{cases} \tag{40}$$

Let $\langle\omega_m\rangle$ and $\langle\psi_m\rangle$ be the partial sum sequences of (40) where expressed as:

$$\begin{cases} \omega_m(x,t)=5-0.2\coth(0.1x+1)+\sum_{n=1}^{m}h_n(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)} \\ \psi_m(x,t)=-0.02\ \mathrm{csch}(0.1x+1)+\sum_{n=1}^{m}\hat{h}_n(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}. \end{cases} \tag{41}$$

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Let Res_ω and Res_ψ be the residual maps of (36) where expressed as:

$$\begin{cases} Res_\omega(x,t) = {}^c\mathcal{D}_{0,t}^{\alpha;\phi}\omega(x,t) + \omega(x,t)\omega_x(x,t) + \psi_x(x,t) \\ Res_\psi(x,t) = {}^c\mathcal{D}_{0,t}^{\alpha;\phi}\psi(x,t) + [\omega(x,t)\psi(x,t)]_x - \omega_{xxx}(x,t). \end{cases} \tag{42}$$

Thus, the residual functions truncated at index m are given by

$$\begin{cases} Res_{\omega_m}(x,t) = {}^c\mathcal{D}_{0,t}^{\alpha;\phi}\omega_m(x,t) + \omega_m(x,t)\omega_{mx}(x,t) + \psi_{mx}(x,t) \\ Res_{\psi_m}(x,t) = {}^c\mathcal{D}_{0,t}^{\alpha;\phi}\psi_m(x,t) + [\omega_m(x,t)\psi_m(x,t)]_x - \omega_{mxxx}(x,t). \end{cases} \tag{43}$$

Put (41) in (43) to get

$$\begin{aligned} Res_{\omega_m}(x,t) &= {}^c\mathcal{D}_{0,t}^{\alpha;\phi}\left[\sum_{n=1}^{m} h_n(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\right] \\ &+ \left[5 - 0.2\coth(0.1x+1) + \sum_{n=1}^{m} h_n(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\right] \\ &\times \left[0.02\,\mathrm{csch}^2(0.1x+1) + \sum_{n=1}^{m} h_n'(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\right] \\ &+ \left[0.002\ \mathrm{csch}(0.1x+1)\coth(0.1x+1) + \sum_{n=1}^{m} \hat{h}_n'(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\right] \end{aligned} \tag{44}$$

and

$$\begin{aligned} Res_{\psi_m}(x,t) &= {}^c\mathcal{D}_{0,t}^{\alpha;\phi}\left[\sum_{n=1}^{m} \hat{h}_n(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\right] \\ &+ \left[5 - 0.2\coth(0.1x+1) + \sum_{n=1}^{m} h_n(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\right] \\ &\times \left[0.002\,\mathrm{csch}(0.1x+1)\coth(0.1x+1) + \sum_{n=1}^{m} \hat{h}_n'(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\right] \\ &+ \left[0.02\,\mathrm{csch}^2(0.1x+1) + \sum_{n=1}^{m} h_n'(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\right] \\ &\times \left[-0.02\ \mathrm{csch}(0.1x+1) + \sum_{n=1}^{m} \hat{h}_n(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\right] \\ &- \Big\{0.0008\,\mathrm{csch}^2(0.1x+1)\coth^2(0.1x+1) + 0.0004\mathrm{csch}^4(0.1x+1) \\ &\qquad + \sum_{n=1}^{m} h_n'''(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\Big\}. \end{aligned} \tag{45}$$

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Take $n = 1$ in (41) to get

$$\begin{cases} \omega_1(x,t) = 5 - 0.2\coth(0.1x+1) + h_1(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\ \psi_1(x,t) = -0.02\ \mathrm{csch}(0.1x+1) + \hat{h}_1(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}. \end{cases} \tag{46}$$

Next, we compute the first residual functions by substituting (46) into (44) and (45).

$$\begin{aligned} Res_{\omega_1}(x,t) &= h_1(x) + \left[5 - 0.2\coth(0.1x+1) + h_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &\times \left[0.02\,\mathrm{csch}^2(0.1x+1) + h_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &+ \left[0.002\ \mathrm{csch}(0.1x+1)\coth(0.1x+1) + \hat{h}_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] \end{aligned} \tag{47}$$

and

$$\begin{aligned} Res_{\psi_1}(x,t) &= \hat{h}_1(x) + \left[5 - 0.2\coth(0.1x+1) + h_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &\times \left[0.002\,\mathrm{csch}(0.1x+1)\coth(0.1x+1) + \hat{h}_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &+ \left[0.02\,\mathrm{csch}^2(0.1x+1) + h_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] \\ &\times \left[-0.02\ \mathrm{csch}(0.1x+1) + \hat{h}_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] - \Big\{0.0008\,\mathrm{csch}^2(0.1x+1) \\ &\times \coth^2(0.1x+1) + 0.0004\mathrm{csch}^4(0.1x+1) + h_1'''(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\Big\}. \end{aligned} \tag{48}$$

In the next step, in (47) and (48) the truncated residual functions are computed at $\phi(t) = 0$, $Res_{\omega_1}(x,t) = 0$ and $Res_{\psi_1}(x,t) = 0$, yielding:

$$\begin{aligned} h_1(x) &= \Big[0.004\ \coth(0.1x+1) - 0.1\Big]\,\mathrm{csch}^2(0.1x+1) \\ &- 0.002\,\mathrm{csch}(0.1x+1)\coth(0.1x+1) \end{aligned} \tag{49}$$

and

$$\begin{aligned} \hat{h}_1(x) &= \Big[0.0004\coth(0.1x+1) - 0.01\Big]\,\mathrm{csch}(0.1x+1)\coth(0.1x+1) \\ &+ 0.0008\,\mathrm{csch}^2(0.1x+1)\coth^2(0.1x+1) + 0.0004\mathrm{csch}^4(0.1x+1) \\ &+ 0.0004\,\mathrm{csch}^3(0.1x+1). \end{aligned} \tag{50}$$

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Take $n = 2$ in (41) to get

$$\begin{cases} \omega_2(x,t) = 5 - 0.2\coth(0.1x+1) + h_1(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\ \quad + h_2(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)} \\ \psi_2(x,t) = -0.02\ \mathrm{csch}(0.1x+1) + \hat{h}_1(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\ \quad + \hat{h}_2(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}. \end{cases} \tag{51}$$

Next, we compute the first residual functions by substituting (51) into (44) and (45).

$$\begin{aligned} Res_{\omega_2}(x,t) &= h_1(x) + h_2(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\ &+ \left[5 - 0.2\coth(0.1x+1) + h_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + h_2(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\ &\times \left[0.02\,\mathrm{csch}^2(0.1x+1) + h_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + h_2'(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\ &+ \left[0.002\ \mathrm{csch}(0.1x+1)\coth(0.1x+1) + \hat{h}_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right. \\ &\left.+\hat{h}_2'(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \end{aligned} \tag{52}$$

and

$$\begin{aligned} Res_{\psi_m}(x,t) &= \hat{h}_1(x) + \hat{h}_2(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\ &+ \left[5 - 0.2\coth(0.1x+1) + h_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + h_2(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\ &\times \left[0.002\,\mathrm{csch}(0.1x+1)\coth(0.1x+1) + \hat{h}_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right. \\ &\left.+\hat{h}_2'(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\ &+ \left[0.02\,\mathrm{csch}^2(0.1x+1) + h_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + h_2'(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\ &\times \left[-0.02\ \mathrm{csch}(0.1x+1) + \hat{h}_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + \hat{h}_2(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \end{aligned}$$

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$$
\begin{aligned}
&- \Big\{0.0008\,\mathrm{csch}^2(0.1x+1)\coth^2(0.1x+1) + 0.0004\mathrm{csch}^4(0.1x+1) \\
&+ h_1'''(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + h_2'''(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\Big\}.
\end{aligned} \tag{53}
$$

In the next step, in (52) and (53) the truncated residual functions are computed at $\phi(t)=0$, ${}^{c}\mathcal{D}_{0,t}^{\alpha;\phi} Res_{\omega_2}(x,t)=0$ and ${}^{c}\mathcal{D}_{0,t}^{\alpha;\phi} Res_{\psi_2}(x,t)=0$, yielding:

$$
h_2(x) = (0.2\,\coth(0.1x+1)-5)h_1' - 0.02\,\mathrm{csch}^2(0.1x+1)h_1 - \hat{h}_1' \tag{54}
$$

and

$$
\begin{aligned}
\hat{h}_2(x) &= 0.02\,\mathrm{csch}(0.1x+1)h_1' - 0.02\,\mathrm{csch}^2(0.1x+1)\hat{h}_1 \\
&- 0.008\,\mathrm{csch}(0.1x+1)\coth(0.1x+1)h_1 - (5-0.02\coth(0.1x+1))\hat{h}_1'.
\end{aligned} \tag{55}
$$

Hence the approximate solution of (36) for three terms is given by

$$
\begin{aligned}
&\omega(x,t) = 5 - 0.2\coth(0.1x+1) \\
&+ \frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\Big[(0.004\,\coth(0.1x+1)-0.1)\,\mathrm{csch}^2(0.1x+1) \\
&- 0.002\,\mathrm{csch}(0.1x+1)\coth(0.1x+1)\Big] \\
&+ \frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\Big[(0.2\,\coth(0.1x+1)-5)h_1' - 0.02\,\mathrm{csch}^2(0.1x+1)h_1 - \hat{h}_1'\Big] + \cdots
\end{aligned} \tag{56}
$$

and

$$
\begin{aligned}
&\psi(x,t) = -0.02\,\mathrm{csch}(0.1x+1) + \frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\Big[(0.0004\coth(0.1x+1)-0.01) \\
&\times \mathrm{csch}(0.1x+1)\coth(0.1x+1) + 0.0008\,\mathrm{csch}^2(0.1x+1)\coth^2(0.1x+1) \\
&+ 0.0004\mathrm{csch}^4(0.1x+1) + 0.0004\,\mathrm{csch}^3(0.1x+1)\Big] \\
&+ \frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\Big[0.02\,\mathrm{csch}(0.1x+1)h_1' - 0.02\,\mathrm{csch}^2(0.1x+1)\hat{h}_1 \\
&- 0.008\,\mathrm{csch}(0.1x+1)\coth(0.1x+1)h_1 - (5-0.02\coth(0.1x+1))\hat{h}_1'\Big] + \cdots
\end{aligned} \tag{57}
$$

If we take $\phi(t)=t$ then the approximate solution of (36) becomes

$$
\begin{aligned}
&\omega(x,t) = 5 - 0.2\coth(0.1x+1) \\
&+ \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big[(0.004\,\coth(0.1x+1)-0.1)\,\mathrm{csch}^2(0.1x+1) \\
&- 0.002\,\mathrm{csch}(0.1x+1)\coth(0.1x+1)\Big] \\
&+ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big[(0.2\,\coth(0.1x+1)-5)h_1' - 0.02\,\mathrm{csch}^2(0.1x+1)h_1 - \hat{h}_1'\Big] + \cdots
\end{aligned} \tag{58}
$$

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and

$$\begin{aligned}\psi(x,t) &= -0.02\ \mathrm{csch}(0.1x+1) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big[(0.0004\coth(0.1x+1) - 0.01)\\ &\times \mathrm{csch}(0.1x+1)\coth(0.1x+1) + 0.0008\,\mathrm{csch}^2(0.1x+1)\coth^2(0.1x+1)\\ &+ 0.0004\mathrm{csch}^4(0.1x+1) + 0.0004\,\mathrm{csch}^3(0.1x+1)\Big]\\ &+ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big[0.02\,\mathrm{csch}(0.1x+1)h_1' - 0.02\,\mathrm{csch}^2(0.1x+1)\hat{h}_1\\ &- 0.008\,\mathrm{csch}(0.1x+1)\coth(0.1x+1)h_1 - (5 - 0.02\coth(0.1x+1))\hat{h}_1'\Big] + \cdots\end{aligned} \tag{59}$$

In Example(4.1) by using the RPS technique, we assume the following series representations for the solution of (36):

$$\begin{cases}\omega(x,t) = 4.74 + 0.0145x + \displaystyle\sum_{n=1}^{\infty} h_n(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\\ \psi(x,t) = -0.017 + 0.00223x + \displaystyle\sum_{n=1}^{\infty} \hat{h}_n(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}.\end{cases} \tag{60}$$

Let $\langle\omega_m\rangle$ and $\langle\psi_m\rangle$ be the partial sum sequences of (40) where expressed as:

$$\begin{cases}\omega_m(x,t) = 4.74 + 0.0145x + \displaystyle\sum_{n=1}^{m} h_n(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}\\ \psi_m(x,t) = -0.017 + 0.00223x + \displaystyle\sum_{n=1}^{m} \hat{h}_n(x)\frac{\big(\phi(t)-\phi(0)\big)^{n\alpha}}{\Gamma(n\alpha+1)}.\end{cases} \tag{61}$$

Take $n = 1$ in (61) to get

$$\begin{cases}\omega_1(x,t) = 4.74 + 0.0145x + h_1(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\\ \psi_1(x,t) = -0.017 + 0.00223x + \hat{h}_1(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}.\end{cases} \tag{62}$$

Next, we compute the first residual functions

$$\begin{aligned}Res_{\omega_1}(x,t) &= h_1(x) + \left[4.74 + 0.0145x + h_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right]\\ &\times \left[0.0145 + h_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] + 0.00223 + \hat{h}_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\end{aligned} \tag{63}$$

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and

$$
\begin{aligned}
Res_{\psi_1}(x,t) &= \hat{h}_1(x) + \left[4.74 + 0.0145x + h_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&\times \left[0.00223 + \hat{h}_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] + \left[0.0145 + h_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] \\
&\times \left[-0.017 + 0.00223x + \hat{h}_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\right] - h_1'''(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned} \tag{64}
$$

In the next step, in (47) and (64) the truncated residual functions are computed at $\phi(t) = 0$, $Res_{\omega_1}(x,t) = 0$ and $Res_{\psi_1}(x,t) = 0$, yielding:

$$
h_1(x) = -0.07096 - 0.00021x \text{ and } \hat{h}_1(x) = -0.008 - 0.002198x. \tag{65}
$$

Take $n = 2$ in (61) to get

$$
\begin{cases}
\omega_2(x,t) = 4.74 + 0.0145x + h_1(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + h_2(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)} \\
\psi_2(x,t) = -0.017 + 0.00223x + \hat{h}_1(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + \hat{h}_2(x)\dfrac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{cases} \tag{66}
$$

Next, we compute the first residual functions by substituting (51) into (**??**) and (**??**).

$$
\begin{aligned}
Res_{\omega_2}(x,t) &= h_1(x) + h_2(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \left[4.74 + 0.0145x + h_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + h_2(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\
&\times \left[0.0145 + h_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + h_2'(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\
&+ 0.00223 + \hat{h}_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + \hat{h}_2'(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}
\end{aligned} \tag{67}
$$

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and

$$
\begin{aligned}
Res_{\psi_m}(x,t) &= \hat{h}_1(x) + \hat{h}_2(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \left[4.74 + 0.0145x + h_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + h_2(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\
&\times \left[0.00223 + \hat{h}_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + \hat{h}_2'(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\
&+ \left[0.0145 + h_1'(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + h_2'(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\
&\times \left[-0.017 + 0.00223x + \hat{h}_1(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} + \hat{h}_2(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\
&- h_1'''(x)\frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)} - h_2'''(x)\frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned} \tag{68}
$$

In the next step, in (67) and (53) the truncated residual functions are computed at $\phi(t) = 0$, ${}^{c}\mathcal{D}_{0,t}^{\alpha;\phi} Res_{\omega_2}(x,t) = 0$ and ${}^{c}\mathcal{D}_{0,t}^{\alpha;\phi} Res_{\psi_2}(x,t) = 0$, yielding:

$$
h_2(x) = 0.0042 + 0.000006x \text{ and } \hat{h}_2(x) = 0.01 + 0.000064x. \tag{69}
$$

Hence the approximate solution of (36) for three terms is given by

$$
\begin{aligned}
\omega(x,t) &= 4.74 + 0.0145x + \frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\Big[-0.07096 - 0.00021x\Big] \\
&+ \frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\Big[0.0042 + 0.000006x\Big] + \cdots
\end{aligned} \tag{70}
$$

and

$$
\begin{aligned}
\psi(x,t) &= -0.017 + 0.00223x + \frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\Big[-0.008 - 0.002198x\Big] \\
&+ \frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\Big[0.01 + 0.000064x\Big] + \cdots
\end{aligned} \tag{71}
$$

If we take $\phi(t) = t$ then the approximate solution of (38) becomes

$$
\begin{aligned}
\omega(x,t) &= 4.74 + 0.0145x + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big[-0.07096 - 0.00021x\Big] \\
&+ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big[0.0042 + 0.000006x\Big] + \cdots
\end{aligned} \tag{72}
$$

and

$$
\begin{aligned}
\psi(x,t) &= -0.017 + 0.00223x + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big[-0.008 - 0.002198x\Big] \\
&+ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big[0.01 + 0.000064x\Big] + \cdots
\end{aligned} \tag{73}
$$

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4.2 Using ϕ-CNIM

In this subsection we will use ϕ-CNIM in solving (36) in Example(4.1) and (38) in Example(4.2). For Example(4.1), the initial conditions are met by ω and ψ, allowing them to be rewritten as:

$$\begin{cases} \omega_0(x,t) = \omega(x,0) = h_0(x) = 5 - 0.2\coth(0.1x+1) \\ \psi_0(x,t) = \psi(x,0) = \hat{h}_0(x) = -0.02\ \mathrm{csch}(0.1x+1). \end{cases} \tag{74}$$

Applying $J_{0t}^{\alpha;\phi}$ to each equation in (36), we obtain

$$\begin{aligned} \omega(x,t) &= 5 - 0.2\coth(0.1x+1) - J_{0t}^{\alpha;\phi}\big[\omega\,\omega_x + \psi_x\big](x,t), \\ \psi(x,t) &= -0.02\ \mathrm{csch}(0.1x+1) - J_{0t}^{\alpha;\phi}\big[\omega_x\psi + \omega\psi_x - \omega_{xxx}\big](x,t). \end{aligned} \tag{75}$$

Hence

$$\begin{aligned} \omega(x,t) &= 5 - 0.2\coth(0.1x+1) + \mathcal{N}_\omega(\omega,\psi)(x,t), \\ \psi(x,t) &= -0.02\ \mathrm{csch}(0.1x+1) + \mathcal{N}_\psi(\omega,\psi)(x,t), \end{aligned} \tag{76}$$

where

$$\begin{aligned} \mathcal{N}_\omega(\omega,\psi)(x,t) &:= -J_{0t}^{\alpha;\phi}\big[\omega\,\omega_x + \psi_x\big](x,t), \\ \mathcal{N}_\psi(\omega,\psi)(x,t) &:= -J_{0t}^{\alpha;\phi}\big[\omega_x\psi + \omega\psi_x - \omega_{xxx}\big](x,t). \end{aligned} \tag{77}$$

Assume the series expansions

$$\omega(x,t) = \sum_{n=0}^{\infty}\omega_n(x,t), \qquad \psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t). \tag{78}$$

For our system, apply (33) and (34) to get

$$\begin{aligned} &\omega_0(x,t) = 5 - 0.2\coth(0.1x+1) \\ &\omega_1(x,t) = \mathcal{N}_\omega(\omega_0,\psi_0)(x,t), \\ &\omega_2(x,t) = \mathcal{N}_\omega(\omega_0+\omega_1,\psi_0+\psi_1)(x,t) - \mathcal{N}_\omega(\omega_0,\psi_0)(x,t), \\ &\vdots \end{aligned} \tag{79}$$

and

$$\begin{aligned} &\psi_0(x,t) = -0.02\ \mathrm{csch}(0.1x+1), \\ &\psi_1(x,t) = \mathcal{N}_\psi(\omega_0,\psi_0)(x,t), \\ &\psi_2(x,t) = \mathcal{N}_\psi(\omega_0+\omega_1,\psi_0+\psi_1)(x,t) - \mathcal{N}_\psi(\omega_0,\psi_0)(x,t) \\ &\vdots \end{aligned} \tag{80}$$

Hence at $n = 1$

$$\begin{aligned} \omega_1(x,t) = \frac{(\phi(t)-\phi(0))^\alpha}{\Gamma(\alpha+1)}\Big[&(0.004\ \coth(0.1x+1) - 0.1)\,\mathrm{csch}^2(0.1x+1) \\ &+ 0.002\,\mathrm{csch}(0.1x+1)\ \coth(0.1x+1)\Big], \end{aligned} \tag{81}$$

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and

$$\psi_1(x,t) = \frac{(\phi(t)-\phi(0))^{\alpha}}{\Gamma(\alpha+1)}\Big[(0.01 - 0.0004\,\coth(0.1x+1))\,\mathrm{csch}(0.1x+1) \times \coth(0.1x+1) - 0.0008\,\mathrm{csch}^2(0.1x+1)\,\coth^2(0.1x+1) - 0.0004\,\mathrm{csch}^4(0.1x+1) - 0.0004\,\mathrm{csch}^3(0.1x+1)\Big]. \tag{82}$$

At $n = 2$ we have

$$\omega_2(x,t) = \frac{(\phi(t)-\phi(0))^{2\alpha}}{\Gamma(2\alpha+1)}\Big[h\,\mathbb{E}'_{\mathcal{U}} + \mathbb{E}_{\mathcal{U}}\,h' + \mathbb{E}'_{\mathcal{V}}\Big](x), \tag{83}$$

and

$$\psi_2(x,t) = \frac{(\phi(t)-\phi(0))^{2\alpha}}{\Gamma(2\alpha+1)}\Big[h'_0\,\mathbb{E}_{\mathcal{V}} + \mathbb{E}'_{\mathcal{U}}\,\hat{h}_0 + h_0\,\mathbb{E}'_{\mathcal{V}} + \mathbb{E}_{\mathcal{U}}\,\hat{h}'_0 - \mathbb{E}'''_{\mathcal{U}}\Big](x) \tag{84}$$

where

$$\mathbb{E}_{\mathcal{U}}(x) := h_0(x)\,h'_0(x) + \hat{h}'_0(x),$$
$$\mathbb{E}_{\mathcal{V}}(x) := \big(h_0(x)\,\hat{h}_0(x)\big)' - h'''_0(x),$$

Hence the approximate solution of (36) for three terms is given by

$$\omega(x,t) = 5 - 0.2\coth(0.1x+1) + \frac{(\phi(t)-\phi(0))^{\alpha}}{\Gamma(\alpha+1)}\Big[(0.004\,\coth(0.1x+1) - 0.1)\,\mathrm{csch}^2(0.1x+1) + 0.002\,\mathrm{csch}(0.1x+1)\,\coth(0.1x+1)\Big] + \frac{(\phi(t)-\phi(0))^{2\alpha}}{\Gamma(2\alpha+1)}\Big[h\,\mathbb{E}'_{\mathcal{U}} + \mathbb{E}_{\mathcal{U}}\,h' + \mathbb{E}'_{\mathcal{V}}\Big](x) + \cdots \tag{85}$$

and

$$\psi(x,t) = -0.02\,\mathrm{csch}(0.1x+1) + \frac{(\phi(t)-\phi(0))^{\alpha}}{\Gamma(\alpha+1)}\Big[(0.01 - 0.0004\,\coth(0.1x+1))\,\mathrm{csch}(0.1x+1) \times \coth(0.1x+1) - 0.0008\,\mathrm{csch}^2(0.1x+1)\,\coth^2(0.1x+1) - 0.0004\,\mathrm{csch}^4(0.1x+1) - 0.0004\,\mathrm{csch}^3(0.1x+1)\Big] + \frac{(\phi(t)-\phi(0))^{2\alpha}}{\Gamma(2\alpha+1)}\Big[h'_0\,\mathbb{E}_{\mathcal{V}} + \mathbb{E}'_{\mathcal{U}}\,\hat{h}_0 + h_0\,\mathbb{E}'_{\mathcal{V}} + \mathbb{E}_{\mathcal{U}}\,\hat{h}'_0 - \mathbb{E}'''_{\mathcal{U}}\Big](x) + \cdots \tag{86}$$

If we take $\phi(t) = t$ then the approximate solution of (36) becomes

$$\omega(x,t) = 5 - 0.2\coth(0.1x+1) + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big[(0.004\,\coth(0.1x+1) - 0.1)\,\mathrm{csch}^2(0.1x+1) + 0.002\,\mathrm{csch}(0.1x+1)\,\coth(0.1x+1)\Big] + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big[h\,\mathbb{E}'_{\mathcal{U}} + \mathbb{E}_{\mathcal{U}}\,h' + \mathbb{E}'_{\mathcal{V}}\Big](x) + \cdots \tag{87}$$

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and

$$\begin{aligned}\psi(x,t) &= -0.02\ \mathrm{csch}(0.1x+1)\\ &+ \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big[(0.01-0.0004\ \coth(0.1x+1))\,\mathrm{csch}(0.1x+1)\\ &\times \coth(0.1x+1) - 0.0008\,\mathrm{csch}^2(0.1x+1)\ \coth^2(0.1x+1)\\ &- 0.0004\,\mathrm{csch}^4(0.1x+1) - 0.0004\,\mathrm{csch}^3(0.1x+1)\Big]\\ &+ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\left[h_0'\,\mathbb{E}_{\mathcal{V}} + \mathbb{E}_{\mathcal{U}}'\,\hat{h}_0 + h_0\,\mathbb{E}_{\mathcal{V}}' + \mathbb{E}_{\mathcal{U}}\,\hat{h}_0' - \mathbb{E}_{\mathcal{U}}'''\right](x) + \cdots\end{aligned} \tag{88}$$

For Example(4.2), the initial conditions are met by ω and ψ, allowing them to be rewritten as:

$$\begin{cases}\omega_0(x,t) = \omega(x,0) = h_0(x) = 4.74 + 0.0145x\\ \psi_0(x,t) = \psi(x,0) = \hat{h}_0(x) = -0.017 + 0.00223x.\end{cases} \tag{89}$$

Applying $J_{0t}^{\alpha;\phi}$ to each equation in (38), we obtain

$$\begin{aligned}\omega(x,t) &= 4.74 + 0.0145x - J_{0t}^{\alpha;\phi}\big[\omega\,\omega_x + \psi_x\big](x,t),\\ \psi(x,t) &= -0.017 + 0.00223x - J_{0t}^{\alpha;\phi}\big[\omega_x\psi + \omega\psi_x - \omega_{xxx}\big](x,t).\end{aligned} \tag{90}$$

Hence

$$\begin{aligned}\omega(x,t) &= 4.74 + 0.0145x + \mathcal{N}_{\omega}(\omega,\psi)(x,t),\\ \psi(x,t) &= -0.017 + 0.00223x + \mathcal{N}_{\psi}(\omega,\psi)(x,t),\end{aligned} \tag{91}$$

where

$$\begin{aligned}\mathcal{N}_{\omega}(\omega,\psi)(x,t) &:= -J_{0t}^{\alpha;\phi}\big[\omega\,\omega_x + \psi_x\big](x,t),\\ \mathcal{N}_{\psi}(\omega,\psi)(x,t) &:= -J_{0t}^{\alpha;\phi}\big[\omega_x\psi + \omega\psi_x - \omega_{xxx}\big](x,t).\end{aligned} \tag{92}$$

Assume the series expansions

$$\omega(x,t) = \sum_{n=0}^{\infty}\omega_n(x,t), \qquad \psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t). \tag{93}$$

For our system, apply (33) and (34) to get

$$\begin{aligned}&\omega_0(x,t) = 4.74 + 0.0145x\\ &\omega_1(x,t) = \mathcal{N}_{\omega}(\omega_0,\psi_0)(x,t),\\ &\omega_2(x,t) = \mathcal{N}_{\omega}(\omega_0+\omega_1,\psi_0+\psi_1)(x,t) - \mathcal{N}_{\omega}(\omega_0,\psi_0)(x,t),\\ &\vdots\end{aligned} \tag{94}$$

and

$$\begin{aligned}&\psi_0(x,t) = -0.017 + 0.00223x,\\ &\psi_1(x,t) = \mathcal{N}_{\psi}(\omega_0,\psi_0)(x,t),\\ &\psi_2(x,t) = \mathcal{N}_{\psi}(\omega_0+\omega_1,\psi_0+\psi_1)(x,t) - \mathcal{N}_{\psi}(\omega_0,\psi_0)(x,t)\\ &\vdots\end{aligned} \tag{95}$$

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Similarly, we have

$$h_1(x) = -0.07096 - 0.00021x \text{ and } \hat{h}_1(x) = -0.008 - 0.002198x \tag{96}$$

and

$$h_2(x) = 0.0042 + 0.000006x \text{ and } \hat{h}_2(x) = 0.01 + 0.000064x. \tag{97}$$

Hence the approximate solution of (36) for three terms is given by

$$\begin{aligned}\omega(x,t) &= 4.74 + 0.0145x + \frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\Big[-0.07096 - 0.00021x\Big] \\ &+ \frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\Big[0.0042 + 0.000006x\Big] + \cdots\end{aligned} \tag{98}$$

and

$$\begin{aligned}\psi(x,t) &= -0.017 + 0.00223x + \frac{\big(\phi(t)-\phi(0)\big)^{\alpha}}{\Gamma(\alpha+1)}\Big[-0.008 - 0.002198x\Big] \\ &+ \frac{\big(\phi(t)-\phi(0)\big)^{2\alpha}}{\Gamma(2\alpha+1)}\Big[0.01 + 0.000064x\Big] + \cdots\end{aligned} \tag{99}$$

If we take $\phi(t) = t$ then the approximate solution of (38) becomes

$$\begin{aligned}\omega(x,t) &= 4.74 + 0.0145x + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big[-0.07096 - 0.00021x\Big] \\ &+ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big[0.0042 + 0.000006x\Big] + \cdots\end{aligned} \tag{100}$$

and

$$\begin{aligned}\psi(x,t) &= -0.017 + 0.00223x + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big[-0.008 - 0.002198x\Big] \\ &+ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big[0.01 + 0.000064x\Big] + \cdots\end{aligned} \tag{101}$$

5 Numerical discussion

In this section, we summarize and interpret the numerical findings presented in Tables 1-2 and Figures 1-6. These results compare the performance of the φ-Residual Power Series Method (φ-RPSM) and the φ-New Iterative Method (φ-NIM) in approximating the solutions of the fractional WBKEs for (36) in Example(4.1) and (38) in Example(4.2). The fractional orders $\alpha = 0.25, 0.50, 0.75$, and 1 are selected to represent different levels of memory effects, ranging from strong fractional behavior to the classical integer-order case, allowing clear observation of the influence of the fractional parameter on the obtained solutions. The tables provide quantitative accuracy assessments, while the figures offer graphical insight into the behavior,

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Table 1: Comparing AE of exact, ϕ-CNIM and ϕ-CRPSM solutions $\omega(x,t)$ of (36) in Example(4.1) at $\alpha = 1$.

x	t	$\omega(x,t)_{\text{Exact}}$	$\omega(x,t)_{\phi\text{-CRPSM}}$	$\omega(x,t)_{\phi\text{-CNIM}}$	$\text{AE}_{\phi\text{-CRPSM}}$	$\text{AE}_{\phi\text{-CNIM}}$
2.00	0.10	4.669076000	4.668609092	4.668141184	0.000466908	0.000934816
	0.20	3.986702000	3.986303330	3.985905000	0.000398670	0.000797000
	0.30	5.686548000	5.685979345	5.685410690	0.000568655	0.001137310
	0.40	5.301188000	5.300657881	5.300127762	0.000530119	0.001060238
	0.50	5.232093000	5.231569791	5.231046582	0.000523209	0.001046418
4.00	0.10	4.720787000	4.720314921	4.719842842	0.000472079	0.000944158
	0.20	4.473614000	4.473166264	4.472718528	0.000447736	0.000895472
	0.30	7.006662000	7.005961334	7.005260668	0.000700666	0.001401332
	0.40	5.372405000	5.371867760	5.371330520	0.000537240	0.001074480
	0.50	5.249844000	5.249319016	5.248794032	0.000524984	0.001049968
6.00	0.10	4.750156000	4.749681984	4.749207968	0.000474016	0.000948032
	0.20	4.627595000	4.627132241	4.626669482	0.000462759	0.000925518
	0.30	2.993338000	2.993038666	2.992739332	0.000299334	0.000598668
	0.40	5.526386000	5.525833736	5.525281472	0.000552264	0.001104528
	0.50	5.279213000	5.278685079	5.278157158	0.000527921	0.001055842
8.00	0.10	4.767907000	4.767430209	4.766953418	0.000476791	0.000953582
	0.20	4.698812000	4.698342119	4.697872238	0.000469881	0.000939762
	0.30	4.313452000	4.313020655	4.312589310	0.000431345	0.000862690
	0.40	6.013298000	6.012696670	6.012095340	0.000601330	0.001202660
	0.50	5.330924000	5.330390908	5.329857816	0.000533092	0.001066184

Table 2: Comparing AE of exact, ϕ-CNIM and ϕ-CRPSM solutions $\psi(x,t)$ of (36) in Example(4.1) at $\alpha = 1$.

x	t	$\psi(x,t)_{\text{Exact}}$	$\psi(x,t)_{\phi\text{-CRPSM}}$	$\psi(x,t)_{\phi\text{-CNIM}}$	$\text{AE}_{\phi\text{-CRPSM}}$	$\text{AE}_{\phi\text{-CNIM}}$
2.00	0.10	-0.026365000	-0.026362364	-0.026359727	0.000002636	0.000005273
	0.20	-0.099336000	-0.099326066	-0.099316133	0.000009934	0.000019867
	0.30	0.065677000	0.065670432	0.065663865	0.000006568	0.000013135
	0.40	0.022520000	0.022517748	0.022515496	0.000002252	0.000004504
	0.50	0.011776000	0.011774823	0.011773646	0.000001176	0.000002352
4.00	0.10	-0.019483000	-0.019481052	-0.019479104	0.000001948	0.000003896
	0.20	-0.048691000	-0.048686131	-0.048681262	0.000004869	0.000009738
	0.30	0.199667000	0.199647033	0.199627067	0.000019967	0.000039934
	0.40	0.031414000	0.031410859	0.031407718	0.000003141	0.000006282
	0.50	0.014974000	0.014972502	0.014971005	0.000001498	0.000002995
6.00	0.10	-0.014974000	-0.014972502	-0.014971005	0.000001498	0.000002995
	0.20	-0.031414000	-0.031410859	-0.031407718	0.000003141	0.000006282
	0.30	-0.199667000	-0.199647033	-0.199627067	0.000019967	0.000039934
	0.40	0.048691000	0.048686131	0.048681262	0.000004869	0.000009738
	0.50	0.019483000	0.019481052	0.019479104	0.000001948	0.000003896
8.00	0.10	-0.011776000	-0.011774823	-0.011773646	0.000001176	0.000002352
	0.20	-0.022520000	-0.022517748	-0.022515496	0.000002252	0.000004504
	0.30	-0.065677000	-0.065670432	-0.065663865	0.000006568	0.000013135
	0.40	0.099336000	0.099326066	0.099316133	0.000009934	0.000019867
	0.50	0.026365000	0.026362364	0.026359727	0.000002636	0.000005273

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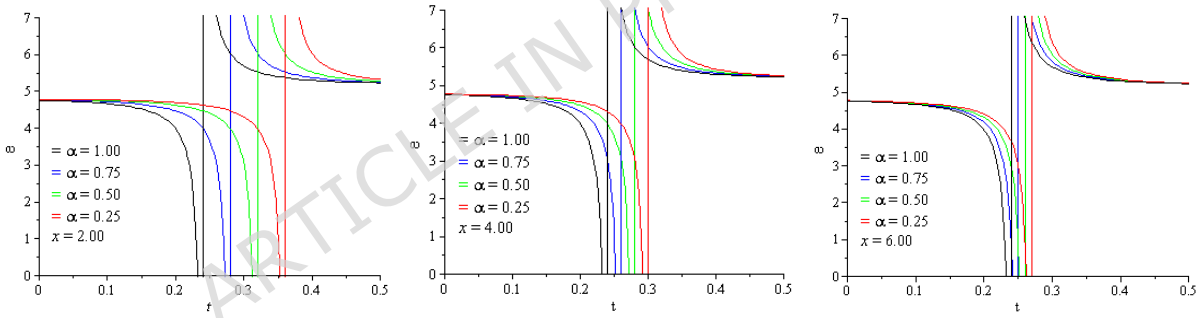
Table 3: Some ϕ-CNIM and ϕ-CRPSM solutions of (38) in Example(4.2) at $\alpha = 1$.

x	t	α	$\omega(x,t)_{\phi\text{-CRPSM}}$	$\psi(x,t)_{\phi\text{-CRPSM}}$	$\omega(x,t)_{\phi\text{-CNIM}}$	$\psi(x,t)_{\phi\text{-CNIM}}$
2.00	0.10	0.25	4.726218000	-0.016617000	4.725745378	-0.016615338
		0.50	4.743951000	-0.015950000	4.743476605	-0.015948405
		0.75	4.755289000	-0.014698000	4.754813471	-0.014696530
	0.20	0.25	4.718462000	-0.016575000	4.717990154	-0.016573342
		0.50	4.733822000	-0.016770000	4.733348618	-0.016768323
		0.75	4.746056000	-0.015892000	4.745581394	-0.015890411
	0.30	0.25	4.713321000	-0.016402000	4.712849668	-0.016400360
		0.50	4.726148000	-0.017163000	4.725675385	-0.017161284
		0.75	4.738038000	-0.016755000	4.737564246	-0.016753325
4.00	0.10	0.25	4.754962000	-0.014838000	4.754486504	-0.014836516
		0.50	4.772802000	-0.013046000	4.772324720	-0.013044695
		0.75	4.784208000	-0.011085000	4.783729579	-0.011083891
	0.20	0.25	4.747158000	-0.015294000	4.746683284	-0.015292471
		0.50	4.762613000	-0.014502000	4.762136739	-0.014500550
		0.75	4.774920000	-0.012854000	4.774442508	-0.012852714
	0.30	0.25	4.741985000	-0.015452000	4.741510801	-0.015450455
		0.50	4.754892000	-0.015381000	4.754416511	-0.015379462
		0.75	4.766854000	-0.014219000	4.766377314	-0.014217578



(a) Curves at $x = 2.00$ (b) Curves at $x = 4.00$ (c) Curves at $t = 6.00$

Figure 1: ϕ-CRPSM curves for the solutions $\omega(x,t)$ of (36) in Example(4.1).

stability, and convergence of each method. Table 1 presents numerical values of the solution $\omega(x,t)$ for (36) in Example(4.1) at $\alpha = 1$. It lists the exact solution alongside the φ-RPSM and φ-NIM approximations for various x and t. The absolute errors show that both methods achieve high accuracy, typically between 10^{-6} and 10^{-4}, with φ-RPSM consistently yielding slightly smaller deviations from the exact solution. Table 2 contains the corresponding numerical comparison for $\psi(x,t)$ in (36) in Example(4.1). As in Table 1, the exact values are contrasted with the approximations obtained by both schemes. The errors remain extremely small, confirming that both methods accurately reproduce the dynamic behavior of ψ. Once again, φ-RPSM demonstrates a marginal accuracy advantage over φ-NIM. Table 3 displays computed values of $\omega(x,t)$ and $\psi(x,t)$ for (38) in Example(4.2) at fractional orders $\alpha = 0.25$ and $\alpha = 0.50$. Since an exact solution is not available for this case, only the approximate solutions from φ-RPSM and φ-NIM are shown. The results exhibit excellent agreement between the two methods, with differences remaining on the order of 10^{-4} to 10^{-6}. The values vary smoothly with x, t, and α, clearly reflecting fractional-order effects on wave propaga-

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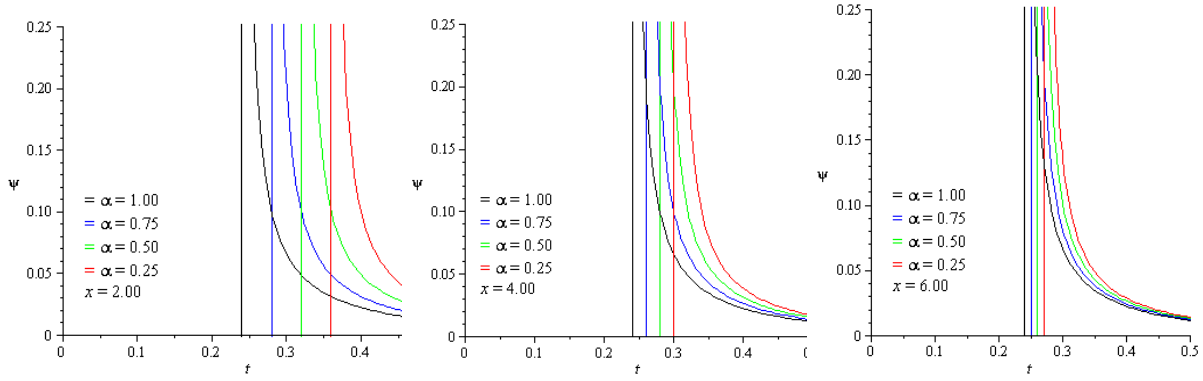
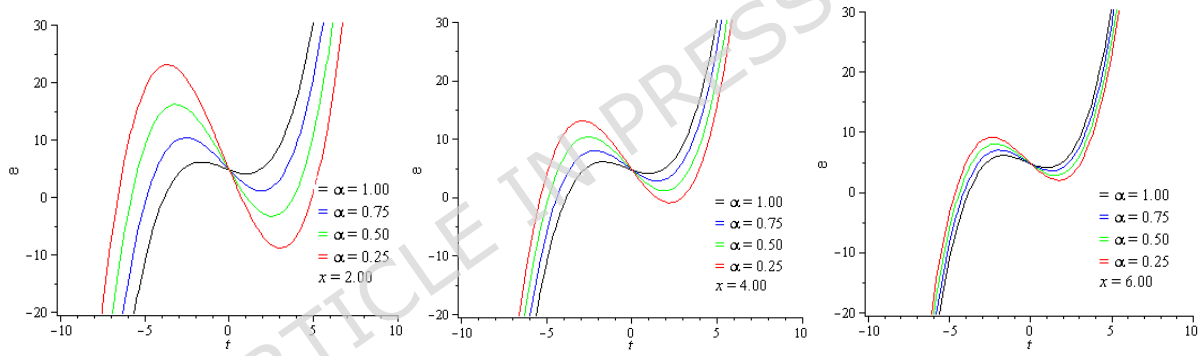


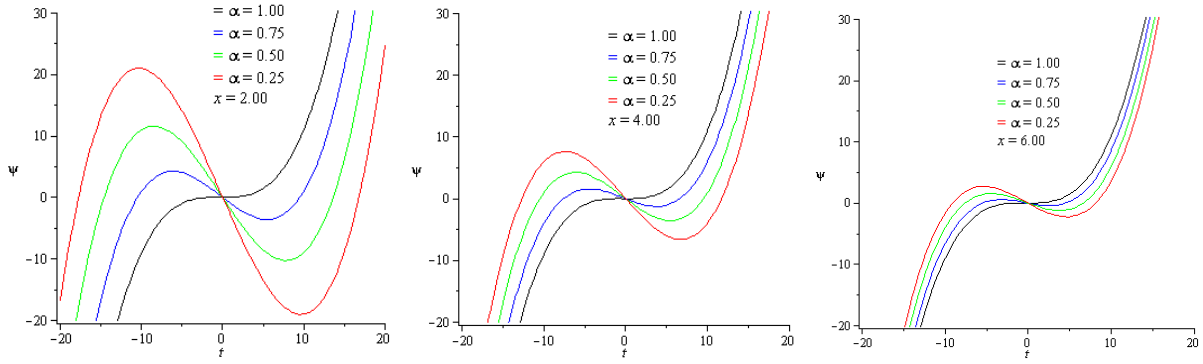
(a) Curves at $x = 2.00$ (b) Curves at $x = 4.00$ (c) Curves at $t = 6.00$

Figure 2: ϕ-CRPSM curves for the solutions $\psi(x,t)$ of (36) in Example(4.1).



(a) Curves at $x = 2.00$ (b) Curves at $x = 4.00$ (c) Curves at $t = 6.00$

Figure 3: ϕ-CRPSM curves for the solutions $\omega(x,t)$ of (38) in Example(4.2).



(a) Curves at $x = 2.00$ (b) Curves at $x = 4.00$ (c) Curves at $t = 6.00$

Figure 4: ϕ-CRPSM curves for the solutions $\psi(x,t)$ of (38) in Example(4.2).

tion. Figure 1 shows φ-RPSM curves of $\omega(x,t)$ for Example(4.1) at $x = 2$, $x = 4$, and fixed $t = 6$. The plots are smooth and stable, illustrating accurate wave profiles at different spatial

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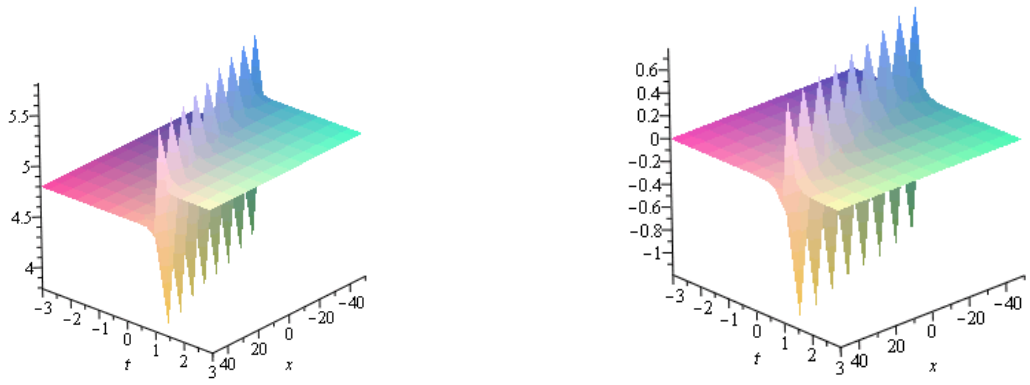
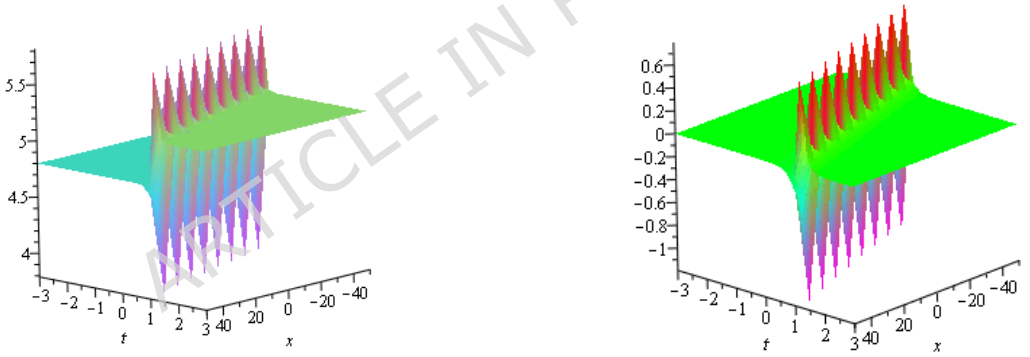


(a) Surface of $\omega(x,t)$ (b) Surface of $\psi(x,t)$

Figure 5: ϕ-CRPSM surface for the solutions of (36) in Example(4.1) at $\alpha = 1.00$.



(a) Surface of $\omega(x,t)$ (b) Surface of $\psi(x,t)$

Figure 6: ϕ-CRPSM surface for the solutions of (38) in Example(4.2) at $\alpha = 1.00$.

positions. In Figure 2, the φ-RPSM approximations for $\psi(x,t)$ are displayed for the same parameter values as in Figure 1. The curves reflect the correct dynamical structure of the coupled system and reveal the stable performance of the method. Figure 3 provides φ-RPSM curves of $\omega(x,t)$ for Example(4.2). Despite different initial conditions, the method maintains its accuracy, producing smooth solution profiles. Figure 4 presents φ-RPSM curves for $\psi(x,t)$ in Example(4.2). The plots confirm the scheme's robustness for both solution components. Figure 5 displays 3D surface plots of $\omega(x,t)$ and $\psi(x,t)$ for Example(4.1) at $\alpha = 1$. The surfaces exhibit stable and physically realistic wave propagation patterns across space and time. In Figure 6, although not included explicitly in the extracted text, this figure follows the pattern of Figure 1 but for φ-NIM, showing curves of $\omega(x,t)$. The method yields results closely matching the φ-RPSM curves. Hence Tables 1-2 and Figures 1-6 collectively demonstrate that both the φ-RPSM and φ-NIM approaches provide highly accurate and stable

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approximations to the fractional WBKE system. The absolute errors reported in the tables are very small, showcasing excellent quantitative performance. The graphical results confirm that both solution components remain smooth and free of numerical artifacts under each method. The φ-RPSM typically exhibits slightly better accuracy, while the φ-NIM remains competitive and robust. Furthermore, the influence of fractional order α is clearly visible in Example(4.2), where lower values lead to more diffusive wave profiles, and higher values approach classical ($\alpha = 1$) behavior. Overall, both methods demonstrate high efficiency, rapid convergence, and suitability for nonlinear fractional partial differential equations. Compared with many existing numerical and semi-analytical techniques, the proposed ϕ-CRPSM and ϕ-CNIM methods provide rapidly convergent solutions without discretization or linearization, resulting in reduced computational effort while maintaining high accuracy. In addition, their formulation within the generalized ϕ-Caputo framework allows wider applicability to different fractional models.

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6 Conclusions

This study investigated the fractional WBKEs using generalized ϕ-Caputo operators together with the RPSM and the NIM. The findings show that both methods generate accurate and smoothly convergent series solutions, effectively capturing the nonlocal, memory dependent, and dispersive behaviors characteristic of fractional shallow water wave dynamics. The results demonstrate that integrating generalized fractional operators with semi-analytical schemes provides a powerful and efficient framework for solving nonlinear fractional partial differential equations. Future work may consider extending these approaches to higher dimensional WBK-type models and other nonlinear dispersive systems, as well as exploring additional fractional operators, including nonsingular or variable-order derivatives. Further theoretical analysis focused on stability, convergence, and error estimation would strengthen the mathematical foundation of the proposed methods. Additional directions include constructing hybrid numerical analytical formulations to enhance computational performance and applying the developed fractional models to experimental or real-world wave data for further validation. The results also have practical significance for real-world nonlinear wave systems where memory and dispersive effects play an important role, such as coastal wave dynamics and fluid transport processes. The proposed methods provide efficient analytical numerical tools for studying such complex fractional models and can assist researchers in analyzing nonlinear dynamical systems governed by fractional equations.

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