Non-Regular Languages

Non-regular languages

$$\{a^n b^n : \quad n \geq 0\}$$

$$\{vv^R : \quad v \in \{a,b\}^*\}$$

Regular languages

$$a^*b$$

$$b^*c+a$$

$$b+c(a+b)^*$$

etc...

How can we prove that a language L is not regular?

Prove that there is no DFA that accepts L

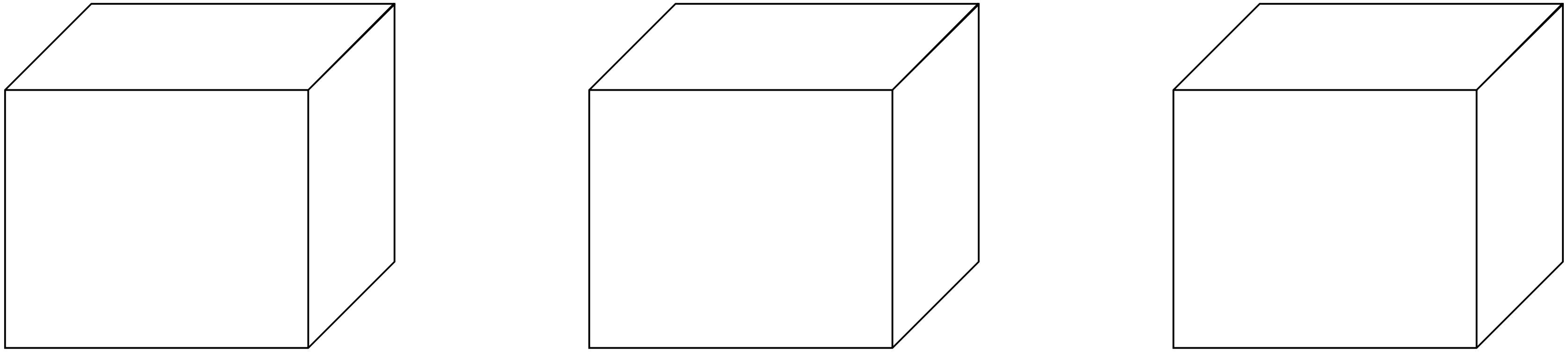
Problem: this is not easy to prove

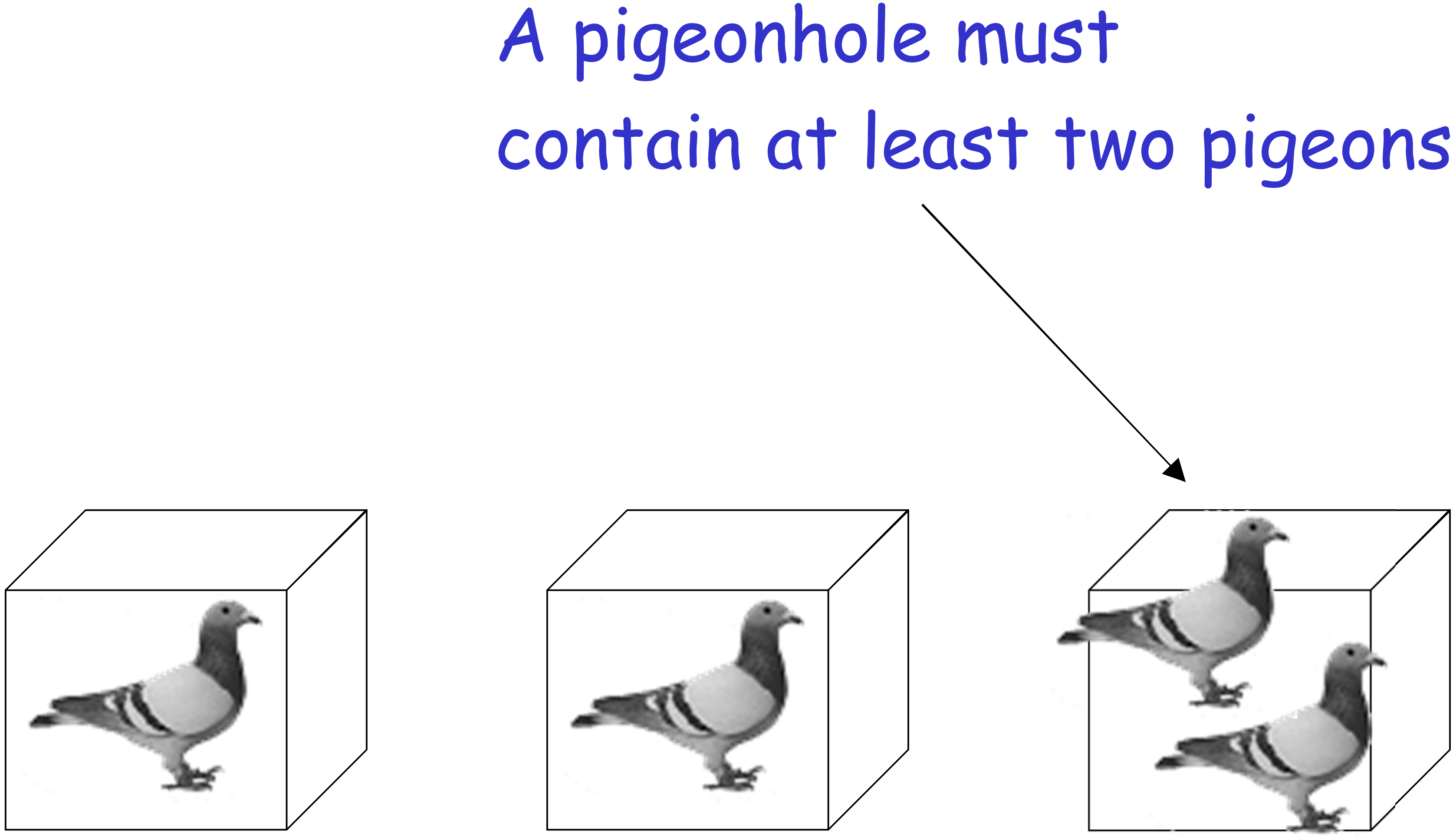
Solution: the Pumping Lemma !!!

The Pigeonhole Principle

4 pigeons

3 pigeonholes

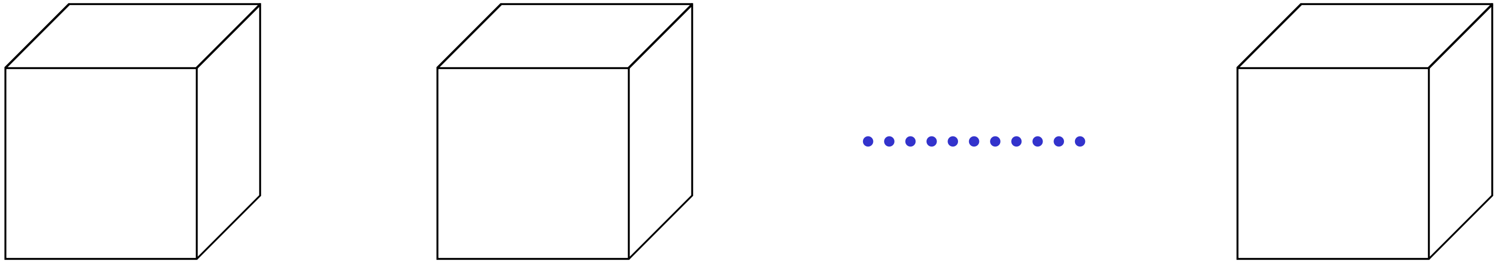




A pigeonhole must contain at least two pigeons

n pigeons

m pigeonholes $n > m$



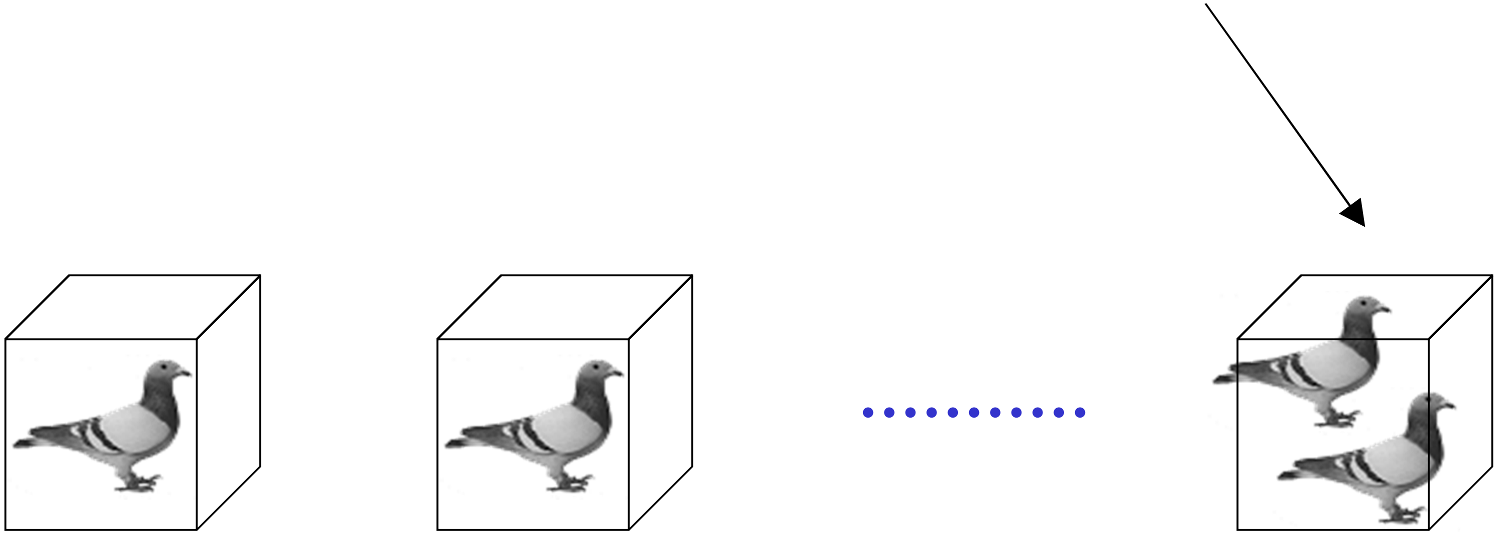
The Pigeonhole Principle

n pigeons

m pigeonholes

$n > m$

There is a pigeonhole with at least 2 pigeons

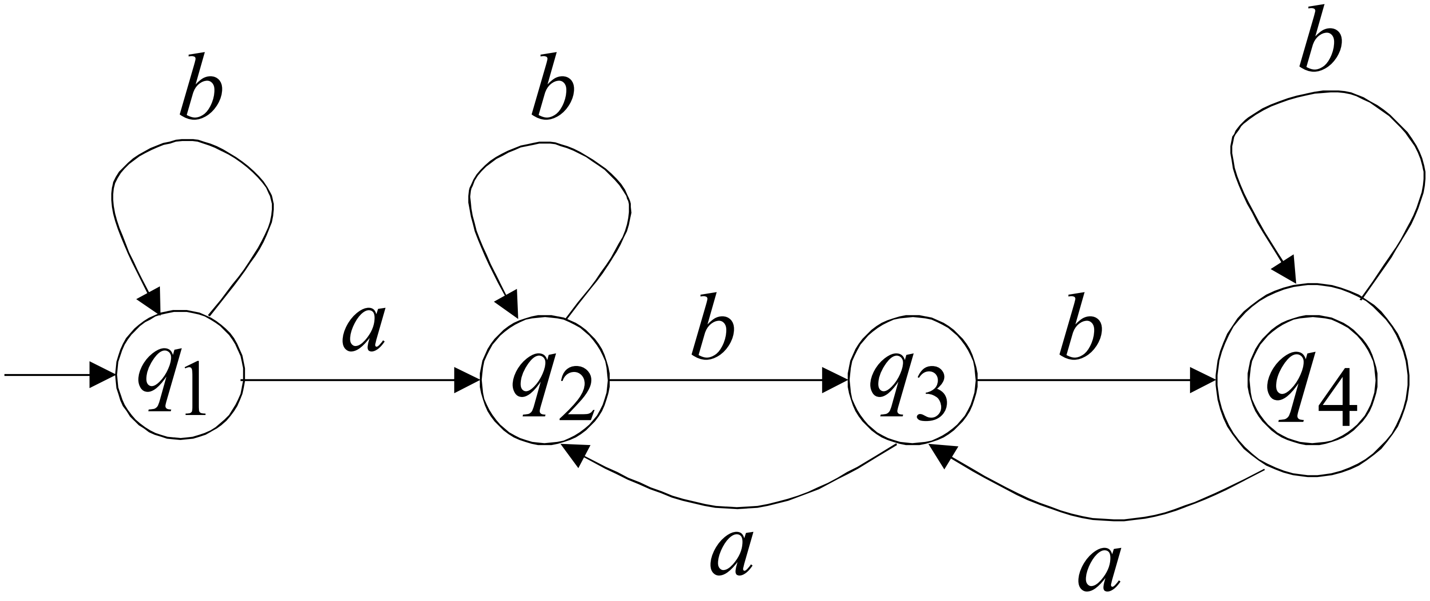


The Pigeonhole Principle

and

DFAs

DFA with 4 states

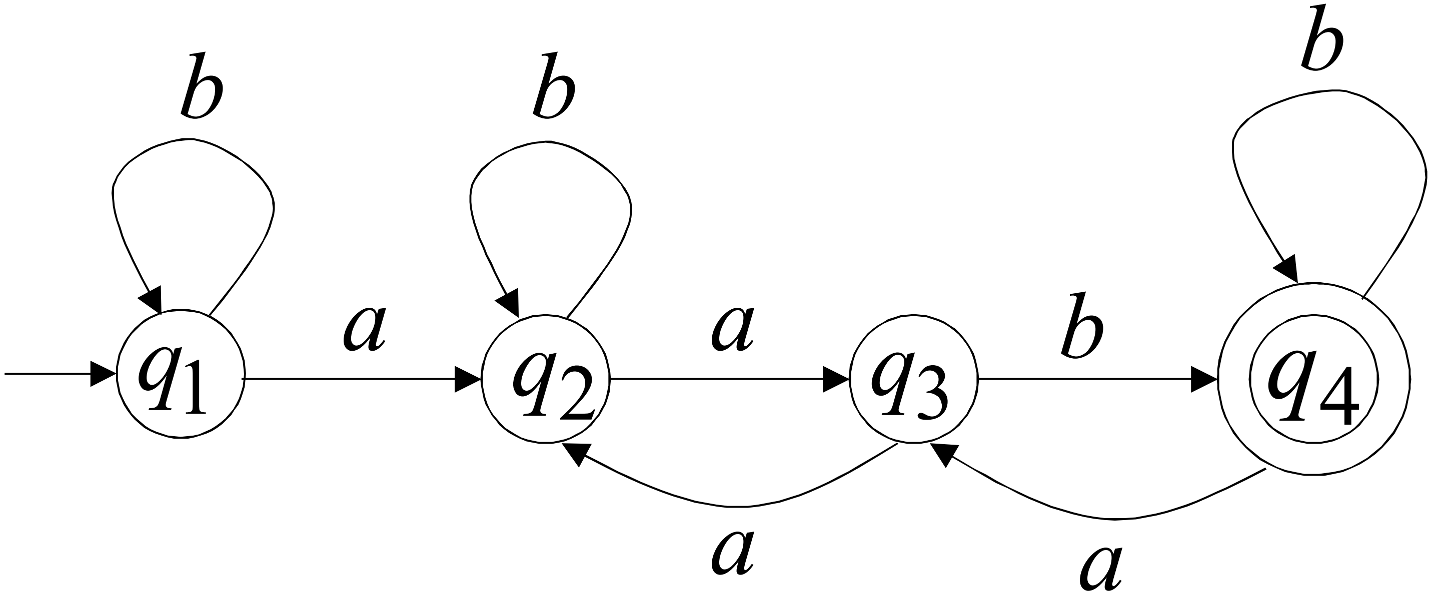


In walks of strings: a

aa

aab

no state
is repeated



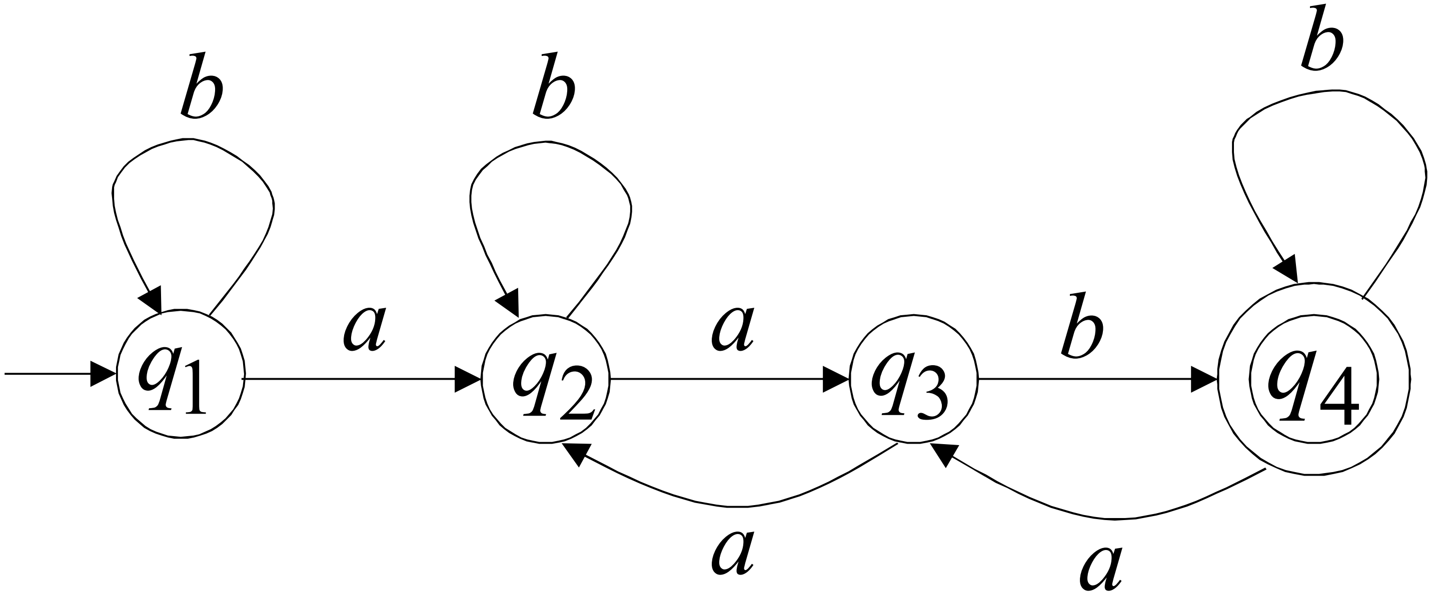
In walks of strings: $aabb$

$bbaa$

$abbabb$

$abbbabbabb...$

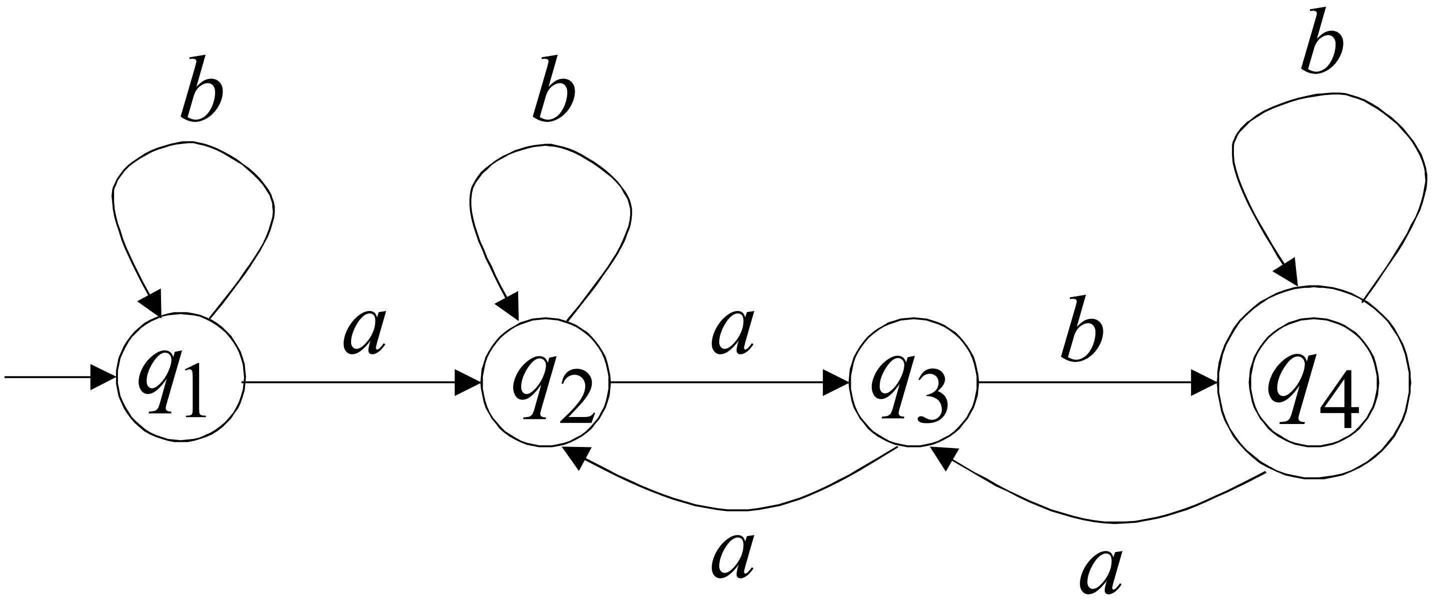
a state
is repeated



If string w has length $|w| \geq 4$:

Then the transitions of string w are more than the states of the DFA

Thus, a state must be repeated

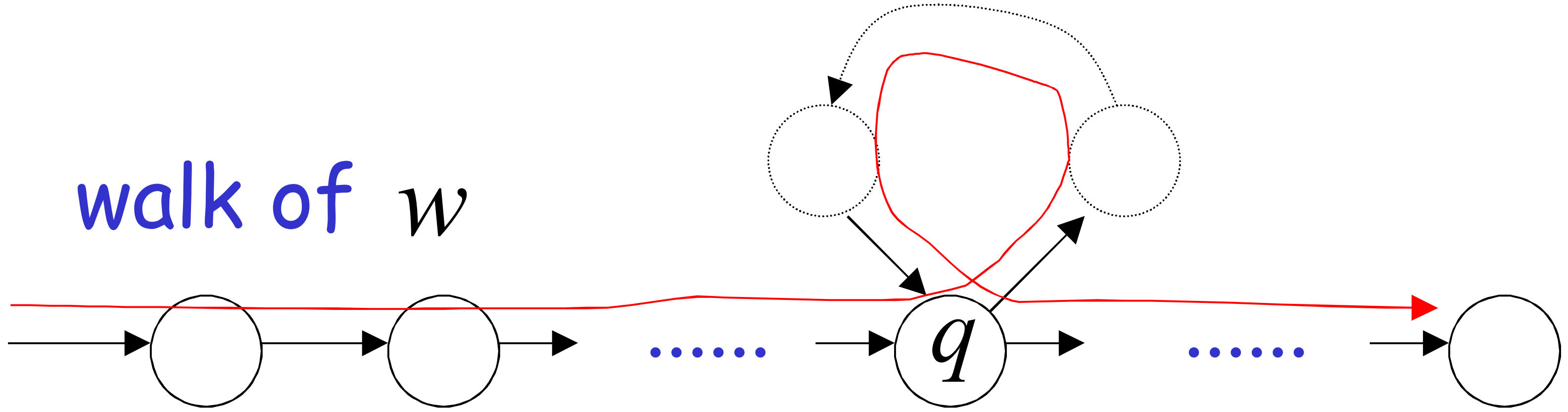


In general, for any DFA:

String w has length $\geq$ number of states

⇓

A state q must be repeated in the walk of w

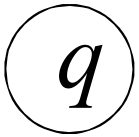


walk of w

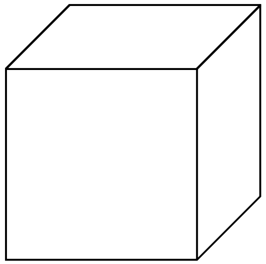
Repeated state

In other words for a string w:

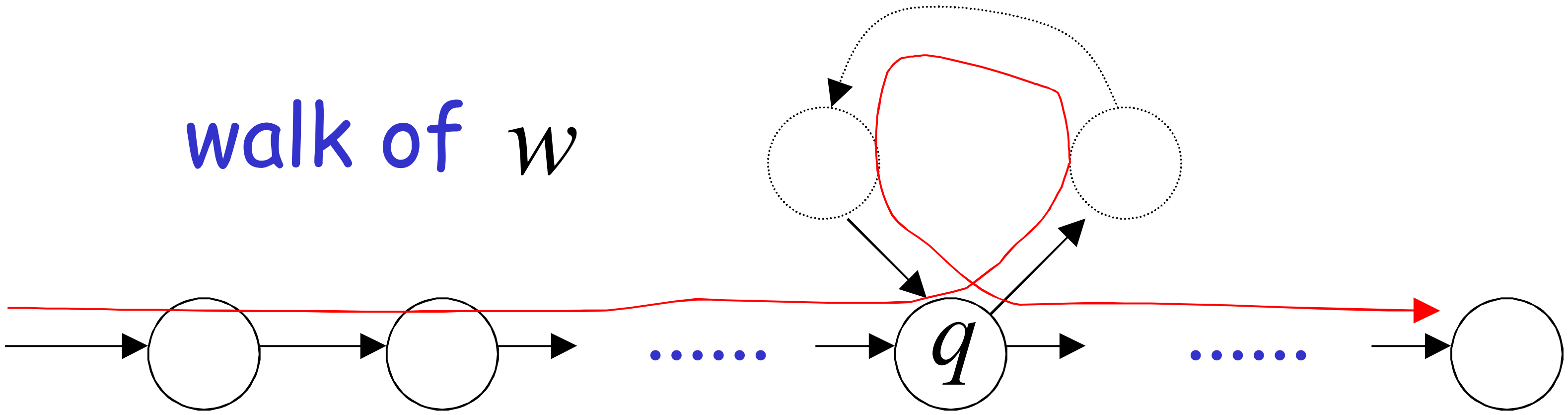
$\xrightarrow{a}$ transitions are pigeons



states are pigeonholes



walk of w

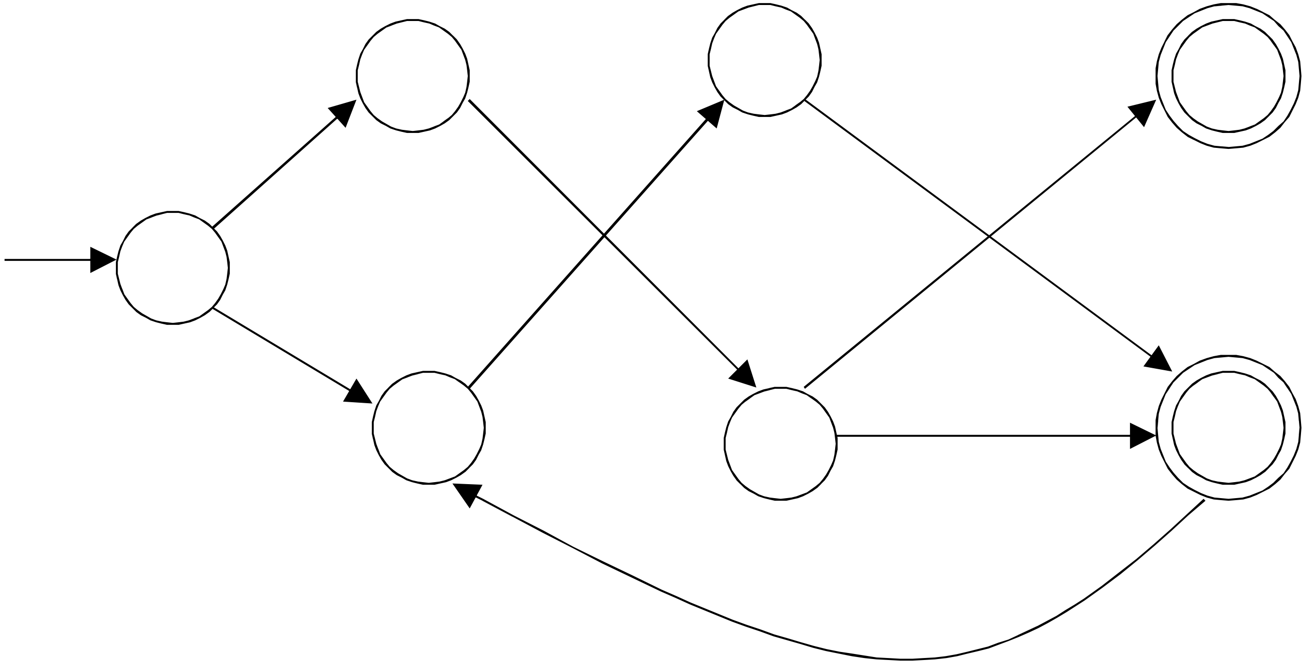


Repeated state

The Pumping Lemma

Take an infinite regular language L

There exists a DFA that accepts L



m
states

Take string w with $w \in L$

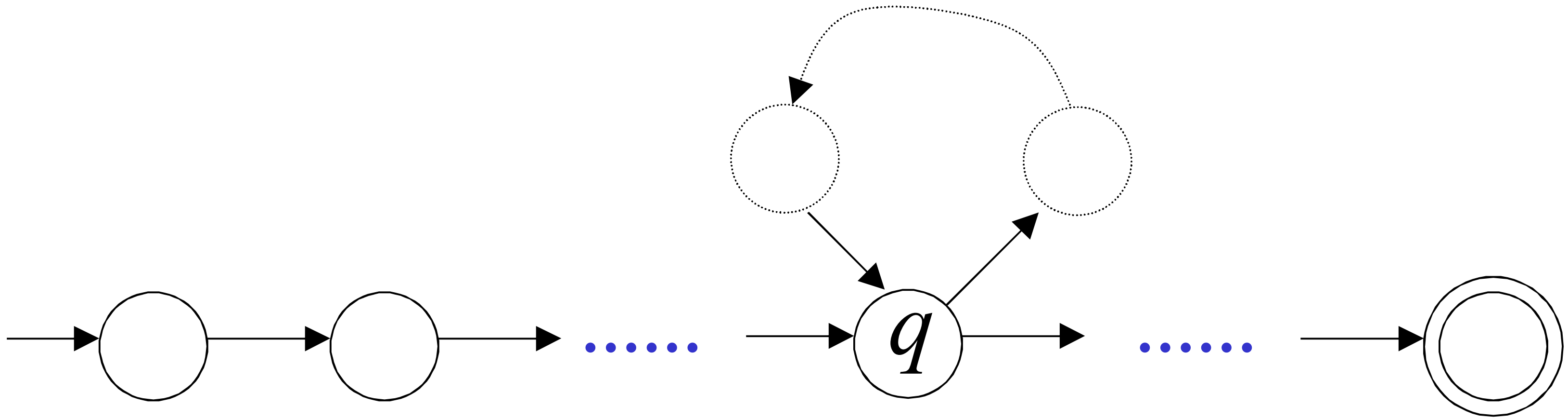
There is a walk with label w:

walk w

If string w has length $|w| \geq m$ (number of states of DFA)

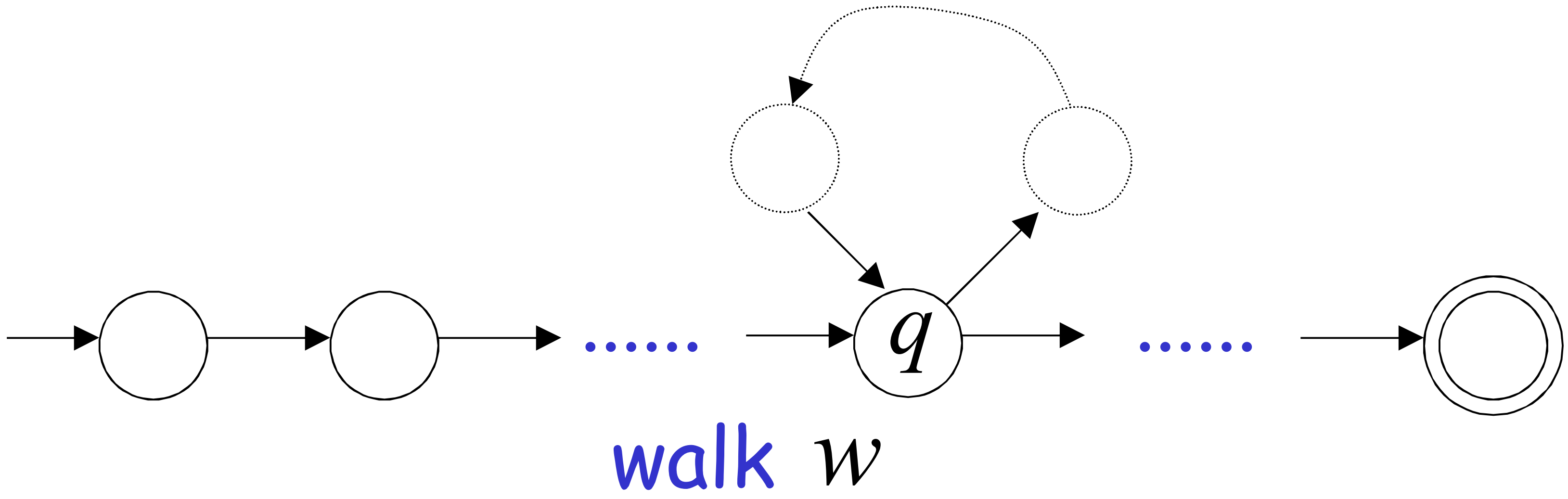
then, from the pigeonhole principle:

a state is repeated in the walk w

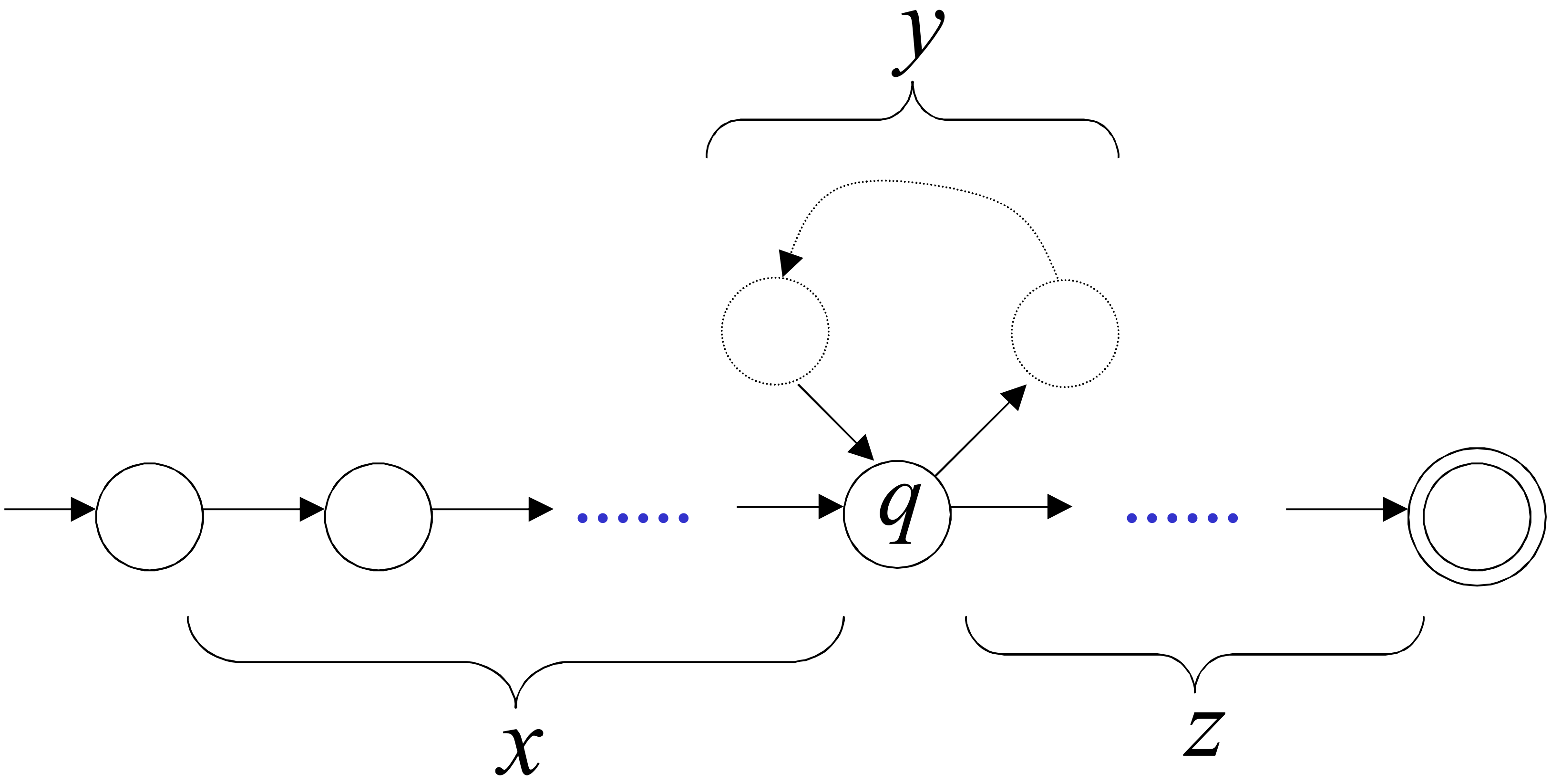


walk w

Let q be the first state repeated in the walk of w

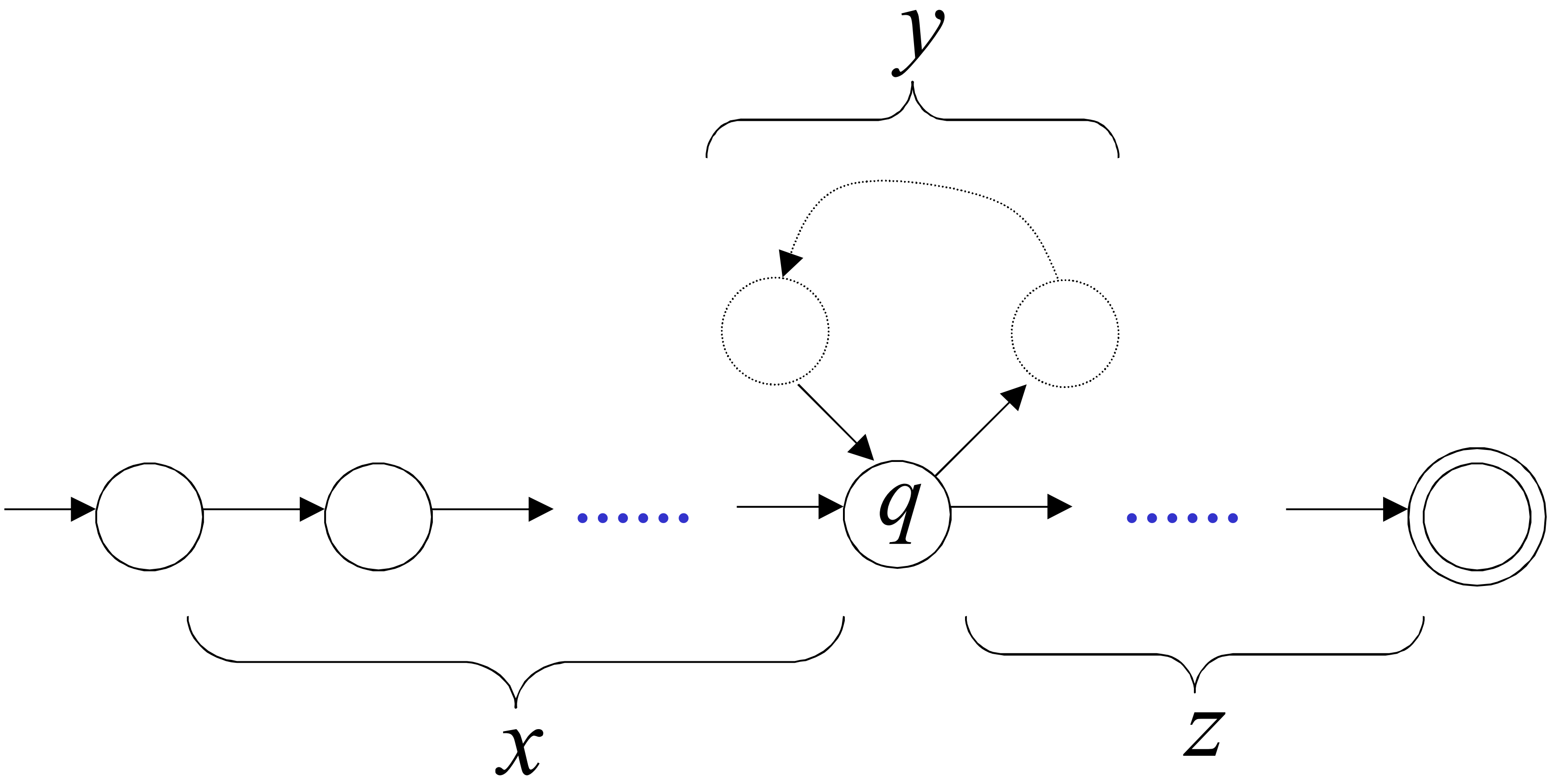


Write $w = x\, y\, z$

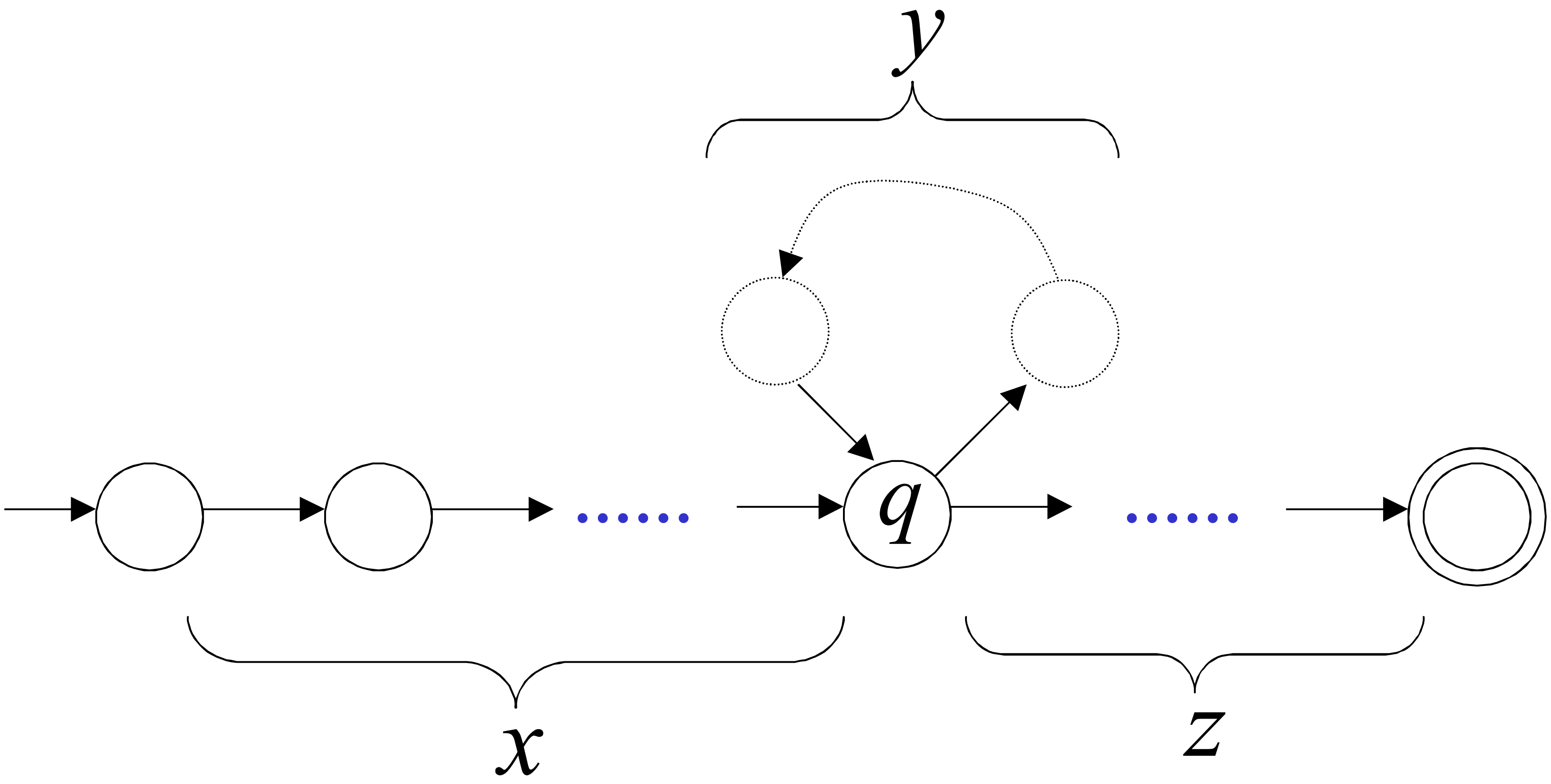


Observations: length $|x\,y| \leq m$ number of states of DFA

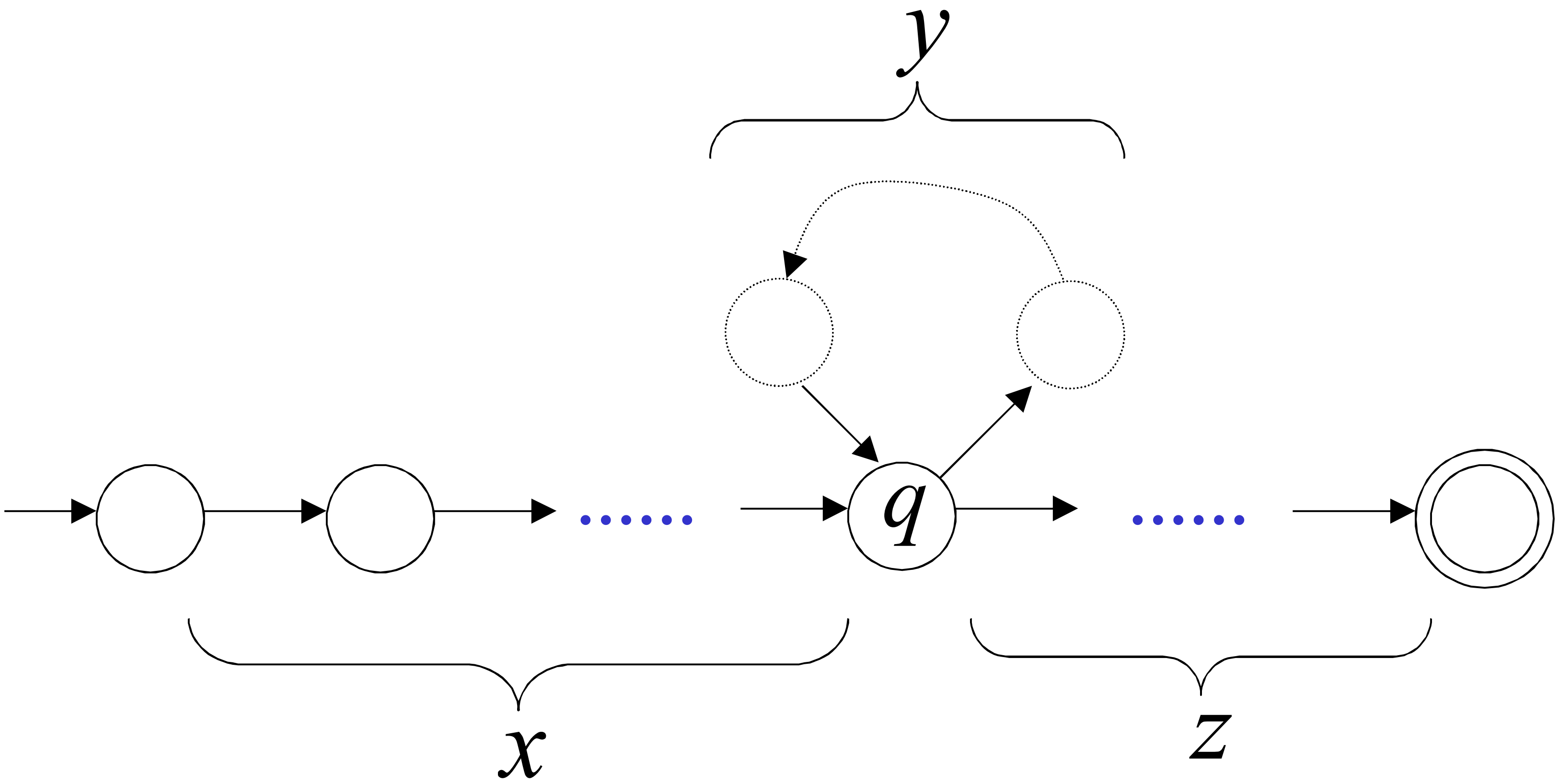
length $|y| \geq 1$



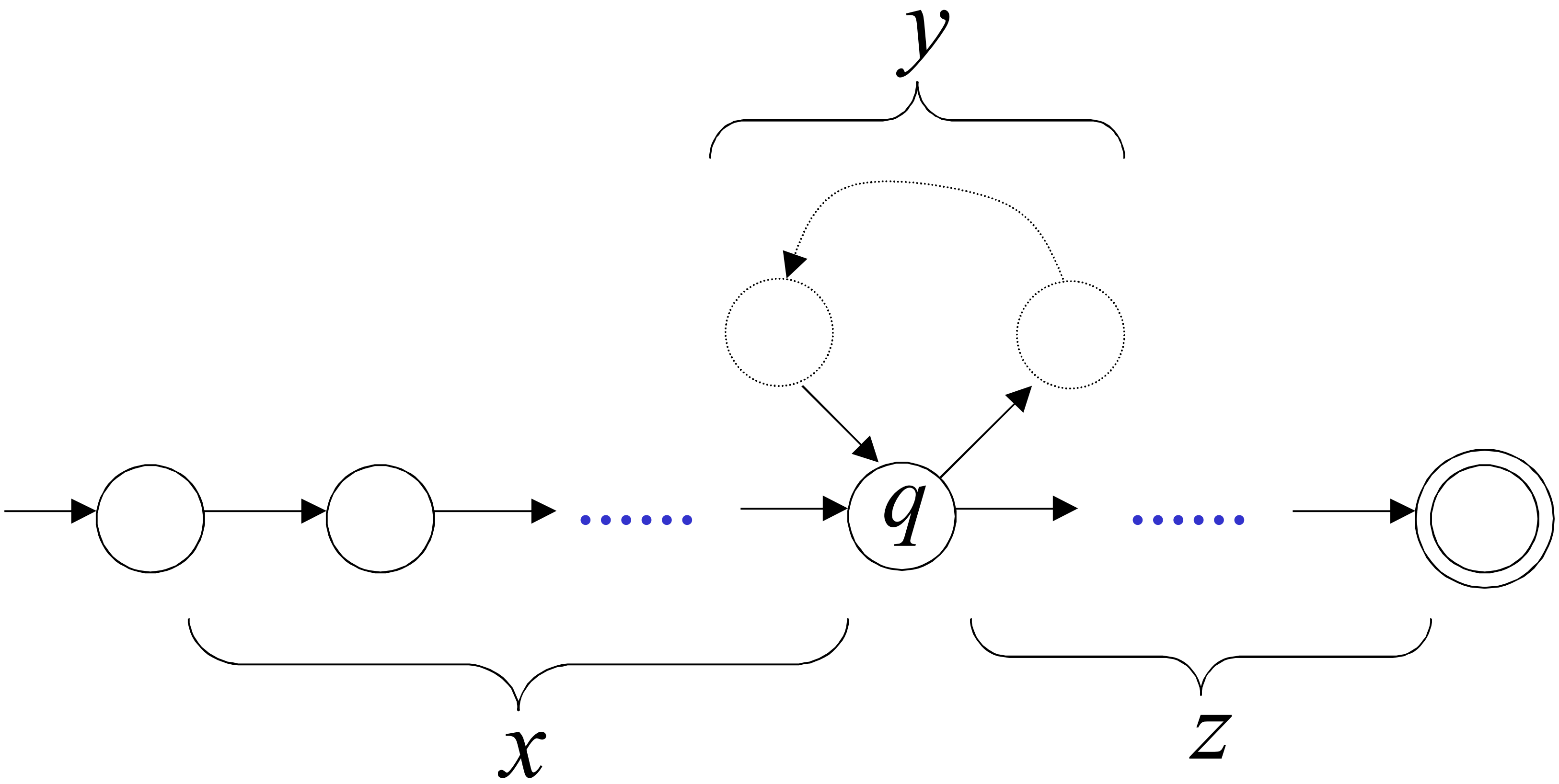
Observation: The string $x\,z$ is accepted



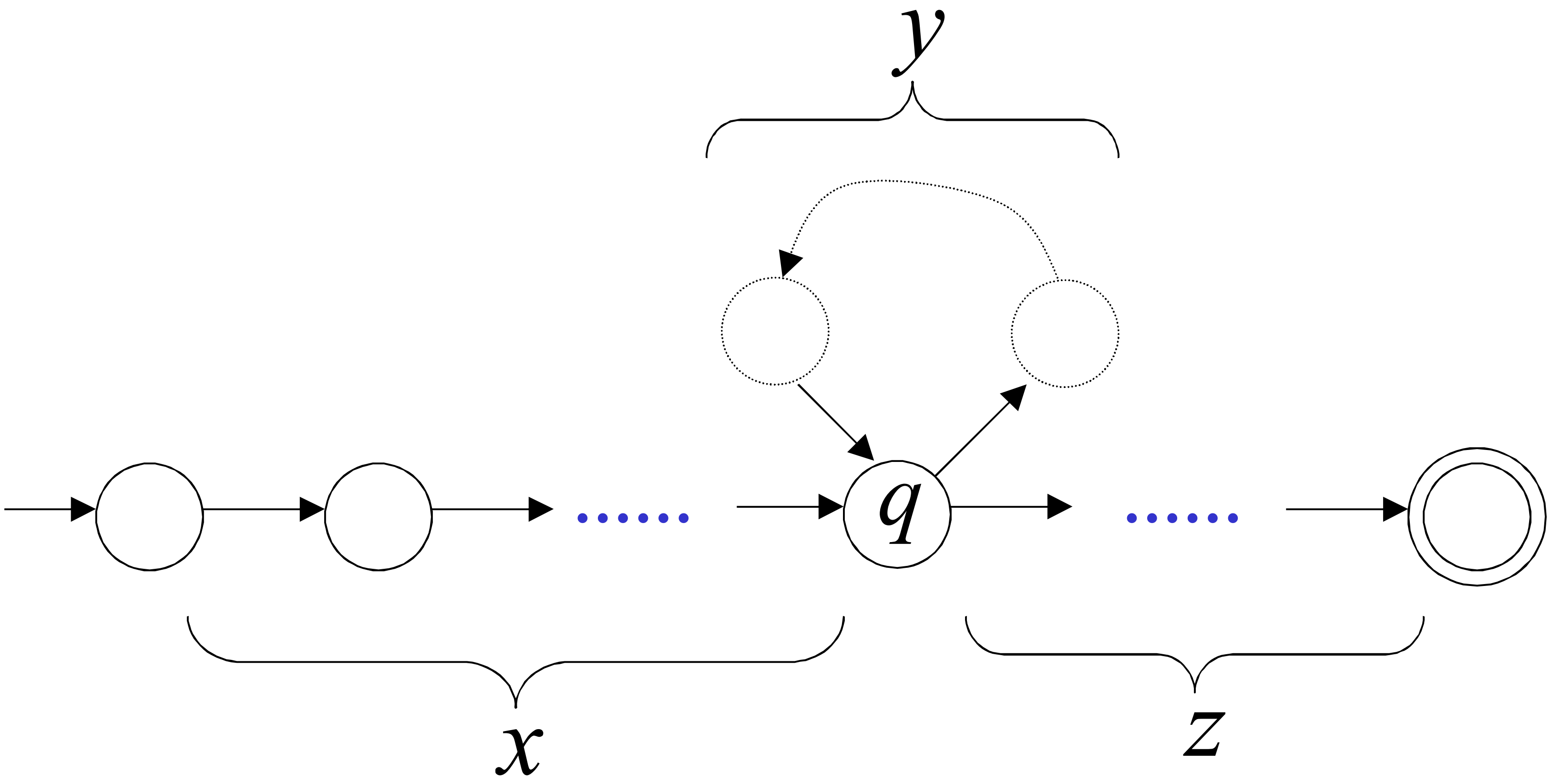
Observation: The string $x\ y\ y\ z$ is accepted



Observation: The string $x\,y\,y\,y\,z$ is accepted

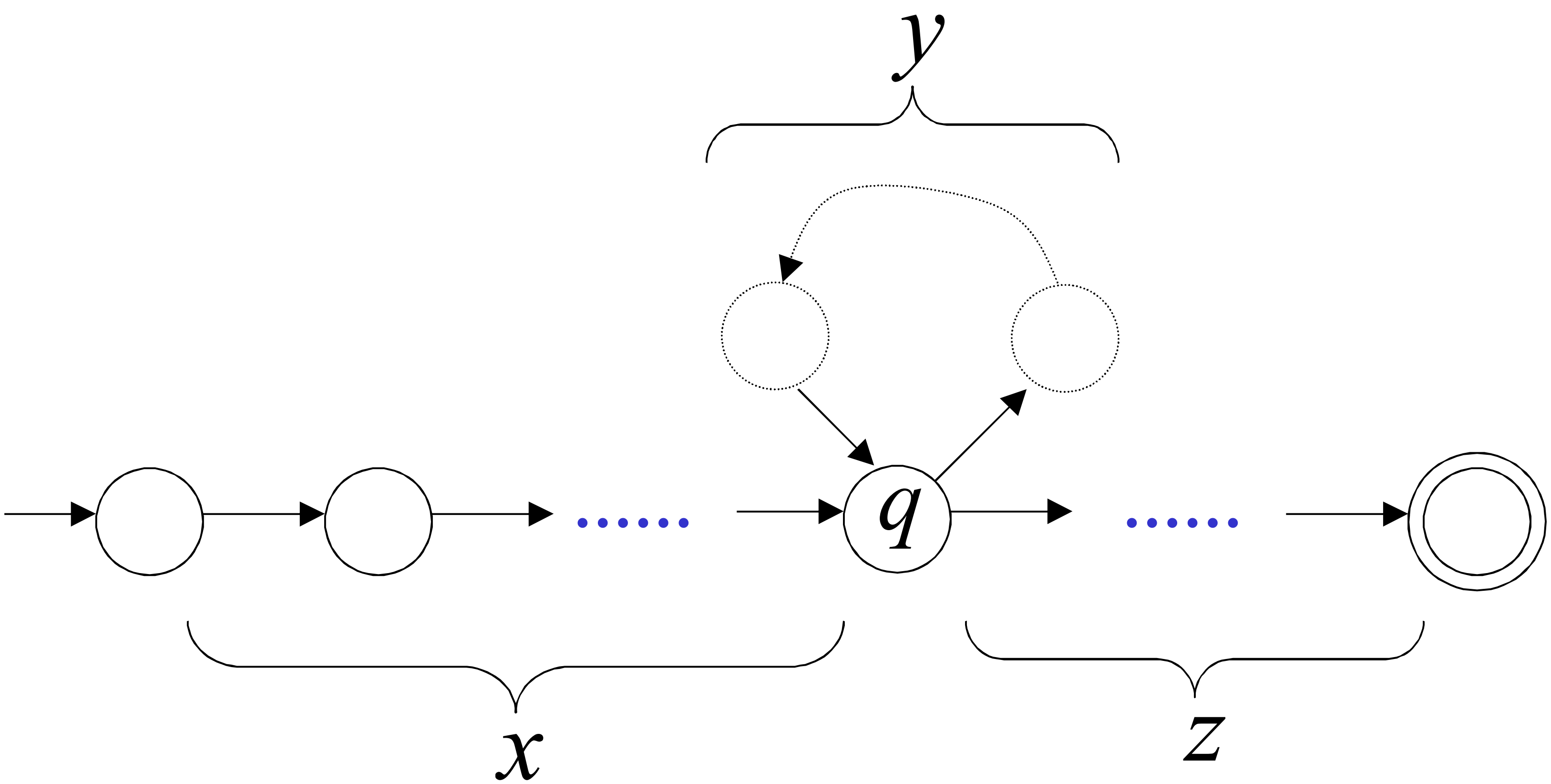


In General: The string $x\,y^i\,z$ is accepted $i = 0, 1, 2, \ldots$



In General: $x\, y^i\, z \in L$ $\quad i = 0, 1, 2, \ldots$

Language accepted by the DFA



In other words, we described:

The Pumping Lemma !!!

The Pumping Lemma:

- Given an infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

Applications

of

the Pumping Lemma

Theorem: The language $L = \{a^n b^n : n \geq 0\}$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^m b^m$

Write: $a^m b^m = x\ y\ z$

From the Pumping Lemma

it must be that length $|x\ y| \leq m, \quad |y| \geq 1$

$$xyz = a^m b^m \quad = \quad \rlap{$\overbrace{}^{m}\overbrace{}^{m}$}\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots ab\ldots b}_{z}$$

Thus: $y = a^k, \quad k \geq 1$

$$x\,y\,z = a^m b^m \qquad y = a^k, \quad k \geq 1$$

From the **Pumping Lemma:** $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \in L$

$$x\ y\ z = a^m b^m \qquad\qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\ y^2\ z \ \in\ L$

$$xy^2z = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}\underbrace{a\ldots a}_{y}\mathrlap{\underbrace{}_{z}}a\ldots a}^{m+k}\overbrace{b\ldots b}^{m} \in L$$

Thus: $a^{m+k}b^m \in L$

$a^{m+k}b^{m} \in L \qquad k \geq 1$

BUT: $L = \{a^{n}b^{n} : n \geq 0\}$

↓

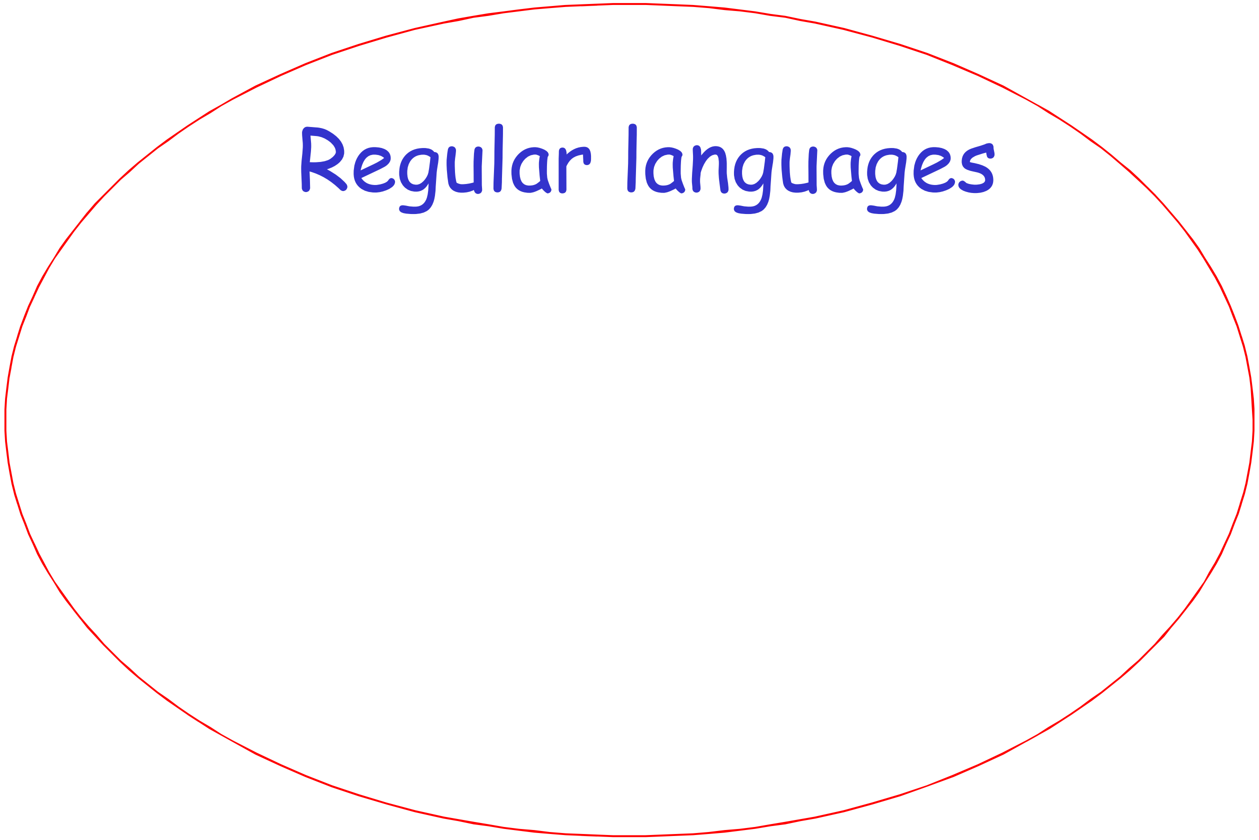
$a^{m+k}b^{m} \notin L$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages $\{a^n b^n : \quad n \geq 0\}$



Regular languages