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A Note on Neutrosophic Soft Set over Hyperalgebras

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Abstract: This research aims to introduce and explore the theory of neutrosophic soft hyperalgebras ($NSHA$s), focusing on their core principles and potential applications in decision-making under uncertainty. By defining key operations such as intersection and union, we clarify the foundational characteristics of $NSHA$s and their relationship to soft hyperalgebras. The concepts of ξ_β-identity $NSHA$ and ξ-absolute $NSHA$ are also examined to better understand their properties. The practical relevance of $NSHA$ is demonstrated through applications in various fields, highlighting its adaptability in addressing complex decision-making scenarios. This approach offers a novel, more precise method for navigating uncertainty in areas such as project methodology selection, sensitivity analysis, and AI chatbot selection.

Keywords: algebraic hyperstructure; homomorphism; hyperalgebra; soft set; neutrosophic soft set

MSC: 03B52; 03E72; 08A72; 14L17

1. Introduction

Algebraic hyperstructure is an algebraic structure type that is a generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, whereas in an algebraic hyperstructure, the composition of two elements is a set. Marty [1] first introduced the concept of hyperstructure at the 8th Congress of Scandinavian Mathematicians in 1934. Algebraic hyperstructure theory has diverse applications across various fields, as reviewed in [2–5]. In 1965, Zadeh pioneered the fuzzy set [6], and then some researchers extended abstract algebra concepts to the fuzzy hyperset structure as in [7]. In 1999, Molodtsov [8] developed the soft set theory as a method for coping with uncertainty that is free from the difficulties associated with the earlier theories. According to this idea, the challenge of determining the membership function does not occur, making the approach applicable to a wide range of domains. A collection of parameterized universe possibilities, known as soft sets, was proposed. The foundation of soft set theory is the parameterization principle, which contends that complex objects should be viewed from a variety of angles and that any single facet can only provide a rough and incomplete description of the whole thing. These results in the rapid development of soft set theory and its related fields (see [9–11]). Smarandache [12] studied new types of soft sets. Soft matrices were introduced by Cagman et al. [13] and used to address the problem of decision-making. In addition, they presented fuzzy soft matrices [14] with basic operations and talked about their desirable characteristics. Three fundamental t-norm operators were used in Mondal et al.'s [15] presentation of a fuzzy and intuitionistic fuzzy soft matrix and multi-criteria decision-making. Neutrosophy, introduced by Smarandache in [16], serves as the foundation of neutrosophic logic. Just as the neutrosophic set extends the fuzzy set, neutrosophic logic generalizes fuzzy logic, intuitionistic logic, and multivalued logic. Each proposition is expected to obtain the proportion of truth in a subset t, the percentage of indeterminacy in a subset i, and the percentage of falsity in a subset f in neutrosophic logic. Because the fuzzy set approach is reduced to modeling a condition that needs ambiguity, the utilization of the neutrosophic approach becomes essential when a condition requiring

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indeterminacy is to be produced. The idea of a neutrosophic set was narrowed down to a single-valued neutrosophic set (SVNS) from an objective standpoint. Additionally, the symmetry between the degrees of truth, indeterminacy, and falsity in a neutrosophic set offers a more balanced and comprehensive representation of uncertainty. Smarandache [16] proposed the SVNS for the advancement. A universalization of the classical set, fuzzy set, and intuitionistic fuzzy set is the SVNS. The development of the neutrosophic concept has led to the creation of neutrosophic algebraic structures [17], sparking a scientific revolution in this field. Numerous scientists have investigated this topic (see [18–24]) for more insights into SVNS. Maji [25] introduced the concept of a neutrosophic soft set by combining soft sets and neutrosophic sets. This idea has since been applied to various algebraic structures by other researchers (see [26–30]). The neutrosophic soft set and neutrosophic statistics are two different areas of neutrosophic mathematics. Neutrosophic soft set theory is used for decision-making [31] and expert systems, while neutrosophic statistics is used for data analysis (see [32,33]), modeling, and inference. The neutrosophic soft set theory provides a mathematical framework for managing uncertainty and indeterminacy, while neutrosophic statistics focuses on statistical methods for analyzing data with neutrosophic elements. Smarandache [34] introduced and developed superhyperalgebra and neutrosophic superhyperalgebra. Recently, Cano [35] utilized Maji's concept to apply *NSS*s to algebraic structures for the first time, focusing on hyper-UP algebras, and described various characterizations of neutrosophic soft hyper-UP algebras.

1.1. Related Work

Recent literature features a multitude of studies investigating the application of neutrosophic structures to address various decision-making challenges in diverse fields. Decision-making, particularly in complex scenarios such as project management or business applications, often involves handling incomplete or uncertain information. Neutrosophic logic, like fuzzy logic, can be employed as an effective tool to navigate such uncertainties and support more informed decisions. It enables the evaluation of potential alternatives based on criteria that may be imprecise, uncertain, or even indeterminate. When the requirements or standards for a given project are not well defined or subject to change, neutrosophic logic can assess the viability of different options by considering varying degrees of truth, indeterminacy, and falsity. This makes it possible to analyze alternatives based on their ability to meet a range of possible scenarios or criteria. By integrating neutrosophic logic into decision-making processes, teams can make more comprehensive assessments, minimizing the risk of selecting options that are unsuitable for the given context. Neutrosophic logic's ability to capture a wider spectrum of uncertainty, as compared to fuzzy logic, provides an enhanced framework for decision-making in complex, real-world applications, demonstrating a greater degree of symmetry in its ability to handle both certain and uncertain information. Symmetry in decision-making processes can be achieved through the application of neutrosophic logic, allowing for a more balanced and comprehensive evaluation of alternatives. Maji et al. [36] were the first to apply soft sets to solve decision problems using the neutrosophic approach. Addressing decision-making in imprecise environments has become increasingly important in recent years. Jha et al. [31] proposed using *NSS* for decision-making in analyzing stock value predictions. Dalkılıc [30] proposed a decision-making algorithm based on soft relations for uncertainty problems. Peng et al. [37] proposed three algorithms to solve single-valued neutrosophic soft decision-making problems and demonstrated the effectiveness and feasibility of approaches. Deli et al. [38] investigated decision-making based on interval-valued *NSS*s by level soft sets. In addition, the article [39] gave a neutrosophic soft matrix decision-making method, based on the end-product of neutrosophic soft matrices. Karaaslan [40] proposed a decision-making method called the neutrosophic soft decision-making possibility method. Uluçay et al. [41] defined a generalized neutrosophic soft expert set specifically for enhancing multiple-criteria decision-making processes. Abu Qamar et al. [42] investigated a novel approach to Q-neutrosophic soft sets and explored its applications in various decision-making

scenarios. Saeed et al. [43] introduced multi-polar neutrosophic soft sets, applying them to improve decision-making in medical diagnostics. Acharjya et al. [44] examined multi-criteria decision-making through the lens of interval-valued neutrosophic soft sets, offering new insights into this methodology. Dong et al. [45] developed distance, similarity, and entropy measures for dynamic interval-valued neutrosophic soft sets, demonstrating their utility in decision-making frameworks. Bui et al. [46] analyzed a sequence of neutrosophic soft sets, focusing on their implications for decision-making in the context of medical diagnosis. Priyadarsin et al. [47] provided a comprehensive review of generalized neutrosophic soft sets, emphasizing their role in addressing multiple expert decision-making challenges. Al-Sharqi et al. [48] explored interval complex neutrosophic soft relations, highlighting their effectiveness in various decision-making applications. Ihsan et al. [49] proposed an innovative decision-making framework for supplier selection that utilizes a hybrid interval-valued neutrosophic soft expert set. In a separate study, Ihsan et al. [50] optimized hard disk selection using a fuzzy parameterized single-valued neutrosophic soft set approach, showcasing its practical applications. Mohanty et al. [51] presented a new approach to neutrosophic soft sets, focusing on their practical applications in decision-making contexts. Karatas et al. [52] highlighted effective theories of neutrosophic soft sets, showcasing their utility in enhancing decision-making processes. Dalkılıç et al. [53] offered a novel perspective on Q-neutrosophic soft relations, emphasizing their applications in various decision-making scenarios. Al-Sharqi et al. [54] investigated decision-making techniques that leverage similarity measures of possibility neutrosophic soft expert sets, demonstrating their effectiveness. Rabunia et al. [55] explored the applications of neutrosophic soft sets. This introduces decision-making, employing an operational approach to facilitate analysis. Ali et al. [56] utilized aggregation operators based on q-rung orthopair neutrosophic soft sets, illustrating their applications in multi-attribute decision-making problems. Zhang et al. [57] developed a parameter reduction-based decision-making method utilizing interval-valued neutrosophic soft sets for selecting bionic thin-wall structures. Jayaprakash et al. [58] introduced the concept of neutrosophic soft set matrices and examined their applications in addressing multi-criteria decision-making challenges. Kaur et al. [59] proposed an innovative approach to multi-criteria decision-making within a neutrosophic picture hyper-soft set environment, utilizing the generalized neutrosophic TOPSIS method. Table 1 presents a review of some papers about NSSs and the extension of NSSs to decision-making in the literature.

Table 1. Review of NSSs and extensions in decision-making.

Reference	Authors	Year	Structure
[36]	Maji et al.	2012	Neutrosophic Soft Set
[28]	M. Ali et al.	2017	Bipolar Neutrosophic Soft Set
[41]	Uluçay et al.	2018	Generalized Neutrosophic Soft Expert Set
[42]	Abu Qamar et al.	2019	Q-neutrosophic Soft Set
[43]	Saeed et al.	2020	Multi-polar Neutrosophic Soft Sets
[44]	Acharjya et al.	2021	Interval Valued Neutrosophic Soft Set
[48]	Al-Sharqi et al.	2022	Interval Complex Neutrosophic Soft Set
[49]	Ihsan et al.	2023	Interval-Valued Neutrosophic Soft Expert Set
[52]	Karatas et al.	2023	Effective Neutrosophic Soft Set
[55]	Rabunia et al.	2023	Neutrosophic Soft Open Sets
[56]	Ali et al.	2024	q-Rung Orthopair Neutrosophic Soft Sets
[59]	Kaur et al.	2024	Neutrosophic Picture Hyper-Soft Set

1.2. Research Gap

In the literature, the integration of NSSs into the realm of algebra has been relatively unexplored. The development of $NSHA$ represents a significant advancement, providing a structured approach to dealing with complex algebraic operations that yield multiple outcomes. This study aims to bridge the gap between soft set theory and hyperalgebra, presenting a novel mathematical framework that accommodates the inherent uncertainties

of various applications. The motivation for this research stems from the pressing need for robust decision-making tools in fields such as artificial intelligence, project management, and data analysis. By leveraging the properties of NSSs and extending them to hyperalgebras, we aim to enhance problem-solving capabilities in environments characterized by incomplete or conflicting information. To our knowledge, there are no existing studies on neutrosophic set theory $NSHA$. To address this gap, we introduce the concept of $NSHA$ in this investigation. Specifically, there is even less research on decision-making concerning algebraic structures. In this work, we introduce a new neutrosophic soft algebraic structure, hyperalgebra, using a distinct approach. We propose the concept of $NSHA$, which handles inaccuracy and uncertainty more effectively than other generalizations of soft sets, such as fuzzy soft sets, intuitionistic fuzzy soft sets, and vague soft sets. Additionally, we present a decision-making approach based on $NSHA$s. It should be noted that the main motivation of the decision algorithm is to find the best parameter for a given NSS.

1.3. Motivational Factors for the Proposed Research

A thorough examination of the existing literature on $NSHA$ reveals that no current model effectively addresses all of the following challenges within a single framework:

1. Existing models fail to comprehensively manage algebraic operations that yield multiple outcomes, which is crucial for representing complex systems.
2. The integration of neutrosophic logic, with its ability to account for truth, indeterminacy, and falsity, is either insufficiently developed or entirely absent in current models of hyperalgebras.
3. No model fully addresses the inherent uncertainty and vagueness within algebraic structures, leaving a gap in accurate representation and problem-solving in uncertain environments.
4. Current approaches do not provide robust decision-making tools capable of functioning in environments characterized by incomplete or conflicting information, particularly within algebraic contexts.
5. There is no unified model that effectively combines the strengths of NSS and hyperalgebras to handle uncertainty, indeterminacy, and complex algebraic operations within a single, cohesive framework.

1.4. Main Contributions

The main contributions of this work are as follows:

(1) Introduction of the $NSHA$ concept.
(2) Presentation of $NSHA$ operations and theorems, including the definitions and identifications of ξ_β-identity $NSHA$ and ξ-absolute $NSHA$.
(3) Proposal of the concept of level sets of $NSHA$ and the introduction of $NSHA$ homomorphism.
(4) Exploration of a novel NSS decision-making method based on hyperalgebra.
(5) Outline of the methodology used in this study.
(6) Demonstration of a practical application for selecting the best project method- ology in real-life scenarios.
(7) Execution of a sensitivity analysis by varying parameter weights.
(8) Evaluation of the current strategy's usefulness and effectiveness through comparative analysis with previous studies.
(9) Evidence from comparative analysis showing that this strategy not only meets but exceeds the performance of earlier approaches.

The organization of this paper is as follows:

Section 2 discusses preliminary concepts with relevant citations.
Section 3 defines $NSHA$ and its various operations.
Section 4 introduces the concept of $NSHA$ level sets.

Section 5 proposes the notion of $NSHA$ homomorphism and provides details on images and inverse images under homomorphism.
Section 6 outlines the methodology used in the study.
Section 7 applies $NSHA$ to a real-life problem, including a decision-making method, sensitivity analysis, and a numerical example for selecting the best artificial intelligence chatbots, with a comparative analysis of previous research.
Section 8 presents the conclusions drawn from the study.

2. Preliminaries

This section provides essential definitions needed for discussing neutrosophic soft hyperalgebras in the subsequent section. First, we will introduce the concept of hyperalgebras.

Definition 1. *Let H be used for the non-void set. $\mathcal{P}^*(H)$ means the family of all non-void subsets of H while H^n is the set of n-tuples over H, where n is a positive integer.*

Let β be an n-ary hyperoperation over H, and let β be a function $\beta : H^n \to \mathcal{P}^*(H)$, for a positive integer n which is called arity of β. When $S \subseteq H$, it is said that S is closed under β if $(x_1, \ldots, x_n) \in S^n$, implying that $\beta(x_1, \ldots, x_n) \subseteq S$. A hyperalgebra $\langle H, (\beta_i)_{i\in I}\rangle$ is a set which is collection $(\beta_i)_{i\in I}$ of hyperoperations with H. $\langle S, (\beta_i)_{i\in I}\rangle$ is called a subhyperalgebra of $\langle H, (\beta_i)_{i\in I}\rangle$ if S is closed under β_i, for any $i \in I$,, which is an index set that represents the collection of hyperoperations associated with the hyperalgebra $\langle H, (\beta_i)_{i\in I}\rangle$. The map from I into the set $\mathbb{N}^*$ of non-negative integers which assigns the arity of β_i to each $i \in I$ is a type of H. If two hyperalgebras are of the same type, they are named similar hyperalgebras [60,61].

The concepts of soft sets and fuzzy soft sets provide a flexible mathematical framework for handling uncertainty and vagueness. Now, the concepts of soft sets and fuzzy soft sets will be described.

Definition 2. *Suppose that $\mathcal{U}$ is the initial universe set and $\mathcal{K}$ is the set of parameters. $(\mathcal{F}, \mathcal{K})$ is named a soft set over $\mathcal{U}$ if and only if $\mathcal{F} : \mathcal{K} \to \mathcal{P}(\mathcal{U})$ is a mapping with $\mathcal{P}(\mathcal{U})$ being the power set of $\mathcal{U}$ [7]. It is written as $(\mathcal{F}, \mathcal{K}) = \{\mathcal{F}(e) \mid e \in \mathcal{K}\}$. A soft set can be used to represent a fuzzy set, as can be seen. Assume that $\mathcal{K}$ is a set of parameters and $\mathcal{X} \subset \mathcal{K}$. A pair $(\mu, \mathcal{X})$ is known as a fuzzy soft set on $\mathcal{U}$ such that $\mu : \mathcal{X} \to I^{\mathcal{U}}$.*

The concept of soft hyperalgebras, by extending classical hyperalgebraic structures to soft set theory, provides a flexible framework for dealing with uncertainty and imprecision in algebraic systems. Then, the concept of soft hyperalgebras will be given.

Definition 3. *Let $H \neq \emptyset$, β_i be a fuzzy n_i-ary hyperoperation over H for every $i \in I$. Then, $\langle H, (\beta_i)_{i\in I}\rangle$ is a fuzzy hyperalgebra. Furthermore, $(n_i : i \in I)$ is a type of $\langle H, (\beta_i)_{i\in I}\rangle$ [60]. Suppose that $\langle H, (\beta_i)_{i\in I}\rangle$ is a hyperalgebra and $(\mathcal{F}, \mathcal{X})$ is a soft set over H. Let $\mathcal{F}(a)$ be a subhyperalgebra for all $a \in Supp(\mathcal{F}, \mathcal{X})$. Then, $(\mathcal{F}, \mathcal{X})$ is a soft hyperalgebra of H [62].*

Now, single-valued neutrosophic sets and the neutrosophic soft sets (NSSs) will be explained so that they can be combined with NSSs and hyperalgebras in the main part of this paper.

Definition 4. *Let $\mathcal{U}$ be a space of objects with a generic element in $\mathcal{U}$ indicated by u. A single-valued neutrosophic set (SVNS) $\mathcal{N}$ in $\mathcal{U}$ is defined by a truth-membership function t_N, an indeterminacy-membership function i_N and a falsity-membership function f_N and it can be represented by*

$$\mathcal{N} = \{\langle u, (t_N(u), i_N(u), f_N(u))\rangle : u \in U,\ t_N(u), i_N(u), f_N(u) \in [0,1]\}$$

such that $0 \leq t_N(u) + i_N(u) + f_N(u) \leq 3$ [16].

The concept of NSS offers a flexible mathematical framework for dealing with uncertainty and incomplete information by extending traditional set theory to accommodate truth, indeterminacy, and falsity in decision-making processes.

Let $\mathcal{U}$ be an initial universe set and E be a set of parameters and $\mathcal{A} \subseteq \mathcal{E}$. Let $\mathcal{P}(\mathcal{N})$ denote the set of all neutrosophic sets of $\mathcal{U}$. The NSS over $\mathcal{U}$ is referred to as the collection $(\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ is a transformation provided by $\mathcal{N} : \mathcal{A} \to \mathcal{P}(\mathcal{N})$ [25].

Let $(\mathcal{N}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be two NSSs over common universe $\mathcal{U}$. $(\mathcal{N}, \mathcal{A})$ is called a neutrosophic soft subset of $(\mathcal{G}, \mathcal{B})$ if $\mathcal{A} \subset \mathcal{B}$, and $t_a(z) \leq t_b(z)$, $i_a(z) \leq i_b(z)$ and $f_a(z) \geq f_b(z)$, $\forall a \in \mathcal{A}$ and $\forall b \in \mathcal{B}$, $z \in \mathcal{U}$. It is represented by $(\mathcal{N}, \mathcal{A}) \subseteq (\mathcal{G}, \mathcal{B})$ [25].

The union of *NSS*s will be introduced as a crucial operation that enhances the ability to combine multiple sets of uncertain, incomplete, or inconsistent information, thus providing a comprehensive tool for more robust decision-making in complex environments.

Definition 5. *Let $(\mathcal{N}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be two NSSs over the common universe $\mathcal{U}$. Then, the union of $(\mathcal{N}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ is symbolized by $(\mathcal{N}, \mathcal{A})\breve{\cup}(\mathcal{G}, \mathcal{B})$ and is characterized by $(\mathcal{N}, \mathcal{A})\breve{\cup}(\mathcal{G}, \mathcal{B}) = (\mathcal{K}, \mathcal{C})$, where $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$*

$$t_c(z) = \left\{ \begin{array}{ll} t_a(z) & ,\text{if } c \in \mathcal{A} - \mathcal{B} \\ t_b(z) & ,\text{if } c \in \mathcal{B} - \mathcal{A} \\ \max(t_a(z), t_b(z)) & ,\text{if } c \in \mathcal{A} \cap \mathcal{B} \end{array} \right\}$$

$$i_c(z) = \left\{ \begin{array}{ll} i_a(z) & ,\text{if } c \in \mathcal{A} - \mathcal{B} \\ i_b(z) & ,\text{if } c \in \mathcal{B} - \mathcal{A} \\ \max(i_a(z), i_b(z)) & ,\text{if } c \in \mathcal{A} \cap \mathcal{B} \end{array} \right\}$$

$$f_c(z) = \left\{ \begin{array}{ll} f_a(z) & ,\text{if } c \in \mathcal{A} - \mathcal{B} \\ f_b(z) & ,\text{if } c \in \mathcal{B} - \mathcal{A} \\ \min(f_a(z), f_b(z)) & ,\text{if } c \in \mathcal{A} \cap \mathcal{B} \end{array} \right\}$$

[25].

In exploring the properties of *NSS*s, it becomes essential to analyze their intersections, as this will enhance our understanding of how these sets interact within the framework of uncertainty and complexity. Then, the intersection of *NSS*s will be presented.

Definition 6. *Let $(\mathcal{N}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be two NSSs over the common universe $\mathcal{U}$. Then, the intersection of $(\mathcal{N}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ is identified by $(\mathcal{N}, \mathcal{A})\tilde{\cap}(\mathcal{G}, \mathcal{B})$ and is characterized by $(\mathcal{N}, \mathcal{A})\tilde{\cap}(\mathcal{G}, \mathcal{B}) = (\mathcal{K}, \mathcal{C})$,*

$$(\mathcal{K}, \mathcal{C}) = \{c, \langle t_c(z), i_c(z), f_c(z)\rangle : z \in \mathcal{U},\ c \in \mathcal{A} \cap \mathcal{B}\},$$

where

$$\begin{aligned} t_c(z) &= t_a(z) \wedge t_b(z) \\ i_c(z) &= i_a(z) \wedge i_b(z) \\ f_c(z) &= f_a(z) \vee f_b(z) \end{aligned}$$

[25].

In this context, the "AND" operation of *NSS*s will be explored, providing a systematic approach to combine multiple sets while preserving their inherent uncertainties and characteristics.

Definition 7. *Let $(\mathcal{N}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{B})$ be two NSSs over the common universe $\mathcal{U}$. Then, the "AND" operation on them is symbolized by $(\mathcal{N}, \mathcal{A})\tilde{\wedge}(\mathcal{G}, \mathcal{B})$, is identified by $(\mathcal{N}, \mathcal{A})\tilde{\wedge}(\mathcal{G}, \mathcal{B}) = (\mathcal{K}, \mathcal{A} \times \mathcal{B})$ and is characterized by*

$$(\mathcal{K}, \mathcal{A} \times \mathcal{B}) = \left\{(a,b), \left\langle t_{(a,b)}(z), i_{(a,b)}(z), f_{(a,b)}(z)\right\rangle : z \in \mathcal{U},\ (a,b) \in \mathcal{A} \times \mathcal{B}\right\},$$

where

$$\begin{aligned} t_{(a,b)}(z) &= t_a(z) \wedge t_b(z) \\ i_{(a,b)}(z) &= i_a(z) \wedge i_b(z) \\ f_{(a,b)}(z) &= f_a(z) \vee f_b(z) \end{aligned}$$

[25].

3. Neutrosophic Soft Hyperalgebras

In this section, neutrosophic soft hyperalgebra ($NSHA$) is defined. An $NSHA$ extends the concept of neutrosophic soft sets into algebraic structures. It integrates neutrosophic soft sets into hyperalgebraic systems, where the operations are generalized to handle hyperoperations rather than simple binary operations.

Definition 8. *Presume that $\langle H, (\beta_i)_{i\in I}\rangle$ is a hyperalgebra. When $(\mathcal{N}, \mathcal{A})$ is an NSS over H, if*

$$\begin{aligned} \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_a(z) &\geq \min\{t_a(x_1), \ldots, t_a(x_{n_i})\}, \\ \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_a(z) &\geq \min\{i_a(x_1), \ldots, i_a(x_{n_i})\}, \\ \bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} f_a(z) &\leq \max\{f_a(x_1), \ldots, f_a(x_{n_i})\} \end{aligned}$$

for all $a \in \mathcal{A}$, $\forall i \in I$, $\forall x_1, \ldots, x_{n_i} \in H$, $\forall z \in \beta_i(x_1, \ldots, x_{n_i})$, then $(\mathcal{N}, \mathcal{A})$ is described as an $NSHA$ over H. In addition, if $\langle H, (\beta_i)_{i\in I}\rangle$ has null hyperoperation β, for all $z \in \beta$ and $a \in \mathcal{A}$, $t_a(z)$, $i_a(z)$ and $f_a(z)$ are constant. Furthermore, $t_a(z)$ and $i_a(z)$ are greatest and $f_a(z)$ is least.

Example 1. *Every soft hyperalgebra is taken into account as a neutrosophic subhyperalgebra for the reason that a subhyperalgebra's characteristic function is a neutrosophic subhyperalgebra. An $NSHA$ $(\mathcal{N}, \mathcal{A})$ with singleton $\mathcal{A}$ is a neutrosophic subhyperalgebra. Hence, $NSHA$s comprise neutrosophic subhyperalgebras and hyperalgebras.*

Example 2. *Every neutrosophic soft hypergroup is an $NSHA$.*

Example 3. *Let $H = \{j, w, v\}$. Consider the following tables:*

$\oplus$	j	w	v
j	j	w	v
w	w	w	H
v	v	H	v

and

$\odot$	j	w	v
j	j	j	j
w	j	w	v
v	j	w	v

.

It really is simply indicated that $(H, \oplus, \odot)$ is a hyperalgebra. Let $x, y \in \mathcal{A}$ be a parameter set and $(\mathcal{N}, \mathcal{A})$ be an NSSs over hyperalgebra H where

$$\begin{matrix} t_x(z) = 0.6/j + 0.7/w + 0.4/v \\ t_y(z) = 0.5/j + 0.8/w + 0.3/v \end{matrix}', \begin{matrix} i_x(z) = 0.5/j + 0.6/w + 0.3/v \\ i_y(z) = 0.4/j + 0.7/w + 0.2/v \end{matrix}', \text{ and } \begin{matrix} f_x(z) = 0.2/j + 0.2/w + 0.6/v \\ f_y(z) = 0.3/j + 0.1/w + 0.4/v \end{matrix}.$$

Then $(\mathcal{N}, \mathcal{A})$ *is an NSHA.*

Example 4. *Let* $S = \{j, w, v, s\}$ *and* $H = \{0, 1, 2, 3\}$ *with the following hyperoperation tables:*

$\oplus$	j	w	v	s
j	j	w	j	s
w	w	$\{w, v\}$	s	$\{j, v\}$
v	j	s	$\{v, s\}$	v
s	s	$\{j, v\}$	v	s

and

$\odot$	j	w	v	s
0	j	w	v	s
1	s	w	v	s
2	j	v	s	s
3	j	j	s	s

Then, S is a hyperalgebra over H. Let $X = \{e_1, e_2, e_3\}$ *be the parameter's set. We identify an NSS* $(\mathcal{N}, \mathcal{A})$:

$\mathcal{N}(e_1) = \{[0.7, 0.78, 0.52]/j, [0.6, 0.64, 0.76]/w, [0.74, 0.83, 0.43]/v, [0.8, 0.81, 0.39]/s\}$
$\mathcal{N}(e_2) = \{[0.3, 0.65, 0.95]/j, [0.6, 0.85, 0.55]/w, [0.45, 0.73, 0.82]/v, [0.01, 0.02, 0.05]/s\}$
$\mathcal{N}(e_3) = \{[0.54, 0.88, 0.58]/j, [0.9, 0.9, 0.2]/w, [0.7, 0.87, 0.43]/v, [0.2, 0.63, 0.81]/s\}$
After that, $(\mathcal{N}, \mathcal{A})$ *is an NSHA.*

Theorem 1. *If* $\langle H, (\beta_i)_{i \in I} \rangle$ *is a hyperalgebra,* $(\mathcal{N}, \mathcal{A})$ *and* $(\mathcal{M}, \mathcal{B})$ *are two NSHAs over H, and then;*

(i) $(\mathcal{N}, \mathcal{A}) \tilde{\cap} (\mathcal{M}, \mathcal{B})$ *is an NSHA over H.*
(ii) $(\mathcal{N}, \mathcal{A}) \tilde{\cup} (\mathcal{M}, \mathcal{B})$ *is an NSHA over H where* $\mathcal{A} \cap \mathcal{B} = \emptyset$.
(iii) $(\mathcal{N}, \mathcal{A}) \tilde{\wedge} (\mathcal{M}, \mathcal{B})$ *is an NSHA over H.*

Proof. (i) Let $(\mathcal{N}, \mathcal{A})$ and $(\mathcal{M}, \mathcal{B})$ be two $NSHA$s. Then, we have

$$\bigwedge_{z \in \beta_i(x_1, \ldots, x_{n_i})} t_a(z) \geq \min\{t_a(x_1), \ldots, t_a(x_{n_i})\}$$
$$\bigwedge_{z \in \beta_i(x_1, \ldots, x_{n_i})} i_a(z) \geq \min\{i_a(x_1), \ldots, i_a(x_{n_i})\}$$
$$\bigvee_{z \in \beta_i(x_1, \ldots, x_{n_i})} f_a(z) \leq \max\{f_a(x_1), \ldots, f_a(x_{n_i})\}$$

and

$$\bigwedge_{z \in \beta_i(x_1, \ldots, x_{n_i})} t_b(z) \geq \min\{t_b(x_1), \ldots, t_b(x_{n_i})\}$$
$$\bigwedge_{z \in \beta_i(x_1, \ldots, x_{n_i})} i_b(z) \geq \min\{i_b(x_1), \ldots, i_b(x_{n_i})\}$$
$$\bigvee_{z \in \beta_i(x_1, \ldots, x_{n_i})} f_b(z) \leq \max\{f_b(x_1), \ldots, f_b(x_{n_i})\}$$

for $\forall i \in I, \forall x_1, \ldots, x_{n_i} \in H, \forall z \in \beta_i(x_1, \ldots, x_{n_i}), a \in \mathcal{A}, b \in \mathcal{B}$. Suppose $(\mathcal{N}, \mathcal{A}) \tilde{\cap} (\tilde{\mathcal{M}}, \mathcal{B})$

$= (\mathcal{K}, \mathcal{C})$. So, for all $c \in C$

$$\begin{aligned}\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_c(z) &= \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} (t_a(z) \wedge t_b(z)) \\ &= \left(\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_a(z)\right) \wedge \left(\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_b(z)\right) \\ &\geq \min\{t_a(x_1),\ldots,t_a(x_{n_i})\} \wedge \min\{t_b(x_1),\ldots,t_b(x_{n_i})\} \\ &= \min\{t_a(x_1) \wedge t_b(x_1),\ldots,t_a(x_{n_i}) \wedge t_b(x_{n_i})\},\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_c(z) &= \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} (i_a(z) \wedge i_b(z)) \\ &= \left(\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_a(z)\right) \wedge \left(\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_b(z)\right) \\ &\geq \min\{i_a(x_1),\ldots,i_a(x_{n_i})\} \wedge \min\{i_b(x_1),\ldots,i_b(x_{n_i})\} \\ &= \min\{i_a(x_1) \wedge i_b(x_1),\ldots,i_a(x_{n_i}) \wedge i_b(x_{n_i})\},\end{aligned}$$

and

$$\begin{aligned}\bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} f_c(z) &= \bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} (f_a(z)) \vee (f_b(z)) \\ &= \left(\bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} (f_a(z))\right) \vee \left(\bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} (f_b(z))\right) \\ &\leq \max\{f_a(x_1),\ldots,f_a(x_{n_i})\} \vee \max\{f_b(x_1),\ldots,f_b(x_{n_i})\} \\ &= \max\{f_a(x_1) \vee f_b(x_1),\ldots,f_a(x_{n_i}) \vee f_b(x_{n_i})\}.\end{aligned}$$

Thus, $(\mathcal{N}, \mathcal{A})\tilde{\cap}(\tilde{\mathcal{M}}, \mathcal{B})$ is an $NSHA$ over H.

Let β be a null hyperoperation over H, for $z \in \beta$. Then,

$$\begin{aligned}(t_a \wedge t_b)(z) &= t_a(z) \wedge t_b(z) \geq t_a(y) \wedge t_b(y) \\ (i_a \wedge i_b)(z) &= i_a(z) \wedge i_b(z) \geq i_a(y) \wedge i_b(y) \\ (f_a \vee f_b)(z) &= f_a(z) \vee f_b(z) \leq f_a(y) \vee f_b(y)\end{aligned}$$

for every $y \in H$. Hence,

$$\begin{aligned}(t_a \wedge t_b)(z) &= (t_a \wedge t_b)(y) \\ (i_a \wedge i_b)(z) &= (i_a \wedge i_b)(y) \\ (f_a \vee f_b)(z) &= (f_a \vee f_b)(y).\end{aligned}$$

(ii) Presume that $\mathcal{A} \cap \mathcal{B} = \emptyset$, $(\mathcal{N}, \mathcal{A})\tilde{\cup}(\mathcal{M}, \mathcal{B}) = (\tilde{\mathcal{Z}}, \mathcal{C})$. Since $\mathcal{A} \cap \mathcal{B} = \emptyset$, for all $c \in C$, we have two cases:

Case 1: If $c \in \mathcal{A} - \mathcal{B}$,

$$\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_c(z) = \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_a(z)$$
$$\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_c(z) = \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_a(z)$$
$$\bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} f_c(z) = \bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} f_a(z)$$

Case 2: If $c \in \mathcal{B} - \mathcal{A}$,

$$\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_c(z) = \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_b(z)$$
$$\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_c(z) = \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_b(z)$$
$$\bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} f_c(z) = \bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} f_b(z).$$

Since $(\mathcal{N},\mathcal{A})$ and $(\mathcal{M},\mathcal{B})$ are two $NSHA$s, the proof is completed.
(iii) Assume $(\mathcal{N},\mathcal{A})\tilde{\wedge}(\mathcal{M},\mathcal{B}) = (\tilde{\mathcal{T}},\mathcal{A}\times\mathcal{B})$. Then,

$$\begin{aligned}\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_{(a,b)}(z) &= \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_a(z)\wedge t_b(z)\\ &= \left(\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_a(z)\right)\wedge\left(\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_b(z)\right)\\ &\geq \min\{t_a(x_1),\ldots,t_a(x_{n_i})\}\wedge\min\{t_b(x_1),\ldots,t_b(x_{n_i})\}\\ &= \min\{t_a(x_1)\wedge t_b(x_1),\ldots,t_a(x_{n_i})\wedge t_b(x_{n_i})\}\\ &= \min\{t_{(a,b)}(x_1),\ldots,t_{(a,b)}(x_{n_i})\}\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_{(a,b)}(z) &= \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_a(z)\wedge i_b(z)\\ &= \left(\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_a(z)\right)\wedge\left(\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_b(z)\right)\\ &\geq \min\{i_a(x_1),\ldots,i_a(x_{n_i})\}\wedge\min\{i_b(x_1),\ldots,i_b(x_{n_i})\}\\ &= \min\{i_a(x_1)\wedge i_b(x_1),\ldots,i_a(x_{n_i})\wedge i_b(x_{n_i})\}\\ &= \min\{i_{(a,b)}(x_1),\ldots,i_{(a,b)}(x_{n_i})\}\end{aligned}$$

and

$$\begin{aligned}\bigvee_{z\in\beta_f(x_1,\ldots,x_{n_f})} f_{(a,b)}(z) &= \bigvee_{z\in\beta_f(x_1,\ldots,x_{n_f})} f_a(z)\vee f_b(z)\\ &= \left(\bigvee_{z\in\beta_f(x_1,\ldots,x_{n_f})} f_a(z)\right)\vee\left(\bigvee_{z\in\beta_f(x_1,\ldots,x_{n_f})} f_b(z)\right)\\ &\leq \max\{f_a(x_1),\ldots,f_a(x_{n_f})\}\vee\max\{f_b(x_1),\ldots,f_b(x_{n_f})\}\\ &= \max\{f_a(x_1)\vee f_b(x_1),\ldots,f_a(x_{n_f})\vee f_b(x_{n_f})\}\\ &= \max\{f_{(a,b)}(x_1),\ldots,f_{(a,b)}(x_{n_f})\}\end{aligned}$$

Thus, $(\mathcal{N},\mathcal{A})\tilde{\wedge}(\mathcal{M},\mathcal{B})$ is an $NSHA$ over H. □

4. Level Set of NSHA

In a manner similar to [63], we define the neutrosophic soft (p,q,r)-*cut* of an NSS over hyperalgebra.

Definition 9. *Suppose $(\mathcal{N},\mathcal{A})$ is an NSS over hyperalgebra $\langle H,(\beta_i)_{i\in I}\rangle$. The soft set $(\mathcal{N},\mathcal{A})_{(p,q,r)} = \{(\mathcal{N}_a)_{(p,q,r)} \mid a\in\mathcal{A}\}$ for all $p,q,r\in[0,1]$, is named the neutrosophic soft (p,q,r)-cut of the NSS $(\mathcal{N},\mathcal{A})$ where $(\mathcal{N}_a)_{(p,q,r)}$ is the (p,q,r)-level set of the neutrosophic set $\mathcal{N}_a$.*

The following theorem shows the relation between $NSHA$ and soft hyperalgebra.

Theorem 2. *Suppose that $(\mathcal{N},\mathcal{A})$ is an NSS over hyperalgebra $\langle H,(\beta_i)_{i\in I}\rangle$. For $(\mathcal{N},\mathcal{A})$ to be an NSHA over H, it is a necessary and sufficient condition that the neutrosophic soft (p,q,r)-cut $(\mathcal{N},\mathcal{A})_{(p,q,r)}$ is a soft hyperalgebra over H for arbitrary $p,q,r\in[0,1]$ with $(\mathcal{N}_a)_{(p,q,r)}$.*

Proof. Assume that $\langle H,(\beta_i)_{i\in I}\rangle$ is a hyperalgebra and $(\mathcal{N},\mathcal{A})$ is an $NSHA$ over H. Then, t_a, i_a, f_a are neutrosophic subhyperalgebras of H for every $a\in\mathcal{A}$. For $p,q,r\in[0,1]$, $(t_a)_p\neq\emptyset$, $(i_a)_q\neq\emptyset$, $(f_a)_r\neq\emptyset$ and $\forall i\in I$, let $x_1,x_2,\ldots,x_{n_i}\in(t_a)_p,(i_a)_q,(f_a)_r$. So,

$$\begin{aligned}&t_a(x_1)\geq p, t_a(x_2)\geq p,\ldots,t_a(x_{n_i})\geq p,\\ &i_a(x_1)\geq q, i_a(x_2)\geq q,\ldots,i_a(x_{n_i})\geq q,\\ &f_a(x_1)\leq r, f_a(x_2)\leq r,\ldots,f_a(x_{n_i})\leq r.\end{aligned}$$

Hence, for all $z\in\beta_i(x_1,\ldots,x_{n_i})$, we have

$$\begin{aligned}\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_a(z)&\geq\min\{t_a(x_1),\ldots,t_a(x_{n_i})\}\geq\min\{p,\ldots,p\}=p\\ \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_a(z)&\geq\min\{i_a(x_1),\ldots,i_a(x_{n_i})\}\geq\min\{q,\ldots,q\}=q\\ \bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} f_a(z)&\leq\max\{f_a(x_1),\ldots,f_a(x_{n_i})\}\leq\max\{r,\ldots,r\}=r.\end{aligned}$$

Hence, $\beta_i(x_1,\ldots,x_{n_i})\subseteq(t_a)_p,(i_a)_q,(f_a)_r$.

If β is a null hyperoperation in $\langle H,(\beta_i)_{i\in I}\rangle$ for all $z\in\beta$, since t_a,i_a, f_a are neutrosophic subhyperalgebras over H, we explore $t_a(z)\geq t_a(y)$, $i_a(z)\geq i_a(y)$, $f_a(z)\leq f_a(y)$, $\forall y\in H$. It means $t_a(z)\geq p$, $i_a(z)\geq q$, $f_a(z)\leq r$. Hence, $\beta\subseteq(t_a)_p,(i_a)_q,(f_a)_r$. We find that $(t_a)_p$, $(i_a)_q$ and $(f_a)_r$ are subhyperalgebras of H for all $a\in\mathcal{A}$.

However, let $(\mathcal{N}, \mathcal{A})$ not be an $NSHA$ over H. In this condition, t_a, i_a, f_a are not $NSHA$s of H. Thus, there is $i \in I$, $x_1, x_2, \ldots, x_{n_i} \in H$ and $z \in \beta_i(x_1, \ldots, x_{n_i})$ such that

$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} t_a(z) < \min\{t_a(x_1), \ldots, t_a(x_{n_i})\}$$
$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} i_a(z) < \min\{i_a(x_1), \ldots, i_a(x_{n_i})\}$$
$$\bigvee_{z \in \beta_i(x_1,\ldots,x_{n_i})} f_a(z) > \max\{f_a(x_1), \ldots, f_a(x_{n_i})\}.$$

Let

$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} t_a(z) = \varepsilon, \text{ and } t_a(x_1) = \alpha_1, \ldots, t_a(x_{n_i}) = \alpha_{n_i}$$
$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} i_a(z) = \upsilon, \text{ and } i_a(x_1) = \lambda_1, \ldots, i_a(x_{n_i}) = \lambda_{n_i}$$
$$\bigvee_{z \in \beta_i(x_1,\ldots,x_{n_i})} f_a(z) = \gamma, \text{ and }, f_a(x_1) = \eta_1, \ldots, f_a(x_{n_i}) = \eta_{n_i}.$$

We obtain

$$\varepsilon < \min\{\alpha_1, \ldots, \alpha_{n_i}\}, \upsilon < \min\{\lambda_1, \ldots, \lambda_{n_i}\} \text{ and } \gamma > \max\{\eta_1, \ldots, \eta_{n_i}\}.$$

Presume that

$$p = \frac{\varepsilon + \min\{\alpha_1, \ldots, \alpha_{n_i}\}}{2}, \text{ and}$$
$$q = \frac{\upsilon + \min\{\lambda_1, \ldots, \lambda_{n_i}\}}{2}, \text{ and}$$
$$r = \frac{\gamma + \max\{\eta_1, \ldots, \eta_{n_i}\}}{2},$$

then

$$\varepsilon < p < \min\{\alpha_1, \ldots, \alpha_{n_i}\}, \upsilon < q < \min\{\lambda_1, \ldots, \lambda_{n_i}\} \text{ and } \gamma > r > \max\{\eta_1, \ldots, \eta_{n_i}\}.$$

Since

$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} t_a(z) = \varepsilon < p,$$
$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} i_a(z) = \upsilon < q$$
$$\bigvee_{z \in \beta_i(x_1,\ldots,x_{n_i})} f_a(z) = \gamma > r,$$

we regard $\beta_i(x_1, \ldots, x_{n_i}) \nsubseteq (t_a)_p$, $\beta_i(x_1, \ldots, x_{n_i}) \nsubseteq (i_a)_q$ and $\beta_i(x_1, \ldots, x_{n_i}) \nsubseteq (f_a)_r$.

Apart from that, $\min\{\alpha_1, \ldots, \alpha_{n_i}\} > p$, $\min\{\lambda_1, \ldots, \lambda_{n_i}\} > q$ and $\min\{\eta_1, \ldots, \eta_{n_i}\} < r$ require $t_a(x_1) \geq p, \ldots, t_a(x_{n_i}) \geq p$, $i_a(x_1) \geq q, \ldots, i_a(x_{n_i}) \geq q$ and $f_a(x_1) \leq r, \ldots, f_a(x_{n_i}) \leq r$; hence, $x_1, x_2, \ldots, x_{n_i} \in (t_a)_p, (i_a)_q, (f_a)_r$ which contradicts $(\mathcal{N}, \mathcal{A})_{(p,q,r)}$ being a soft hy-

peralgebra over H. Also, for all $i \in I$, $x_1, x_2, \ldots, x_{n_i} \in H$ and $z \in \beta_i(x_1, \ldots, x_{n_i})$, we consider

$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} t_a(z) \geq \min\{t_a(x_1), \ldots, t_a(x_{n_i})\}$$
$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} i_a(z) \geq \min\{i_a(x_1), \ldots, i_a(x_{n_i})\}$$
$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} f_a(z) \leq \max\{f_a(x_1), \ldots, f_a(x_{n_i})\}.$$

Moreover, let β be a null hyperoperation in $\langle H, (\beta_i)_{i \in I} \rangle$ for all $z \in \beta$ such that $t_a(z) < t_a(y)$, $i_a(z) < i_a(y)$ and $f_a(z) > f_a(y)$, for some $y \in H$, $p = t_a(y)$, $q = i_a(y)$ and $r = f_a(y)$. Since $(t_a)_p, (i_a)_q$ and $(f_a)_r$ are subhyperalgebras of H, we obtain $\beta \subseteq (t_a)_p$, $\beta \subseteq (i_a)_q$ and $\beta \subseteq (f_a)_r$, which is a contradiction. Hence, we obtain $t_a(y) \leq t_a(z)$, $i_a(y) \leq i_a(z)$ and $f_a(y) \geq f_a(z)$ for all $y \in H$. If $z_1, z_2 \in \beta$ and $t_a(z_1) \neq t_a(z_2)$, $i_a(z_1) \neq i_a(z_2)$ and $f_a(z_1) \neq f_a(z_2)$, we cannot reach the conclusion that for all $y \in H$, $z \in \beta$, $t_a(y) \leq t_a(z)$, $i_a(y) \leq i_a(z)$ and $f_a(y) \geq f_a(z)$. □

Theorem 3. *Suppose $\langle H, (\beta_i)_{i \in I} \rangle$ is a hyperalgebra and $(\mathcal{N}, \mathcal{A})$ is an NSHA over H. Let β be a null hyperoperation in H. The soft set*

$$\begin{aligned}(\mathcal{N}, \mathcal{A})\,|_\beta &= \left(t_a\,|_\beta, i_a\,|_\beta,\ f_a\,|_\beta\right) \\ &= \{y \in H : t_a(y) = t_a(\beta),\ i_a(y) = i_a(\beta),\ f_a(y) = f_a(\beta), a \in \mathcal{A}\}\end{aligned}$$

is a soft hyperalgebra over H.

Proof. For $i \in I$, $x_1, \ldots, x_{n_i} \in t_a\,|_\beta, i_a\,|_\beta, f_a\,|_\beta$, $a \in \mathcal{A}$ and $z \in \beta_i(x_1, \ldots, x_{n_i})$,

$$\begin{aligned}\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} t_a(z) &\geq \min\{t_a(x_1), \ldots, t_a(x_{n_i})\} \\ &= \min\{t_a(\beta), \ldots, t_a(\beta)\} = t_a(\beta),\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} i_a(z) &\geq \min\{i_a(x_1), \ldots, i_a(x_{n_i})\} \\ &= \min\{i_a(\beta), \ldots, i_a(\beta)\} = i_a(\beta),\end{aligned}$$

and

$$\begin{aligned}\bigvee_{z \in \beta_i(x_1,\ldots,x_{n_i})} f_a(z) &\leq \max\{f_a(x_1), \ldots, f_a(x_{n_i})\} \\ &= \max\{f_a(\beta), \ldots, f_a(\beta)\} = f_a(\beta).\end{aligned}$$

On the other hand, we always obtain

$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} t_a(z) \leq t_a(\beta),$$
$$\bigwedge_{z \in \beta_i(x_1,\ldots,x_{n_i})} i_a(z) \leq i_a(\beta),$$
$$\bigvee_{z \in \beta_i(x_1,\ldots,x_{n_i})} f_a(z) \geq f_a(\beta).$$

It means that

$$\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_a(z) = t_a(\beta), \quad \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_a(z) = i_a(\beta) \text{ and } \bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} f_a(z) = f_a(\beta).$$

Thus, $\beta_i(x_1,\ldots,x_{n_i}) \subseteq t_a\mid_\beta, i_a\mid_\beta, f_a\mid_\beta$. It goes without saying that, for every $z \in \beta'$ where β' is a null hyperoperation, $z \in t_a\mid_\beta, i_a\mid_\beta, f_a\mid_\beta$. Additionally, $t_a\mid_\beta, i_a\mid_\beta$ and $f_a\mid_\beta$ are subhyperalgebras of H and $(\mathcal{N},\mathcal{A})\mid_\beta$ is soft hyperalgebra over H. □

5. Homomorphism of *NSHA*

Now, we will identify ξ_β-identity $NSHA$s and ξ-absolute $NSHA$s over H. We will then show the conditions under which the image of an $NSHA$ is a ξ_β-identity $NSHA$, and how a ξ-absolute $NSHA$ remains ξ-absolute under homomorphism.

Definition 10. *Assume that $\langle H,(\beta_i)_{i\in I}\rangle$ is a hyperalgebra. Let $(\mathcal{N},\mathcal{A})$ be an NSHA over H and β be a null hyperoperation in H. For $\xi \in (0,1]$, every $y \in H$ and $a \in \mathcal{A}$;*

(1) If

$$t_a(y) = \begin{cases} \xi, & if\ y \in \beta \\ 0, & otherwise \end{cases},\ i_a(y) = \begin{cases} \xi, & if\ y \in \beta \\ 0, & otherwise \end{cases},\ f_a(y) = \begin{cases} 1-\xi, & if\ y \in \beta \\ 1, & otherwise \end{cases},$$

then $(\mathcal{N},\mathcal{A})$ is called a ξ_β-identity NSHA over H.

(2) If $t_a(y) = \xi, i_a(y) = \xi$ and $f_a(y) = 1-\xi$, then $(\mathcal{N},\mathcal{A})$ is called a ξ-absolute NSHA over H.

It is noted that $\sigma(\mathcal{N})$ implies $(\sigma(t_a),\sigma(i_a),\sigma(f_a))$ for $a \in \mathcal{A}$.

Theorem 4. *Suppose that $\langle H,(\beta_i)_{i\in I}\rangle$ and $\langle H',(\beta'_i)_{i\in I}\rangle$ are similar hyperalgebras; $\sigma : H \to H'$ is a homomorphism and β' is a null hyperoperation in H'. We characterize $(\ker\sigma)_{\beta'} = \{y \in H : \sigma(y) \in \beta'\}$.*

(i) If for $a \in \mathcal{A}$, $y \in H$,

$$t_a(y) = \begin{cases} \xi, & if\ y \in (\ker\sigma)_{\beta'} \\ 0, & otherwise \end{cases},$$

$$i_a(y) = \begin{cases} \xi, & if\ y \in (\ker\sigma)_{\beta'} \\ 0, & otherwise \end{cases}, \text{ and}$$

$$f_a(y) = \begin{cases} 1-\xi, & if\ y \in (\ker\sigma)_{\beta'} \\ 1, & otherwise \end{cases}$$

where $(\mathcal{N},\mathcal{A})$ is an NSHA over H, then $\sigma(\mathcal{N})$ is a $\xi_{\beta'}$-identity NSHA over H'.

(ii) Let $(\mathcal{N},\mathcal{A})$ be a ξ-absolute NSHA over H. Then, $\sigma(\mathcal{N})$ is a ξ-absolute NSHA over H'.

Proof. (i) For every $z \in \beta'$ and $y \in H$,

$$\sigma(t_a)(z) = \bigvee_{y\in\sigma^{-1}(z)} t_a(y) = \bigvee_{y\in(\ker\sigma)_{\beta'}} t_a(y) = \xi,$$

$$\sigma(i_a)(z) = \bigvee_{y\in\sigma^{-1}(z)} i_a(y) = \bigvee_{y\in(\ker\sigma)_{\beta'}} i_a(y) = \xi,$$

$$\sigma(f_a)(z) = \bigwedge_{y\in\sigma^{-1}(z)} f_a(y) = \bigwedge_{y\in(\ker\sigma)_{\beta'}} f_a(y) = 1-\xi.$$

Let $k \in H'$ and $k \notin \beta'$. We have $\sigma(t_a)(k) = 0$, $\sigma(i_a)(k) = 0$ and $\sigma(f_a)(k) = 1$. Then, $\sigma(\mathcal{N})$ is a $\xi_{\beta'}$-identity $NSHA$ over H'.

(ii) Let $(\mathcal{N}, \mathcal{A})$ be a ξ-absolute $NSHA$ over H. For all $k \in H'$,

$$\sigma(t_a)(k) = \bigvee_{y \in \sigma^{-1}(k)} t_a(y) = \xi,$$
$$\sigma(i_a)(k) = \bigvee_{y \in \sigma^{-1}(k)} i_a(y) = \xi,$$
$$\sigma(f_a)(k) = \bigwedge_{y \in \sigma^{-1}(k)} f_a(y) = 1 - \xi.$$

Thus, we find that $\sigma(\mathcal{N})$ is a ξ-absolute $NSHA$ over H'. □

We will propose $NSHA$ homomorphism and isomorphism. Following this, under homomorphism, we will demonstrate if $(\mathcal{N}, \mathcal{A})$ is an $NSHA$ over H, then the image and the pre-image image of $(\mathcal{N}, \mathcal{A})$ are also $NSHA$s over H' where (σ, ρ) assigns H to H'.

Definition 11. *Presume that $(\mathcal{N}, \mathcal{A})$ and $(\mathcal{M}, \mathcal{B})$ are two NSHAs over H and H', respectively. (σ, ρ) is a neutrosophic soft function from H to H' :*

(1) Let σ be a homomorphism. Then, (σ, ρ) is called an $NSHA$ homomorphism.

(2) Let σ be an isomorphism and ρ be a one-to-one mapping from $(\mathcal{N}, \mathcal{A})$ onto $(\mathcal{M}, \mathcal{B})$. Then, (σ, ρ) is called an $NSHA$ isomorphism.

Theorem 5. *Suppose that $\langle H, (\beta_i)_{i \in I} \rangle$ and $\langle H', (\beta'_i)_{i \in I} \rangle$ are similar hyperalgebras. If $(\mathcal{N}, \mathcal{A})$ is an $NSHA$ over H, then $(\sigma, \rho)(\mathcal{N}, \mathcal{A})$ is an $NSHA$ over H' where (σ, ρ) is a neutrosophic soft homomorphism from H to H'.*

Proof. Let $i \in I$ and $y_1, y_2, \ldots, y_{n_i} \in H'$. If $\sigma^{-1}(y_1) = \emptyset$ or $\sigma^{-1}(y_2) = \emptyset \ldots$ or $\sigma^{-1}(y_{n_i}) = \emptyset$, the proof is clear. Assume that $m \in \rho(\mathcal{A})$ and $x_1, x_2, \ldots, x_{n_i} \in H$ such that $\sigma(x_1) = y_1, \sigma(x_2) = y_2, \ldots, \sigma(x_{n_i}) = y_{n_i}$. For all $z \in \beta_i(x_1, \ldots, x_{n_i})$ and $z' \in \beta'_i(y_1, y_2, \ldots, y_{n_i})$ such that $\sigma(z) = z'$, we obtain

$$\begin{aligned}
\bigwedge_{z' \in \beta'_i(y_1, y_2, \ldots, y_{n_i})} \sigma(t_m)(z') &= \bigwedge_{z' \in \beta'_i(y_1, y_2, \ldots, y_{n_i})} \left(\bigvee_{\sigma(z) = z'} \bigvee_{\rho(a) = m} t_a(z) \right) \\
&\geq \bigvee_{\sigma(z) = z'} \bigvee_{\rho(a) = m} \left(\bigwedge_{z \in \beta_i(x_1, x_2, \ldots, x_{n_i})} t_a(z) \right) \\
&\geq \bigvee_{\rho(a) = m} \left(\bigwedge_{z \in \beta_i(x_1, x_2, \ldots, x_{n_i})} t_a(z) \right) \\
&\geq \bigvee_{\rho(a) = m} \min\{t_a(x_1), \ldots, t_a(x_{n_i})\} \\
&= \min\left\{ \bigvee_{\rho(a) = m} t_a(x_1), \ldots, \bigvee_{\rho(a) = m} t_a(x_{n_i}) \right\}
\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{z'\in\beta_i'(y_1,y_2,\ldots,y_{n_i})}\sigma(i_m)(z') &= \bigwedge_{z'\in\beta_i'(y_1,y_2,\ldots,y_{n_i})}\left(\bigvee_{\sigma(z)=z'}\bigvee_{\rho(a)=m} i_a(z)\right)\\ &\geq \bigvee_{\sigma(z)=z'}\bigvee_{\rho(a)=m}\left(\bigwedge_{z\in\beta_i(x_1,x_2,\ldots,x_{n_i})} i_a(z)\right)\\ &\geq \bigvee_{\rho(a)=m}\left(\bigwedge_{z\in\beta_i(x_1,x_2,\ldots,x_{n_i})} i_a(z)\right)\\ &\geq \bigvee_{\rho(a)=m}\min\{i_a(x_1),\ldots,i_a(x_{n_i})\}\\ &= \min\left\{\bigvee_{\rho(a)=m} i_a(x_1),\ldots,\bigvee_{\rho(a)=m} i_a(x_{n_i})\right\}\end{aligned}$$

and

$$\begin{aligned}\bigvee_{z'\in\beta_f'(y_1,y_2,\ldots,y_{n_f})}\sigma(f_m)(z') &= \bigvee_{z'\in\beta_f'(y_1,y_2,\ldots,y_{n_f})}\left(\bigwedge_{\sigma(z)=z'}\bigwedge_{\rho(a)=m} f_a(z)\right)\\ &\leq \bigwedge_{\sigma(z)=z'}\bigwedge_{\rho(a)=m}\left(\bigvee_{z\in\beta_f(x_1,x_2,\ldots,x_{n_f})} f_a(z)\right)\\ &\leq \bigwedge_{\rho(a)=m}\left(\bigvee_{z\in\beta_f(x_1,x_2,\ldots,x_{n_f})} f_a(z)\right)\\ &\leq \bigwedge_{\rho(a)=m}\max\{f_a(x_1),\ldots,f_a(x_{n_f})\}\\ &= \max\left\{\bigwedge_{\rho(a)=m} f_a(x_1),\ldots,\bigwedge_{\rho(a)=m} f_a(x_{n_f})\right\}.\end{aligned}$$

These inequalities are valid for each $x_1,\ldots,x_{n_i}\in H$ such that $\sigma(x_1)=y_1,\ldots,\sigma(x_{n_i})=y_{n_i}$. We put

$$\begin{aligned}\alpha_1 &= \bigvee_{\rho(a)=m} t_a(x_1),\ldots,\alpha_{n_i} = \bigvee_{\rho(a)=m} t_a(x_{n_i})\\ \lambda_1 &= \bigvee_{\rho(a)=m} i_a(x_1),\ldots,\lambda_{n_i} = \bigvee_{\rho(a)=m} i_a(x_{n_i})\\ \eta_1 &= \bigwedge_{\rho(a)=m} f_a(x_1),\ldots,\eta_{n_i} = \bigwedge_{\rho(a)=m} f_a(x_{n_i})\end{aligned}$$

and we obtain

$$\begin{aligned}\bigwedge_{z'\in\beta_i'(y_1,y_2,\ldots,y_{n_i})}\sigma(t)_m(z') &\geq \bigvee_{\sigma(x_{n_i})=y_{n_i}}\left(\ldots\bigvee_{\sigma(x_1)=y_1}\min\{\alpha_1,\ldots,\alpha_{n_i}\}\right)\\ &=\min\left\{\bigvee_{\sigma(x_1)=y_1}\alpha_1,\ldots,\bigvee_{\sigma(x_1)=y_1}\alpha_{n_i}\right\}\\ &=\min\{\sigma(t)_m(y_1),\ldots,\sigma(t)_m(y_{n_i})\}\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{z'\in\beta_i'(y_1,y_2,\ldots,y_{n_i})}\sigma(i)_m(z') &\geq \bigvee_{\sigma(x_{n_i})=y_{n_i}}\left(\ldots\bigvee_{\sigma(x_1)=y_1}\min\{\lambda_1,\ldots,\lambda_{n_i}\}\right)\\ &=\min\left\{\bigvee_{\sigma(x_1)=y_1}\lambda_1,\ldots,\bigvee_{\sigma(x_1)=y_1}\lambda_{n_i}\right\}\\ &=\min\{\sigma(i)_m(y_1),\ldots,\sigma(i)_m(y_{n_i})\}\end{aligned}$$

and

$$\begin{aligned}\bigvee_{z'\in\beta_f'(y_1,y_2,\ldots,y_{n_f})}\sigma(f)_m(z') &\leq \bigwedge_{\sigma(x_{n_f})=y_{n_f}}\left(\ldots\bigwedge_{\sigma(x_1)=y_1}\max\{\eta_1,\ldots,\eta_{n_f}\}\right)\\ &=\max\left\{\bigwedge_{\sigma(x_1)=y_1}\eta_1,\ldots,\bigwedge_{\sigma(x_1)=y_1}\eta_{n_f}\right\}\\ &=\max\{\sigma(f)_m(y_1),\ldots,\sigma(f)_m(y_{n_f})\}.\end{aligned}$$

Let β' be a null hyperoperation in H'. For $z\in\beta'$, we obtain

$$\begin{aligned}\sigma(t_m)(z) &= \bigvee_{\sigma(x)=z\rho(a)=m}\bigvee t_a(x)\\ \sigma(i_m)(z) &= \bigvee_{\sigma(x)=z\rho(a)=m}\bigvee i_a(x)\\ \sigma(f_m)(z) &= \bigwedge_{\sigma(x)=z\rho(a)=m}\bigwedge f_a(x).\end{aligned}$$

Since $\sigma^{-1}(\beta')$ is a null hyperoperation in H, $t_a(x)$, $i_a(x)$ and $f_a(x)$ are constant where $\sigma(x)=z$. Also, $\sigma(t_m)(z)$, $\sigma(i_m)(z)$ and $\sigma(f_m)(z)$ are constant.

Since $t_a(x)\geq t_a(l)$, $i_a(x)\geq i_a(l)$ and $f_a(x)\leq f_a(l)$ for all $l\in H$, we obtain $\sigma(t_m)(z)\geq\sigma(t_m)(s)$, $\sigma(i_m)(z)\geq\sigma(i_m)(s)$ and $\sigma(f_m)(z)\leq\sigma(f_m)(s)$ for all $s\in H'$. The proof is completed. □

Theorem 6. *Presume that $\langle H,(\beta_i)_{i\in I}\rangle$ and $\langle H',(\beta_i')_{i\in I}\rangle$ are similar hyperalgebras and (σ,ρ) is a neutrosophic soft homomorphism from H to H'. If $(\mathcal{M},\mathcal{B})$ is an NSHA over H', then $(\sigma,\rho)^{-1}(\mathcal{M},\mathcal{B})$ is an NSHA over H.*

Proof. Let $a \in \rho^{-1}(\mathcal{B})$, $i \in I$ and $x_1, \ldots, x_{n_i} \in H$. For all $z \in \beta_i(x_1, \ldots, x_{n_i})$,

$$\begin{aligned}\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} \sigma^{-1}(t_a)(z) &= \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} t_{\rho(a)}(\sigma(z)) \\ &\geq \min\{t_{\rho(a)}(\sigma(x_1)), \ldots, t_{\rho(a)}(\sigma(x_{n_i}))\} \\ &= \min\{\sigma^{-1}(t_a)(x_1), \ldots, \sigma^{-1}(t_a)(x_{n_i})\}\end{aligned}$$

and

$$\begin{aligned}\bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} \sigma^{-1}(i_a)(z) &= \bigwedge_{z\in\beta_i(x_1,\ldots,x_{n_i})} i_{\rho(a)}(\sigma(z)) \\ &\geq \min\{i_{\rho(a)}(\sigma(x_1)), \ldots, i_{\rho(a)}(\sigma(x_{n_i}))\} \\ &= \min\{\sigma^{-1}(i_a)(x_1), \ldots, \sigma^{-1}(i_a)(x_{n_i})\}\end{aligned}$$

and

$$\begin{aligned}\bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} \sigma^{-1}(f_a)(z) &= \bigvee_{z\in\beta_i(x_1,\ldots,x_{n_i})} f_{\rho(a)}(\sigma(z)) \\ &\leq \max\{f_{\rho(a)}(\sigma(x_1)), \ldots, f_{\rho(a)}(\sigma(x_{n_i}))\} \\ &= \max\{\sigma^{-1}(f_a)(x_1), \ldots, \sigma^{-1}(f_a)(x_{n_i})\}.\end{aligned}$$

Let β be a null hyperoperation in H. For $z \in \beta$,

$$\begin{aligned}\sigma^{-1}(t_a)(z) &= t_{\rho(a)}(\sigma(z)), \text{ and} \\ \sigma^{-1}(i_a)(z) &= i_{\rho(a)}(\sigma(z)), \text{ and} \\ \sigma^{-1}(f_a)(z) &= f_{\rho(a)}(\sigma(z)),\end{aligned}$$

$t_{\rho(a)}(\sigma(z))$, $i_{\rho(a)}(\sigma(z))$ and $f_{\rho(a)}(\sigma(z))$ are constant, for all $z \in \beta$ since $\sigma(\beta)$ is a null hyperoperation in H and $\sigma(z) \in \sigma(\beta)$. Hence, as $t_{\rho(a)}(\sigma(z))$ and $i_{\rho(a)}(\sigma(z))$ have greatest degrees and $f_{\rho(a)}(\sigma(z))$ has greatest degrees, the same applies to $\sigma^{-1}(t_a)(z)$, $\sigma^{-1}(i_a)(z)$ and $\sigma^{-1}(f_a)(z)$. □

6. Methodology

The approach is scientifically developed by looking at the algebraic proofs of the theorems and propositions, identifying the symbols and notations used in the proofs, and using axiom assumptions. This approach allows for a rigorous analysis of mathematical concepts and their relationships. By breaking down the algebraic proofs, researchers can gain a deeper understanding of the underlying principles and extend them to new problem-solving techniques. Additionally, this scientific approach ensures that mathematical arguments are based on solid foundations, enhancing the reliability and validity of mathematical theories. As a result, the approach is built on direct evidence, construction techniques, and mathematical induction to guide the process and apply the findings to particular project methodologies and applications for artificially intelligent chatbots. This approach ensures a systematic and rigorous analysis of the mathematical foundations behind the development of artificially intelligent chatbots. It enables the development of reliable and effective chatbot systems by utilizing algebraic proofs and axiom assumptions, which allows for a deeper understanding of the underlying principles. Additionally, this approach facilitates the integration of advanced mathematical concepts into the design and implementation of chatbot algorithms, leading to enhanced performance and improved user experiences.

7. Application

In this section, a new approach to NSS decision-making based on hyperalgebra is presented. A practical application based on the selection of the best project method in the real world is given.

Let $\Delta_k = \left[\Delta_{ij}^k\right]_{mn}$ be an $NSHA$ where $\Delta_{ij}^k = \left(t_{ij}^{\Delta_k}, i_{ij}^{\Delta_k}, f_{ij}^{\Delta_k}\right)$. Then, the value of the matrix Δ_k is denoted by $\delta(\Delta_k)$ and is defined as

$\delta(\Delta_k) = \left[\Delta_{ij}^k\right]_{mn}$ where $\Delta_{ij}^k = \omega_i.\left(t_{ij}^{\Delta_k} + i_{ij}^{\Delta_k} - f_{ij}^{\Delta_k}\right)$ for all i, j and where weights ω_i such that $\sum_{i=1}^{n} \omega_i = 1$.

The score of two $NSHA$s Δ_k is defined as $\sigma = \left[\sigma_{ij}^{\Delta_k}\right]_{mn}$ where $\sigma_{ij}^{\Delta_k} = \sum \delta_{ij}(\Delta_k)$. Then, $\sigma = \sum \delta(\Delta_k)$.

The total score for each object in U is $\eta = \sum_{j=1}^{n} \sigma_{ij}$.

Let us assume in the methodology that k decision-makers want to collectively choose one object out of m objects, i.e., hyperalgebra H with respect to n characteristics, i.e., parameter set $\mathcal{A}$. Each decision-maker forms an NSS over $(\mathcal{N}, \mathcal{A})$ and corresponding to each NSS, each receives a matrix of order $m \times n$. The value matrix corresponding to each matrix must be calculated. Then, the rating matrix and finally the overall rating of each object are calculated.

Algorithm 1 represents a structured method for determining the optimal solution by analyzing NSS and $NSHA$. It proceeds step by step, constructing $NSHAs$, computing value matrices, and deriving a score matrix. The algorithm identifies the object with the highest score, offering a decision-making approach that ensures the selection of the most suitable option based on computed criteria. If more than one object has the highest score, further calculations are made to break the tie and select the optimal solution.

Algorithm 1 Optimal Object Selection

Step 1: Input NSSs.
Step 2: Construct the $NSHA$s and write them in tabular form as Δ_k.
Step 3: Compute the value matrices $\delta(\Delta_k)$ of corresponding Δ_k.
Step 4: Calculate the score matrix σ from the value matrices and the total score η for each object in U.
Step 5: Find the maximum object score, which is the optimal solution.
Step 6: If the score is maximum for more than one object, then find $\eta = \sum_{j=1}^{n} (\sigma_{ij})^k$, $k \geq 2$. Choose the object with the maximum number of points and thus the optimal solution.

Figure 1 provides the flowchart for the suggested algorithm.

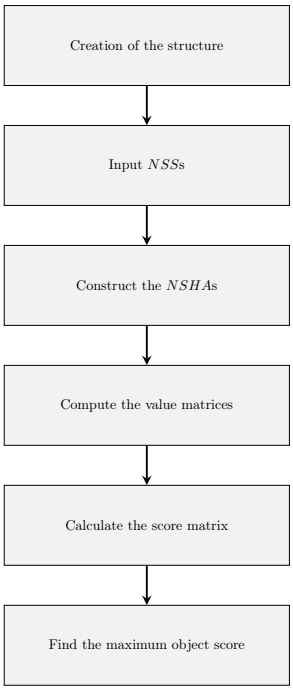


Figure 1. Flowchart of the suggested method.

7.1. A Numerical Example

IT software development processes and companies' operational processes ensure customer satisfaction and are of great importance in order to compete with their competitors. Therefore, our analysis is detailed and appropriate to the scope of the software process. Prepared in a timely manner, with the latest techniques and complete testing of the process, the project will achieve success. These processes are correct, and a good team, successful planning, effective resources, and project management are required.

It is very important to determine the appropriate project management, as every project undertaken today has its own characteristics. The success of the project is ensured by choosing and applying the right project management method. Successful project management is essential for companies to make the right decisions and progress. Figure 2 visualizes the effects of different project management methods on various performance indicators.

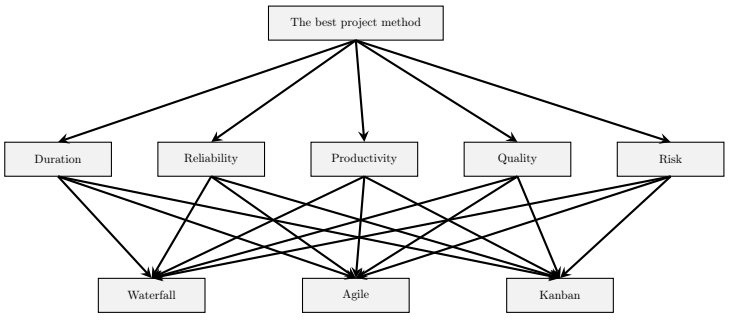


Figure 2. Structure of the Application

Three methods $\{w, a, k\}$ from project methodologies, indicating Waterfall, Agile, and Kanban in a company, have been selected to determine the best method for a year. A team of three project leaders $\{P_1, P_2, P_3\}$ has been formed by the Project Management Department of that company for this purpose. The final selection is based on the set of parameters $\{e_1, e_2, e_3, e_4, e_5\}$ indicating project duration, reliability on the project, productivity, quality of project, and risk of the project, respectively. Project leaders have given their valuable opinions by the following $NSHA$s separately, i.e., the first NSS given by P_1, and so on.

$\mathcal{N}_1 = \{\mathcal{N}_1(e_1), \mathcal{N}_1(e_2), \mathcal{N}_1(e_3), \mathcal{N}_1(e_4), \mathcal{N}_1(e_5)\}$ where
$\mathcal{N}_1(e_1) = \{[0.8, 0.3, 0.7]/w, [0.7, 0.4, 0.6]/a, [0.9, 0.4, 0.6]/k\}$
$\mathcal{N}_1(e_2) = \{[0.5, 0.7, 0.8]/w, [0.8, 0.7, 0.4]/a, [0.6, 0.6, 0.5]/k\}$
$\mathcal{N}_1(e_3) = \{[0.6, 0.5, 0.4]/w, [0.8, 0.5, 0.5]/a, [0.7, 0.5, 0.7]/k\}$
$\mathcal{N}_1(e_4) = \{[0.7, 0.7, 0.6]/w, [0.6, 0.9, 0.7]/a, [0.5, 0.8, 0.5]/k\}$
$\mathcal{N}_1(e_5) = \{[0.9, 0.4, 0.5]/w, [0.8, 0.3, 0.7]/a, [1.0, 0.2, 0.3]/k\}$
$\mathcal{N}_2 = \{\mathcal{N}_2(e_1), \mathcal{N}_2(e_2), \mathcal{N}_2(e_3), \mathcal{N}_2(e_4), \mathcal{N}_2(e_5)\}$ where
$\mathcal{N}_2(e_1) = \{[0.7, 0.5, 0.6]/w, [0.8, 0.5, 0.3]/a, [1.0, 0.5, 0.3]/k\}$
$\mathcal{N}_2(e_2) = \{[0.6, 0.6, 0.7]/w, [0.9, 0.6, 0.2]/a, [0.7, 0.8, 0.6]/k\}$
$\mathcal{N}_2(e_3) = \{[0.8, 0.4, 0.5]/w, [0.9, 0.6, 0.4]/a, [0.6, 0.7, 0.8]/k\}$
$\mathcal{N}_2(e_4) = \{[0.8, 0.6, 0.4]/w, [0.7, 0.8, 0.6]/a, [0.6, 0.6, 0.6]/k\}$
$\mathcal{N}_2(e_5) = \{[0.7, 0.5, 0.7]/w, [0.7, 0.4, 0.8]/a, [0.9, 0.4, 0.4]/k\}$
$\mathcal{N}_3 = \{\mathcal{N}_3(e_1), \mathcal{N}_3(e_2), \mathcal{N}_3(e_3), \mathcal{N}_3(e_4), \mathcal{N}_3(e_5)\}$ where
$\mathcal{N}_3(e_1) = \{[0.9, 0.4, 0.4]/w, [0.9, 0.6, 0.5]/a, [1.0, 0.5, 0.3]/k\}$
$\mathcal{N}_3(e_2) = \{[0.7, 0.5, 0.6]/w, [0.8, 0.7, 0.3]/a, [0.9, 0.6, 0.5]/k\}$
$\mathcal{N}_3(e_3) = \{[0.9, 0.5, 0.2]/w, [0.8, 0.6, 0.6]/a, [0.7, 0.8, 0.4]/k\}$
$\mathcal{N}_3(e_4) = \{[0.7, 0.7, 0.3]/w, [0.9, 0.7, 0.5]/a, [0.8, 0.4, 0.7]/k\}$
$\mathcal{N}_3(e_5) = \{[0.9, 0.5, 0.3]/w, [0.7, 0.5, 0.4]/a, [0.8, 0.6, 0.5]/k\}$

7.1.1. Case 1

Suppose that the project leaders have not given an order of importance to the elements of the parameters. The above three *NSS*s are represented by Δ_1, Δ_2 and Δ_3, respectively, as follows:

Table 2 presents the results of $NSHA$, represented as Δ_1, as given by the first project leader.

Table 2. Δ_1-$NSHA$ given by P_1.

	e_1			e_2			e_3			e_4			e_5		
Methods	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**
Waterfall	0.8	0.3	0.7	0.5	0.7	0.8	0.6	0.5	0.4	0.7	0.7	0.6	0.9	0.4	0.5
Agile	0.7	0.4	0.6	0.8	0.7	0.4	0.8	0.5	0.5	0.6	0.9	0.7	0.8	0.3	0.7
Kanban	0.9	0.4	0.6	0.6	0.6	0.5	0.7	0.5	0.7	0.5	0.8	0.5	1.0	0.2	0.3

Table 3 displays the outcomes of $NSHA$, indicated as Δ_2, based on the second project leader.

Table 3. Δ_2-$NSHA$ given by P_2.

	e_1			e_2			e_3			e_4			e_5		
Methods	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**
Waterfall	0.7	0.5	0.6	0.6	0.6	0.7	0.8	0.4	0.5	0.8	0.6	0.4	0.7	0.5	0.7
Agile	0.8	0.5	0.3	0.9	0.6	0.2	0.9	0.6	0.4	0.7	0.8	0.6	0.7	0.4	0.8
Kanban	1.0	0.5	0.3	0.7	0.8	0.6	0.6	0.7	0.8	0.6	0.6	0.6	0.9	0.4	0.4

Table 4 shows the findings of $NSHA$, denoted as Δ_3, as reported by the third project leader.

Table 4. Δ_3-$NSHA$ given by P_3.

	e_1			e_2			e_3			e_4			e_5		
Methods	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**
Waterfall	0.9	0.4	0.4	0.7	0.5	0.6	0.9	0.5	0.2	0.7	0.7	0.3	0.9	0.5	0.3
Agile	0.9	0.6	0.5	0.8	0.7	0.3	0.8	0.6	0.6	0.9	0.7	0.5	0.7	0.5	0.4
Kanban	1.0	0.5	0.3	0.9	0.6	0.5	0.7	0.8	0.4	0.8	0.4	0.7	0.8	0.6	0.5

Now, we perform calculations, and the corresponding value matrices are as follows:

$$\delta(\Delta_1) = \begin{bmatrix} 0.4 & 0.4 & 0.7 & 0.7 & 0.8 \\ 0.5 & 1.1 & 0.8 & 0.8 & 0.4 \\ 0.7 & 0.7 & 0.5 & 0.8 & 0.9 \end{bmatrix}$$

$$\delta(\Delta_2) = \begin{bmatrix} 0.6 & 0.5 & 0.7 & 1.0 & 0.5 \\ 1.0 & 1.3 & 1.1 & 0.9 & 0.3 \\ 1.2 & 0.9 & 0.5 & 0.6 & 0.9 \end{bmatrix}$$

$$\delta(\Delta_3) = \begin{bmatrix} 0.9 & 0.6 & 1.2 & 1.1 & 1.1 \\ 1.0 & 1.2 & 0.8 & 1.1 & 0.8 \\ 1.2 & 1.0 & 1.1 & 0.5 & 0.9 \end{bmatrix}$$

The score matrix is

$$\sigma = \begin{bmatrix} 1.9 & 1.5 & 2.6 & 2.9 & 2.4 \\ 2.5 & 3.6 & 2.7 & 2.8 & 1.5 \\ 3.1 & 2.6 & 2.1 & 1.9 & 2.7 \end{bmatrix}$$

and the total score is

$$\eta = \begin{bmatrix} 11.3 \\ 13.1 \\ 12.4 \end{bmatrix}$$

The second Agile method will be selected as the best project method for the company in that year.

7.1.2. Case 2

Assume the project leaders have given an order of importance to the elements of the parameter. Then, $NSHAs$ are more helpful in dealing with decision-making problems involving linguistic phrases. Let the weights $\omega_1 = 0.32$, $\omega_2 = 0.15$, $\omega_3 = 0.08$, $\omega_4 = 0.2$ and $\omega_5 = 0.25$ corresponding to the parameters e_1, e_2, e_3, e_4 and e_5 of the project be given separately by the project leaders such that $\sum_{i=1}^{n} \omega_i = 1$. Thus, results obtained by using $NSHA$s are taken into account.

The above three NSSs are represented by the Δ_1, Δ_2 and Δ_3, respectively, as follows:

Table 5 displays the outcomes of $NSHA$ with weights, denoted as Δ_1, as provided by the first project leader.

Table 5. Δ_1-$NSHA$ with weights given by P_1.

	$e_1, \omega_1 = 0.32$			$e_2, \omega_2 = 0.15$			$e_3, \omega_3 = 0.08$			$e_4, \omega_4 = 0.2$			$e_5, \omega_5 = 0.25$		
Methods	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	f	**t**	**i**	**f**
Waterfall	0.8	0.3	0.7	0.5	0.7	0.8	0.6	0.5	0.4	0.7	0.7	0.6	0.9	0.4	0.5
Agile	0.7	0.4	0.6	0.8	0.7	0.4	0.8	0.5	0.5	0.6	0.9	0.7	0.8	0.3	0.7
Kanban	0.9	0.4	0.6	0.6	0.6	0.5	0.7	0.5	0.7	0.5	0.8	0.5	1.0	0.2	0.3

Table 6 presents the results of $NSHA$ with weights, referred to as Δ_2, as supplied by the second project leader.

Table 6. Δ_2-$NSHA$ with weights given by P_2.

	$e_1, \omega_1 = 0.32$			$e_2, \omega_2 = 0.15$			$e_3, \omega_3 = 0.08$			$e_4, \omega_4 = 0.2$			$e_5, \omega_5 = 0.25$		
Methods	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	f	**t**	**i**	**f**
Waterfall	0.7	0.5	0.6	0.6	0.6	0.7	0.8	0.4	0.5	0.8	0.6	0.4	0.7	0.5	0.7
Agile	0.8	0.5	0.3	0.9	0.6	0.2	0.9	0.6	0.4	0.7	0.8	0.6	0.7	0.4	0.8
Kanban	1.0	0.5	0.3	0.7	0.8	0.6	0.6	0.7	0.8	0.6	0.6	0.6	0.9	0.4	0.4

Table 7 shows the results of $NSHA$ with weights, referred to as Δ_3, as supplied by the third project leader. Now, we perform calculations, and the corresponding value matrices are as follows:

Table 7. Δ_3-$NSHA$ with weights given by P_3.

	$e_1, \omega_1 = 0.32$			$e_2, \omega_2 = 0.15$			$e_3, \omega_3 = 0.08$			$e_4, \omega_4 = 0.2$			$e_5, \omega_5 = 0.25$		
Methods	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	f	**t**	**i**	**f**
Waterfall	0.9	0.4	0.4	0.7	0.5	0.6	0.9	0.5	0.2	0.7	0.7	0.3	0.9	0.5	0.3
Agile	0.9	0.6	0.5	0.8	0.7	0.3	0.8	0.6	0.6	0.9	0.7	0.5	0.7	0.5	0.4
Kanban	1.0	0.5	0.3	0.9	0.6	0.5	0.7	0.8	0.4	0.8	0.4	0.7	0.8	0.6	0.5

$$\delta(\Delta_1) = \begin{bmatrix} 0.128 & 0.06 & 0.056 & 0.16 & 0.2 \\ 0.16 & 0.165 & 0.064 & 0.16 & 0.1 \\ 0.224 & 0.105 & 0.04 & 0.16 & 0.225 \end{bmatrix}$$

$$\delta(\Delta_2) = \begin{bmatrix} 0.192 & 0.075 & 0.056 & 0.2 & 0.125 \\ 0.32 & 0.195 & 0.088 & 0.18 & 0.075 \\ 0.384 & 0.135 & 0.04 & 0.12 & 0.225 \end{bmatrix}$$

$$\delta(\Delta_3) = \begin{bmatrix} 0.288 & 0.09 & 0.096 & 0.22 & 0.275 \\ 0.32 & 0.18 & 0.064 & 0.22 & 0.2 \\ 0.384 & 0.15 & 0.088 & 0.1 & 0.225 \end{bmatrix}$$

The score matrix is

$$\sigma = \begin{bmatrix} 0.608 & 0.225 & 0.208 & 0.58 & 0.6 \\ 0.8 & 0.54 & 0.216 & 0.56 & 0.375 \\ 0.992 & 0.39 & 0.168 & 0.38 & 0.675 \end{bmatrix}$$

and the total score is

$$\eta = \begin{bmatrix} 2.221 \\ 2.491 \\ 2.605 \end{bmatrix}$$

Then, the third method Kanban will be selected as the best project method from the company in that year.

7.1.3. Sensitivity Analysis

This study ended with a sensitivity analysis. It was important to switch between parameters that were weighted quite differently when building the scenarios. The following are the scenarios that were developed:

- Scenario 1: Interchange the weights between "Duration" and "Reliability".
- Scenario 2: Interchange the weights between "Duration" and "Productivity".
- Scenario 3: Interchange the weights between "Duration" and "Quality".
- Scenario 4: Interchange the weights between "Duration" and "Risk".
- Scenario 5: Interchange the weights between "Reliability" and "Productivity".
- Scenario 6: Interchange the weights between "Reliability" and "Quality".
- Scenario 7: Interchange the weights between "Reliability" and "Risk".
- Scenario 8: Interchange the weights between "Productivity" and "Quality".
- Scenario 9: Interchange the weights between "Productivity" and "Risk".
- Scenario 10: Interchange the weights between "Quality" and "Risk".

The revised scores were derived considering various situations and method assessments, and they are provided in Table 8 and visually in Figure 3. The numerous rankings obtained show that Kanban tends to be in the first, or in some cases second, place in most of the alternated cases, while the final positions for Waterfall and Agile may differ.

Based on these findings, it is clear that the ordering of methods is sensitive to the importance of the parameter. Regarding these outcomes, the remark that comes out is that data that are obtained through the decision-making framework need to be interpreted delicately, and by a successful process. Hence, it is highly important to apply the novel algorithm with the assistance of elaborated number structures such as $NSHA$s.

Table 8. Sensitivity analysis for project methods.

	Waterfall	Agile	Kanban	Ranking
Original	2.221	2.491	2.605	$k \gg a \gg w$
Scenario 1	2.068	2.457	2.367	$a \gg k \gg w$
Scenario 2	2.389	2.539	2.365	$a \gg w \gg k$
Scenario 3	2.341	2.527	2.461	$a \gg k \gg w$
Scenario 4	2.256	2.421	2.577	$k \gg a \gg w$
Scenario 5	2.298	2.428	2.570	$k \gg a \gg w$
Scenario 6	2.151	2.531	2.640	$k \gg a \gg w$
Scenario 7	2.131	2.701	2.595	$a \gg a \gg w$
Scenario 8	2.185	2.479	2.629	$k \gg a \gg w$
Scenario 9	2.255	2.695	2.503	$a \gg k \gg w$
Scenario 10	2.246	2.556	2.565	$k \gg a \gg w$

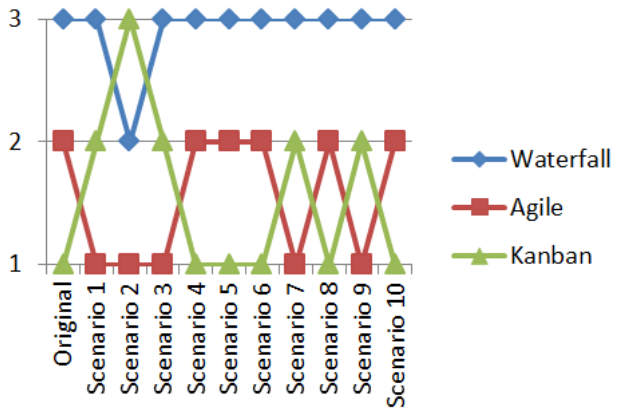


Figure 3. Ranking of project methods according to different scenarios.

7.2. Another Example

Chatbots are now widely used by businesses to interact with their customers, thanks to developments in artificial intelligence and the growth of messaging. Today, chatbots are used extensively, particularly in the service industry, from finance to airlines, and from e-commerce to health. Users can use chatbots for a variety of purposes, such as obtaining information about a topic and taking action, by interacting with them as if they were writing to a human through messaging in a digital environment. The ability to have human-like conversations is crucial because it enables users to communicate in a more natural and comfortable way. Figure 4 shows the impact of artificially intelligent chatbots on various factors.

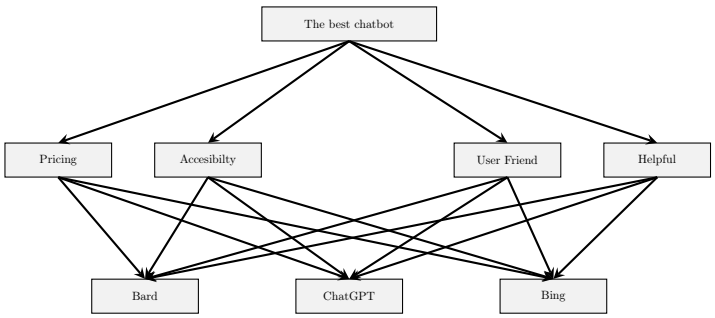


Figure 4. Framework of Implementation

The three artificially intelligent chatbots $\{b, c, g\}$, i.e., Google Bard, ChatGPT, and Bing Chat-x, have been shortlisted to choose the best chatbot of the year. The Chatbot Management Department of that website has organized a team of three software engineers $\{S_1, S_2, S_3\}$ specifically for this purpose. The final decision is based on a series of parameters, which

are $\{e_1, e_2, e_3, e_4\}$, which stand for pricing, accessibility, user-friendliness, and helpfulness, respectively. The first NSS provided by S_1 was followed by the second $NSHA$s, etc., where software engineers provided their insightful thoughts.

$\mathcal{N}_1 = \{\mathcal{N}_1(e_1), \mathcal{N}_1(e_2), \mathcal{N}_1(e_3), \mathcal{N}_1(e_4)\}$, where
$\mathcal{N}_1(e_1) = \{[0.7, 0.2, 0.6]/b, [0.6, 0.3, 0.5]/c, [0.8, 0.3, 0.5]/g\}$
$\mathcal{N}_1(e_2) = \{[0.4, 0.6, 0.7]/b, [0.7, 0.6, 0.3]/c, [0.5, 0.5, 0.4]/g\}$
$\mathcal{N}_1(e_3) = \{[0.5, 0.4, 0.3]/b, [0.7, 0.4, 0.4]/c, [0.6, 0.4, 0.6]/g\}$
$\mathcal{N}_1(e_4) = \{[0.6, 0.6, 0.5]/b, [0.5, 0.8, 0.6]/c, [0.6, 0.7, 0.4]/g\}$
$\mathcal{N}_2 = \{\mathcal{N}_2(e_1), \mathcal{N}_2(e_2), \mathcal{N}_2(e_3), \mathcal{N}_2(e_4)\}$, where
$\mathcal{N}_2(e_1) = \{[0.6, 0.4, 0.5]/b, [0.7, 0.4, 0.2]/c, [0.9, 0.4, 0.2]/g\}$
$\mathcal{N}_2(e_2) = \{[0.5, 0.5, 0.6]/b, [0.8, 0.5, 0.1]/c, [0.6, 0.7, 0.5]/g\}$
$\mathcal{N}_2(e_3) = \{[0.7, 0.3, 0.4]/b, [0.8, 0.5, 0.3]/c, [0.5, 0.6, 0.7]/g\}$
$\mathcal{N}_2(e_4) = \{[0.7, 0.5, 0.3]/b, [0.6, 0.7, 0.5]/c, [0.5, 0.5, 0.5]/g\}$
$\mathcal{N}_3 = \{\mathcal{N}_3(e_1), \mathcal{N}_3(e_2), \mathcal{N}_3(e_3), \mathcal{N}_3(e_4)\}$, where
$\mathcal{N}_3(e_1) = \{[0.8, 0.3, 0.3]/b, [0.8, 0.5, 0.4]/c, [0.9, 0.4, 0.2]/g\}$
$\mathcal{N}_3(e_2) = \{[0.6, 0.4, 0.5]/b, [0.7, 0.6, 0.2]/c, [0.8, 0.5, 0.4]/g\}$
$\mathcal{N}_3(e_3) = \{[0.8, 0.4, 0.1]/b, [0.7, 0.5, 0.5]/c, [0.6, 0.7, 0.3]/g\}$
$\mathcal{N}_3(e_4) = \{[0.6, 0.6, 0.2]/b, [0.8, 0.6, 0.4]/c, [0.7, 0.3, 0.6]/g\}$

Assume that the software engineers did not assign the parameter's components a priority ranking. Δ_1, Δ_2 and Δ_3, respectively, stand in for the three NSSs mentioned above:

Table 9, Table 10, and Table 11 present the results of $NSHA$ with weights, as reported by S_1, S_2 and S_3, respectively.

Table 9. Δ_1-$NSHA$ given by S_1.

	e_1			e_2			e_3			e_4		
Chatbots	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**
Bard	0.7	0.2	0.6	0.4	0.6	0.7	0.5	0.4	0.3	0.6	0.6	0.5
ChatGPT	0.6	0.3	0.5	0.7	0.6	0.3	0.7	0.4	0.4	0.5	0.8	0.6
Bing	0.8	0.3	0.5	0.5	0.5	0.4	0.6	0.4	0.6	0.4	0.7	0.4

Table 10. Δ_2-$NSHA$ given by S_2.

	e_1			e_2			e_3			e_4		
Chatbots	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**
Bard	0.6	0.4	0.5	0.5	0.5	0.6	0.7	0.3	0.4	0.7	0.5	0.3
ChatGPT	0.7	0.4	0.2	0.8	0.5	0.1	0.8	0.5	0.3	0.6	0.7	0.5
Bing	0.9	0.4	0.2	0.6	0.7	0.5	0.5	0.6	0.7	0.5	0.5	0.5

Table 11. Δ_3-$NSHA$ given by S_3.

	e_1			e_2			e_3			e_4		
Chatbots	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**	**t**	**i**	**f**
Bard	0.8	0.3	0.3	0.6	0.4	0.5	0.8	0.4	0.1	0.6	0.6	0.2
ChatGPT	0.8	0.5	0.4	0.7	0.6	0.2	0.7	0.5	0.5	0.8	0.6	0.4
Bing	0.9	0.4	0.2	0.8	0.5	0.4	0.6	0.7	0.3	0.7	0.3	0.6

Following our calculations, the value matrices that correspond are as follows:

$$\delta(\Delta_1) = \begin{bmatrix} 0.3 & 0.3 & 0.6 & 0.7 \\ 0.4 & 1.0 & 0.7 & 0.7 \\ 0.6 & 0.6 & 0.4 & 0.7 \end{bmatrix}$$

$$\delta(\Delta_2) = \begin{bmatrix} 0.5 & 0.4 & 0.6 & 0.9 \\ 0.9 & 1.2 & 1.0 & 0.8 \\ 1.1 & 0.8 & 0.4 & 0.5 \end{bmatrix}$$

$$\delta(\Delta_3) = \begin{bmatrix} 0.8 & 0.5 & 1.1 & 1.0 \\ 0.9 & 1.1 & 0.7 & 1.0 \\ 1.1 & 0.9 & 1.0 & 0.4 \end{bmatrix}$$

The score matrix is

$$\sigma = \begin{bmatrix} 1.6 & 1.2 & 2.3 & 2.6 \\ 2.2 & 3.3 & 2.4 & 2.5 \\ 2.8 & 2.3 & 1.8 & 1.6 \end{bmatrix}$$

and the total score

$$\eta = \begin{bmatrix} 7.7 \\ 10.4 \\ 8.5 \end{bmatrix}$$

The second chatbot, ChatGPT, will be selected as the best chatbot for the website in that year.

7.2.1. Discussion and Comparative Analysis

This subsection will compare and contrast our recommended methodology for making decisions with a few currently used methods. This allows us to highlight the advantages and disadvantages of each strategy and conduct a thorough analysis of various decision-making methods. Additionally, this comparison will help identify areas where our recommended methodology can be further improved or integrated with existing methods to enhance decision-making processes. By examining the strengths and weaknesses of different decision-making methods, we can gain valuable insights into how our recommended methodology can be tailored to specific situations or industries. Furthermore, understanding the limitations of existing methods will enable us to develop a more comprehensive and effective approach that addresses these shortcomings and maximizes decision-making outcomes.

Comparative Analysis:

It may be inferred from the current investigation and comparison analysis that the results obtained by the suggested methodology overlap with the available methodologies. The fundamental advantage of the suggested method, however, is that it includes more information and addresses data uncertainty by taking into account the membership, non-membership, and indeterminacy of sub-characteristics of considered characteristics. Among them, the information related to the object can be analyzed more precisely and impartially. It can also be used to deal with faulty and erroneous data when making decisions. As a result, the motivation for the score value assigned to each parameter will not affect the other values, resulting in the predicted loss of information. On the contrary, our suggested approach does not result in any significant information loss. The planned approach has an advantage over current methods since it not only detects the degree of discrimination but also the degree of similarity between observations, preventing decisions from being made for unfavorable reasons. Therefore, it is a useful tool to combine ambiguous and unreliable information to make decisions. Considering discrimination and similarities between observations, our approach ensures that decisions are based on a comprehensive analysis of the available information. This mitigates the risk of biased or unfair decision-making, making our suggested approach a valuable tool for decision-makers in various fields. Additionally, the ability to combine unreliable and ambiguous information further enhances the robustness and accuracy of the decision-making process. By incorporating multiple sources of information, our approach allows decision-makers to gain a more holistic understanding of the situation at hand. This enables them to consider different perspectives and potential biases, leading to more informed and well-rounded decisions. Furthermore, the flexibility of our approach to handling unreliable and ambiguous information empowers decision-makers to navigate complex scenarios with confidence and adaptability. In addition, our approach also emphasizes the importance of considering both qualitative and quantitative data in the decision-making process. This ensures that decision-makers have a comprehensive view of the situation, taking into account not only

numbers and statistics but also subjective factors such as opinions and emotions. Ultimately, our approach aims to provide decision-makers with a robust framework that promotes effective and thoughtful decision-making in any given context.

Discussion:

The parametric values of the alternative can be analyzed using Cagman and Enginoglu's methods [13], but soft sets lack knowledge of the multiple sub-characteristics of the parameters under consideration. Although Cagman et al. [14] address membership degree uncertainty, the fuzzy soft sets are unable to predict the non-membership degree of any attribute of the alternative. This limitation hinders the ability of the decision-making process to fully capture the complexity and uncertainty associated with the parameters. Therefore, incorporating additional techniques that can address uncertainty about non-membership degrees would be beneficial in enhancing the robustness and precision of the decision-making process. Rajarajeswari et al. [15] used intuitionistic fuzzy soft sets with the membership degree of the parametrized family of attributes being more explicit to get around these issues, but intuitionistic fuzzy soft sets do not reveal anything about the object's indeterminacy. Mustafa et al. [64] introduced bipolar fuzzy soft sets that incorporate both membership and non-membership degrees, along with the concept of indeterminacy. By considering the object's indeterminacy, these bipolar fuzzy soft sets provide a more comprehensive representation of uncertainty in decision-making processes. They also provide a more flexible framework for handling complex and ambiguous information, leading to improved robustness, precision, and symmetry in decision-making. Wang et al. [65] offered a hesitant fuzzy soft set as an extension of bipolar fuzzy soft sets. Hesitant fuzzy soft sets allow decision-makers to express their uncertainty and hesitation in a more explicit manner, enabling a more accurate reflection of their preferences and beliefs. This extension further enhances the applicability and effectiveness of soft set theory in real-world decision-making scenarios. Deli et al. [39] developed neutrosophic soft sets that effectively manage the truthfulness, indeterminacy, and falsity objects of the attributes, but none of the studies previously mentioned have information on the sub-characteristics of the attributes they are considering. When attributes have corresponding sub-characteristics, all of the theories previously mentioned are unable to manage the situation. In such cases, it becomes necessary to explore new approaches that can handle the complexities introduced by sub-characteristics. One potential solution could be the development of a comprehensive framework that integrates both neutrosophic soft matrices and sub-characteristic analysis, allowing for a more accurate representation and management of the underlying indeterminacy and truth values associated with each attribute and its sub-characteristics. This integrated approach would provide a more nuanced understanding of the object's characteristics, enabling better decision-making in real-world scenarios. Under the constraint that $0 \leq t_N(u) + i_N(u) + f_N(u) \leq 3$, our proposed idea successfully addresses such challenges when evaluating the truthfulness, indeterminacy, and falsity objects of any sub-characteristic of the investigated parameters. Therefore, it can be inferred with confidence that the proposed technique suggests improved stability and usability for decision-makers in the decision-making procedure based on the acquired results. When the information is supplied in an interval form, the $NSHA$ that is currently being used cannot handle the issue.

Table 12 displays the final rankings of the suggested techniques and popular methodologies. The final rankings in Table 12 provide a comprehensive comparison between the suggested techniques and popular methodologies. This allows decision-makers to make informed choices based on the improved stability and usability offered by the proposed technique. However, it is important to note that the current $NSHA$ approach is unable to effectively handle interval form information, highlighting the need for alternative methods in such cases.

Table 12. Analysis of the final rankings in light of prior research.

Research	Ranking	Final
Cagman et al. [13]	$c > g > b$	c
Enginoglu et al. [14]	$c > g > b$	c
Rajarajeswari et al. [15]	$c > b > g$	c
Mustafa et al. [64]	$c > g > b$	c
Wang et al. [65]	$c > g > b$	c
Deli et al. [39]	$c > b > g$	c
Suggested method	$c > g > b$	c

8. Conclusions

In the realm of contemporary challenges, decision-making stands out as a formidable task, involving the selection of optimal choices tailored to specific purposes. Traditionally, it was assumed that decision-related information could be distilled into precise numerical values. However, the real world frequently presents us with aggregated facts and data, often marred by inadequacies and inaccuracies. In response to this inherent uncertainty, various mathematical constructs have been introduced. Fuzzy sets were an initial attempt, but for certain parameters, they proved insufficient. Subsequently, intuitionistic fuzzy sets and, more recently, neutrosophic sets emerged as viable tools for modeling uncertainty. To further empower and streamline decision-making in such complex environments, we introduced a novel research strategy rooted in the foundational principles of neutrosophic sets and their inherent properties, ensuring a symmetry between theory and practical application. Some parts of this work were presented at ICOMAA-2022. In this paper, we extended classical algebraic structures into the realm of algebraic hyperalgebras through the utilization of neutrosophic soft sets ($NSSs$).

In this study, we have thoroughly investigated and clarified the theory of $NSHA$, aiming to reveal the fundamental principles and behaviors of this mathematical construct. The impetus for this research is driven by the current demand for effective decision-making tools amidst complexity and uncertainty. While established mathematical frameworks like fuzzy sets are well known, our focus is on exploring the distinctive features and potential applications of $NSHA$ to meet this challenge. We introduced $NSHA$ and defined essential operations such as intersection and union within this framework. By elucidating the relationship between soft hyperalgebras and $NSHA$, we provided a clearer understanding of their foundational aspects. We also examined key concepts such as ξ_β-identity $NSHA$ and ξ-absolute $NSHA$, offering insights into their characteristics and functionalities. Our research highlights the practical applications of $NSHA$ across various domains, demonstrating their potential to transform decision-making processes under conditions of uncertainty. Through our characterization of $NSHA$ and the illustration of their practical use in real-world scenarios, we equip decision-makers with robust tools for making well-informed decisions. Our comparative analysis, informed by previous studies, reinforces the effectiveness and versatility of $NSHA$, presenting new opportunities for informed decision-making in complex situations.

8.1. Limitations

It is important to acknowledge that while our approach shows promise and innovation, it is not without limitations. Some of the key limitations include the following: The proposed neutrosophic soft hyperalgebra ($NSHA$) framework, while powerful, may introduce complexity in practical applications. Handling and interpreting neutrosophic sets and hyperalgebraic operations can be challenging, particularly in large-scale decision-making problems. The effectiveness of the approach relies on the availability of accurate and comprehensive data. In situations where data quality is compromised, the reliability of the results may be affected. Implementing the methodology may require substantial computational resources, particularly for extensive sensitivity analyses or complex decision scenarios. This could limit its accessibility in resource-constrained environments.

8.2. Future Works

Despite these limitations, our work lays the foundation for future research endeavors. The structures and methodologies introduced in this work hold promise for diverse applications in decision-making processes across various domains, including supply chain management, operations research, and medical diagnosis. Future research endeavors will explore the multifaceted applications of our newly developed structures in addressing real-world problems within uncertain environments. Additionally, we encourage researchers to delve into several open problems related to neutrosophic soft hyperalgebras ($NSHAs$) within this comprehensive framework, including the following:

(1) Investigating complex neutrosophic soft hyperalgebras.
(2) Advancing the theory of generalized neutrosophic soft hyperalgebras.
(3) Exploring the concept of interval-valued neutrosophic soft hyperalgebras.
(4) Examining the view of bipolar neutrosophic soft hyperalgebras.
(5) Scrutinizing the approach of neutrosophic soft superhyperalgebras.

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