REPUBLIC OF TURKEY
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UNBOUNDED CONVERGENCE STRUCTURE PROPERTIES IN RIESZ SPACES

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Dedicated to my Safe place...

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TABLE OF CONTENTS

LIST OF SYMBOLS vii

LIST OF ABBREVIATIONS viii

ABSTRACT ix

ÖZET xi

1 INTRODUCTION 1

1.1 Literature Review 1

1.2 Objective of the Thesis 3

1.3 Hypothesis 3

2 PRELIMINERIES 5

2.1 Vector Lattices 5

2.2 Baire Spaces and Some Topological Issues 11

3 CONVERGENCE TYPES AND THEIR STRUCTURES 14

3.1 Net Convergence Structure 14

3.2 Order Convergence 19

3.3 Unbounded Order Convergence 25

4 ORDER AND UO-CONVERGENCE IN CONTINUOUS FUNCTION SPACE 28

4.1 Unbounded Order Convergence in $C(X)$ 29

5 UO-CONVERGENCE OF OPERATOR NETS 32

5.1 Preliminary Notes 32

5.2 Unbounded Order Convergence in $B(X')_+$ 32

6 CONCLUSION 35

REFERENCES 36

PUBLICATIONS FROM THE THESIS 39

LIST OF SYMBOLS

$\overline{A}$	Closure of A
$\downarrow$	Decreasing
$\forall$	For All
$\Leftrightarrow$	If And Only If
$\uparrow$	Increasing
$\wedge$	Infimum
T^{-1}	Inverse of An Operator T
$\xrightarrow{\eta}$	Net Convergence
$\xrightarrow{o}$	Order Convergence / Convergence in Order
X_+	Positive Cone of X
$\mathcal{P}(X)$	Power Set of X
$\preceq$	Quasi-subnet
$\{x_\alpha\}$	Range Set of A Net x_α
Range T	Range Set of T
$\mathfrak{N}(X)$	Set of All Admissible Nets in X
$\langle x_\alpha \rangle$	Set of All Tail Sets of (x_α)
$B(X')_+$	Space of All Continuous Positive Operators on X'
$C(X)$	Space of All Real Valued Continuous Functions on X
$\approx$	Strongly Tail Equivalence
$\vee$	Supremum
$\sim$	Tail Equivalence
$\exists$	There Exists
X'	Topological Dual of X

LIST OF ABBREVIATIONS

NCS	Net Convergence Structure
NCVS	Net Convergence Vector Space
o-Cauchy	Order Cauchy
o-closed	Order Closed
o-conv.	Order Convergent/ Convergence
o-dense	Order Dense
o-null	Order Convergent to Zero
s.t.	Such That
TFAE	The Following Are Equivalent
uo-Cauchy	Unbounded Order Cauchy
uo-conv.	Unbounded Order Convergence/ Convergent
WLOG	Without Loss of Generality
w.r.t	With Respect to

ABSTRACT

Unbounded Convergence Structure Properties In Riesz Spaces

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Unbounded order convergence which is a part of unbounded convergences has been studied for 20 years. Characterizations were made on several spaces. Charalambos D. Aliprantis showed that the convergences on Riesz spaces were applicable to economics. That is why, most of the researches about this subject can be applicable and researches still continue.

Order and unbounded order convergences were characterized on the family of real valued continuous functions. In this thesis, we studied this characterization on the family of continuous positive operators. In this study, we defined the operators on the topological dual of a Tychonoff space. This definition allowed us to use the characterization at the beginning of this paragraph. So, we studied this characterization in continuous positive non-real operator nets. While introducing this study, we sectioned this thesis into four chapters except Introduction and Conclusion parts.

In the first chapter, we gave the preliminary notes about vector lattices and their structures. In addition, we gave some topological properties and information about Baire spaces.

In the second chapter, we gave detailed information about net convergence structure, order convergence and unbounded order convergence. The information that we gave in the first two sections is preliminary for the rest of the thesis.

In the third chapter, we examined order and unbounded order convergence of real

valued continuous functions.

In the last and our main chapter, we defined and characterized the unbounded order convergence of a family of operator nets defined on the topological dual of a Tychonoff space. We wrote a lemma, a theorem and a corollary that will be required for this characterization.

Keywords: Riesz space, unbounded order convergence, Tychonoff space, operator nets.

ÖZET

Riesz Uzaylarında Sınırsız Yakınsama Yapı Özellikleri

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Sınırsız yakınsamaların bir kolu olan sınırsız sıralı yakınsamalar son 20 senedir aktif olarak çalışılmaktadır. Çeşitli uzaylarda karakterizasyonları yapılmıştır. Charalambos D. Aliprantis, Riesz uzayları üzerindeki yakınsamaların ekonomiye uygulanışını göstermiştir. Bundan ötürü, bu konudaki çoğu çalışma ekonomiye uygulanabilmekte ve bu alanda çalışmalar devam etmektedir.

Sıralı ve sınırsız sıralı yakınsamalar, gerçel değerli sürekli fonksiyon ailesi üzerinde karakterize edilmiştir. Bu tezde, bahsi geçen karakterizasyon sürekli ve pozitif operatör ailelerinde çalışılmıştır. Bu çalışmada operatör sınıfı, Tychonoff uzayının topolojik dualinde tanımlanmıştır. Bu tanımlama bize, paragrafın başındaki karakterizasyonu kullanmamıza olanak sağlamıştır. Böylece, bu karakterizasyon sürekli pozitif gerçel olmayan operatör netlerinde çalışılmıştır. Bu çalışma,giriş ve sonuç kısmı hariç 4 bölümdür.

Birinci bölümde, vektör latisleri (kafes, örgü) ve bunların özellikleri ile ilgili temel tanım ve teoremler verilmiştir. Bunlara ek olarak, bazı topolojik özellikler ve Baire uzayları hakkında bilgilere yer verilmiştir.

İkinci bölümde, net yakınsaklık yapısı, sıralı yakınsaklık ve sınırsız sıralı yakınsaklık hakkında detaylı bilgiler verilmiştir. İlk iki bölümde aktarılan bilgiler, tezin geri kalan bölümleri için ön hazırlık niteliğindedir.

Üçüncü bölümde, reel değerli sürekli fonksiyon ailesinin sıralı ve sınırsız sıralı yakınsaması incelenmiştir.

Son ve ana bölümde ise, Tychonoff uzayının topolojik duali üzerinde tanımlanan operatörlerin oluşturduğu netlerin sınıfı üzerindeki sınırsız sıralı yakınsama tanımlanmış ve karakterize edilmiştir. Bu karakterizasyon için gerekli olacak bir yardımcı teorem, ana teorem ve bir sonuç teoremi verilmiştir.

Anahtar Kelimeler: Riesz uzayı, sınırsız sıralı yakınsama, Tychonoff uzayı, operator ağları.

1
INTRODUCTION

1.1 Literature Review

The history of unbounded order convergence begins from 1940s. Earlier 2000s, this topic became more popular and nowadays it still continues to be studied by researchers.

Unbounded order convergence was firstly studied by Hidegoro Nakano in [1], 1948. The term "individual convergence" was used for "unbounded order convergence". Actually, Nakano's aim was to identify the almost everywhere (a.e.) convergence from the point of lattice operations except the first-hand usage of measure theory. Then, the mathematician Ralph DeMaar was the person who used the term "unboundedly order convergence" for the first time, in the paper [2] in 1964.

After couple of years, some mathematicians began studying this topic such as Anthony Wickstead, Jan Harm van der Walt -who is still studying- ([3], [4], [5]) and Samuel Kaplan ([6]). In [6], Kaplan characterized the usage of weak order units to define unbounded order convergence. Walt studied order convergence on the family of continuous functions in [3] and Wickstead stated the relationship between weak convergence and uo-convergence in [7].

Important books have been written about order structures. Wilhelmus Anthonius Josephus Luxemburg and Adriaan Cornelis Zaanen published the book "Riesz Spaces I" in the year 1971 [8] and followingly Zaanen wrote "Riesz Spaces II" in the year 1983 [9]. D. H. Fremlin has been studied measure theory. His book "Topological Riesz Spaces and Measure Theory" is about the relationship between measure theory and order structure [10]. It was written in 1974. Helmut H. Schaefer and Hans-Ulrich Schwarz have been studied operators on Riesz spaces and they wrote two books on this subject: "Banach Lattices and Positive Operators [11]" in 1975 and "Banach Lattices and Operators [12]" in 1984, respectively. Another book by Zaanen is "Introduction to Operator Theory in Riesz Spaces". It was

written in 1997 [13]. In 2000, S. S. Kutateladze edited a book "Nonstandard Analysis and Vector Lattices" in Russian [14]. Actually, this book continues the series which has been started by 1990. It includes several applications of Riesz spaces such as algebraic structures, functional analysis, operator theory, etc. This book was translated by A. E. Gutman, E. Yu. Emelyanov, A. G. Kusraev and the author. Charalambos Dionisios Aliprantis was the person who found that studies on Riesz spaces could be applied to economics. Charalambos Dionisios Aliprantis, Wilhelmus Anthonius Josephus Luxemburg and Kim C. Border published the study "Positive op., Riesz sp., and economics" in 1991 [15]. Then, the studies about the applicability of Riesz spaces to economics became a book. Aliprantis and Owen Burkinsaw wrote a book called "Locally Solid Riesz Sp. with App. to Economics" in the year 2003 [16]. This book was the revision of the book "Locally Solid Riesz Spaces [17]" which was written by the same authors in 1978.

In 2014, the mathematicians Nishuan Gao, Vladimir G. Troitsky and Foivos Xanthos published several papers which include important generalizations to different families, spaces and algebras. If we wish to highlight, they published papers about characterizations in dual spaces [18], on martingales [19] and in Cesaro means [20].

In addition to these studies, different types of unbounded convergence were founded such as unbounded norm convergence, unbounded absolute weak and weak* convergences, unbounded $\tau-$ convergence, etc.

In 2021, the matematicians Michael O'Brien, Vladimir G. Troitsky, and Jan Harm Van Der Walt published a paper named as "Net Conv. Structures With App. to Vector Lattices" [21]. This paper shows that the filter convergence structure and net convergence structure agrees. Also, the paper includes topological and non-topological characterizations and topological modifications. Thanks to these characterizations, nets become easier way to define convergences that defined by filters on order structure.

Some popular mathematicians who study this topic from worldwide are M. Kandic ([22]), O. Zabeti ([23]), M. O'Brien ([21]) and Mitchell Taylor ([24]).

In Türkiye, this topic is started to study around the same time. Leads of the subject in Türkiye are Şafak Alpay, Eduard Emelyanov and Nazife Erkurşun Özcan. The characterizations of several types of unbounded convergences to classes of special operators are studied by these mathematicians in general. [25], [26], [27] and [28] are some of the studies about the issue that covers this thesis. After the study [29] generated from [30], the topic became more popular in Türkiye.

Some of other researchers studying this topic are: Abdullah Aydın [31], Niyazi Anıl Gezer [27], Ömer Gök and Elif Demir [32].

1.2 Objective of the Thesis

As we saw in the literature, unbounded order convergence of spesific classes of operators were characterized. However, there is no study that generalizes operator nets in the sense of unbounded order convergence. The aim of this thesis is to generalize unbounded order convergence to more general class of operators which are defined on function spaces.

1.3 Hypothesis

Order and unbounded order convergences are used in economy. They can be applicable to algebraic structures, such as Boolean algebra, stochastic process, measure theory. Our study may also be useful in economy. Additionally, this study can be developed and maintained by many researchers studying on this subject.

The present thesis has mainly four chapters.

In chapter 2, we gave some preliminary notes about vector lattices and its properties. Additionally, we gave some topological properties and informations about Baire spaces. This chapter and chapter 3 are important to understand the rest of the thesis. Also the reader should refer to the references which are given in the chapters.

In chapter 3, we introduced some of convergence types on vector lattices: Net convergence structure, order convergence and unbounded order convergence. We gave background of these convergences that we need. Our study is based on unbounded order convergence, so the reader should focus on this chapter to understand our main result.

In chapter 4, we gave a place to the article [33]. This chapter includes tools that we use in chapter 5. Chapter 4 provides some definitions and theorems about order and unbounded order convergences defined on a continuous function space. It generalizes order and uo-convergences to a real valued continuous function space which is completely regular. This is a general definition in the literature.

In chapter 5, we investigated a class of operator nets that converges unboundedly in order. The operators are defined on a topological dual of real valued continuous functions which allows us to characterize the unbounded order convergence on operator nets. As a result of the study, we gave the definiton of unbounded order convergence on this class and got some properties; the chapter ends with them.

2
PRELIMINERIES

2.1 Vector Lattices

Let $\Upsilon \neq \varnothing$ be a set with relation $\leq$. Υ is called an ordered set if the conditions below hold:

i. $u \leq u$, $\forall u \in \Upsilon$ *(Reflexivity)*,

ii. $u \leq v$ and $v \leq u$ implies that $u = v$ *(Anti-symmetry)*,

iii. $u \leq v$ and $v \leq w$ implies that $u \leq w$ *(Transitivity)*.

Assume that Λ is a subset of an ordered set Υ. The element $u \in \Upsilon$ is said to be an upper bound of Λ if $v \leq u$, for all $v \in \Lambda$. $w \in \Upsilon$ is said to be a lower bound of Λ if $v \geq w$, for every $v \in \Lambda$. Additionally, Λ is called bounded from above if it has an upper bound. Similarly, if Λ has a lower bound, Λ is said to be bounded from below. Λ is said to be order bounded (or o-bounded), if Λ is bounded from above and from below at the same time. $\Lambda \subset \Upsilon$ is upwards directed if for all $u, v \in \Lambda$, we can find $w \in \Lambda$ such that $w \geq u, v$.

Definition 2.1. An ordered set $(\Upsilon, \leq)$ is said to be a lattice if any $u, v \in \Upsilon$ have a least upper bound shown as

$$u \vee v = \sup\{u, v\}$$

and a greatest lower bound shown as

$$u \wedge v = \inf\{u, v\}.$$

Definition 2.2. For an ordered set Υ, $u, v \in \Upsilon$ such that $u \leq v$, the order interval between u and v is denoted by

$$[u, v] = \{w \in \Upsilon : u \leq w \leq v\}.$$

Definition 2.3. Consider that Υ is an ordered set and also is a real vector space. Then Υ is said to be an ordered vector space if the vector space structure and the order structure are compossible in the following concept:

i. If $u, v \in \Upsilon$ such that $u \leq v$, then $u + w \leq v + w$, $\forall w \in \Upsilon$,

ii. $au \leq av$, $\forall a \in \mathbb{R}$ with $a \geq 0$.

Here, if $(\Upsilon, \leq)$ is a lattice, then Υ is said to be a vector lattice (or a Riesz space).

Definition 2.4. Let Υ be a Riesz space. The positive cone Υ_+ includes all $u \in \Upsilon$ such that $u \geq 0$. Furthermore, $u^+ = u \vee 0$ denotes the the positive part of u and $u^- = (-u) \vee 0$ denotes the negative part of u. Finally, the term that will be used to describe $|u| = u \vee (-u)$ is "the absolute value of u".

$u, v \in \Upsilon$ are said to be disjoint if $|u| \wedge |v| = 0$. Disjoint elements are denoted by $u \perp v$.

Proposition 2.1 (Proposition 1.1.4, [34])**.** *Consider that Υ is an ordered vector space. Then Υ is a Riesz space providing that $\Upsilon = \Upsilon_+ - \Upsilon_+$, $\sup(u, v) = u \vee v$ exists for all $u, v \in \Upsilon_+$.*

The following theorem can be checked from [16] and [34].

Theorem 2.1. *Let Υ be a Riesz space and $u, v, w \in \Upsilon$, $\alpha \in \mathbb{R}$. Then we have the following assertions:*

i. $u + v = u \vee v + u \wedge v$, $u \vee v = -(-u) \wedge (-v)$.

ii. $u + v \vee w = (u + v) \vee (u + w)$, $u + v \wedge w = (u + v) \wedge (u + w)$,
$u - v \vee w = (u - v) \wedge (u - w)$ *and* $u - v \wedge w = (u - v) \vee (w - u)$.

iii. $u = u^+ - u^-$.

iv. $|u| = u^+ + u^-$, $|\alpha u| = |\alpha||u|$.

v. $|u - v| = u \vee v - u \wedge v$.

vi. $u \vee v = \frac{1}{2}(u + v + |u - v|)$ *and* $u \wedge v = \frac{1}{2}(u + v - |u - v|)$.

vii. $||u| - |v|| \leq |u + v| \leq |u| + |v|$ *(Triangle Inequality).*

Proof. i. Consider that $w = (-v) \vee (-u)$ and $t = u \vee v$. Then we get $w = -v \wedge u$. Since $-v \leq w$ and $-u \leq w$ then we have $w + u + v \geq u$, $w + u + v \leq v$. Followingly, $w + u + v \geq u \vee v = t$.

ii. Assume that $s = v \vee w$ and fix u. Then we have, $u+v \leq u+s$ and $u+w \leq u+s$. Consider that $u + v \leq t$ and $u + w \leq t$. It yields that

$$v \leq t - u \text{ and } w \leq t - u \Rightarrow s = v \vee w \leq t - u \text{ or } u + s \leq t.$$

This gives us the remaining result that $u+s$ is the supremum of the set $\{u+v, u+w\}$. This means $(u + v) \vee (u + w) = u + s = u + v \vee w$.

The same steps are used for $\wedge$:

Assume that $s = v \wedge w$ and fix u. Then we have, $u + v \geq u + s$ and $u + w \geq u + s$. Consider that $u + v \geq t$ and $u + w \geq t$. It yields that

$$v \geq t - u \text{ and } w \geq t - u \Rightarrow s = v \wedge w \geq t - u \text{ or } u + s \geq t.$$

This gives us the remaining result that $u+s$ is the infimum of the set $\{u+v, u+w\}$. This means $(u + v) \wedge (u + w) = u + s = u + v \wedge w$.

By (i) and the first part of the (ii), we get the remaining result directly:
Let $v \vee w = -[(-v) \wedge (-w)]$, $v \wedge w = -[(-v) \vee (-w)]$. Then,

$$\begin{aligned} u - v \vee w &= u + [(-v) \wedge (-w)] = (u - v) \wedge (u - w) \\ u - v \wedge w &= u + [(-v) \vee (-w)] = (u - v) \vee (u - w). \end{aligned}$$

iii. Let $v = 0$ in (i). Then we achive that

$$u = u \vee 0 + u \wedge 0 = u \vee 0 - (-u) \vee 0 = u^+ - u^-.$$

iv. By the definition of the absolute value, we have $|u| = (-u) \vee (u)$. Therefore, by (i),

$$\begin{aligned} |u| = u + (-2u) \vee 0 &= u^+ - u^- + 2u^- = u^+ + u^- \text{ and} \\ |u + v| = (u + v) \vee (-u - v) &\leq (u^+ + v^+) \vee (u^- + v^-) \\ &\leq u^+ + v^+ + u^- + v^- \\ &= u^+ + u^- + v^+ + v^- \\ &= |u| + |v|. \end{aligned}$$

v. By the definition of $|.|$, we get

$$\begin{aligned} |u-v| &= (u-v)^+ + (u-v)^- \\ &= u \vee v - v - (u \wedge v - v) \\ &= u \vee v - u \wedge v \end{aligned}$$

for all $u, v \in \Upsilon$.

vi. We use the definition of $|.|$ and (ii). Let $|u-v| = (u-v) \vee [-(u-v)]$. From this, we get

$$\begin{aligned} \frac{1}{2}(u+v+|u-v|) &= \frac{1}{2}(u+v+(u-v) \vee (-u+v)) \\ &= \frac{1}{2}(u+(v+u-v) \vee (v-u+v)) \\ &= \frac{1}{2}((u+u) \vee (u+2v-u)) \\ &= \frac{1}{2}((2u) \vee (2v)) \\ &= u \vee v. \end{aligned}$$

For the rest, we use the same steps:

$$\begin{aligned} \frac{1}{2}(u+v-|u-v|) &= \frac{1}{2}(u+v-[(u-v) \vee (-u+v)]) \\ &= \frac{1}{2}(u+(v-u+v) \wedge (v+u-v)) \\ &= \frac{1}{2}((u+2v-u) \wedge (u+u)) \\ &= \frac{1}{2}((2v) \wedge (2u)) \\ &= v \wedge u. \end{aligned}$$

vii. We know that, for all $u, v \in \Upsilon$,

$$u+v \leq |u|+|v| \text{ and } -(u+v) = -u-v \leq |u|+|v|.$$

It implies that

$$|u+v| = (u+v) \vee [-(u+v)] \leq |u|+|v|. \tag{2.1}$$

Now we prove the beginning of the inequality. Fixing $u = u+v-v$ will help us to

use the Triangle Inequality. Now we get the following result:

$$|u| = |(u+v) - v| \leq |u+v| + |v|$$

and so

$$|u| - |v| \leq |u+v|. \tag{2.2}$$

In the same way,

$$-(|u| - |v|) = |v| - |u| \leq |u+v|. \tag{2.3}$$

From 2.1, 2.2 and 2.3;

$$||u| - |v|| = (|u| - |v|) \vee (|v| - |u|) \leq |u+v|.$$

The proof is completed. ■

Definition 2.5. Assume that ρ is a seminorm on Υ such that $\rho(u) \leq \rho(v)$ implies $|u| \leq |v|$. Under this circumstance ρ is said to be a lattice seminorm. If ρ is a norm then it implies that it is a lattice norm. In this situation, $(\Upsilon, \|.\|)$ is said to be a normed vector lattice. A normed vector lattice is said to be a Banach lattice if it is complete w.r.t the norm.

Proposition 2.2 (Proposition 1.1.6, [34])**.** *Let Υ be a normed vector lattice. Then the assertions below hold:*

i. The operations $\wedge$ and $\vee$ are continuous.

ii. Υ_+ is closed.

iii. $\lim_{n\to\infty} u_n = \sup\{u_n : n \in \mathbb{N}\}$ for all increasing convergent sequence $(u_n)_{n=1}^{\infty} \subset \Upsilon$.

Proof. i. Consider that $u_n \to u$ and $v_n \to v$ when $n \to \infty$. $\forall n \in \mathbb{N}$ we get

$$\begin{aligned} |u_n \wedge v_n - u \wedge v| &= |u_n \wedge v_n - u_n \wedge v + u_n \wedge v - u \wedge v| \\ &\leq |u_n \wedge v_n - u_n \wedge v| + |u_n \wedge v - u \wedge v|. \qquad (\bigstar) \end{aligned}$$

Followingly, from Theorem 2.1 (iv),

$$\bigstar \leq |v_n - v| + |u_n - u|.$$

The conclusion follows directly because $\|.\|$ is a lattice norm.
ii. Consider that $u_n \in \Upsilon_+$, $\forall n \in \mathbb{N}$ and $u_n \to u$ when $n \to \infty$. By (i), we get

$$u_n = u_n \vee 0 \to u \vee 0 \text{ when } n \to \infty \text{ and } u = u \vee 0 \in \Upsilon_+.$$

iii. Assume that (u_n) is an increasing convergent sequence. Let $u = \lim_{n\to\infty} u_n$. Given that $u_m - u_n \in \Upsilon_+$, $\forall n \in \mathbb{N}$, $\forall m \geq n$, by virtue of (ii) that $u - u_n \geq 0$, $\forall n$. Suppose that $v \in \Upsilon$ satisfying $u_n \leq v \leq u$, $\forall n \in \mathbb{N}$. By (i), we get the following:

$$u_n = u_n \wedge v \to u \wedge v \text{ as } n \to \infty.$$

Therefore $v = u$. ■

Definition 2.6. *i.* Υ is said to be Archimedean if $u \leq 0$ ensures when the set $\{mu : m \in \mathbb{N}\}$ is bounded from above.

ii. If each o-bounded set that is non-empty has a supremum and an infimum in Υ, then Υ is referred to as Dedekind complete.

iii. If each o-bounded sequence has a supremum and an infimum in Υ, in the present case Υ is said to be σ-Dedekind complete.

Example 2.1. Let $n \geq 2$ and $u = (u_1, u_2, \cdots, u_n)$, $v = (v_1, v_2, \cdots, v_n)$ be in $\mathbb{R}^n$. The lexicographical order on $\mathbb{R}^n$ is defined by $u \leq v$ if there is $m \in \{0, \cdots, n\}$ such that $u_1 = v_1, \cdots, u_m = v_m$ and $u_{m+1} < v_{m+1}$.

Here, $\mathbb{R}^n$ with the lexicographical order is a vector lattice. Moreover, it is totally ordered s.t. this order is non-Archimedean. Let choose $u = (0, 1, 0, \ldots, 0)$, $v = (1, 0, 0, \ldots, 0)$. Then we get $nu \leq v$ for all $n \in \mathbb{N}$.

Definition 2.7. *i.* A function defined from a directed set Λ to an arbitrary set Υ is said to be a net in Υ indexed by Λ. It is denoted by $(u_\alpha)_{\alpha\in\Lambda}$.

ii. A net $(u_\alpha)_{\alpha\in\Lambda}$ is said to be increasing whenever $u_\alpha \leq u_\beta$, $\forall\, \alpha, \beta \in \Lambda$ such that $\alpha \leq \beta$. If $(u_\alpha)_{\alpha\in\Lambda}$ is an increasing net and $u = \sup\{u_\alpha : \alpha \in \Lambda\}$, then we write $u_\alpha \uparrow u$, $\alpha \in \Lambda$.

iii. A net $(u_\alpha)_{\alpha\in\Lambda}$ is said to be decreasing whenever $u_\alpha \geq u_\beta$, $\forall \alpha, \beta \in \Lambda$ such that $\alpha \leq \beta$. If $(u_\alpha)_{\alpha\in\Lambda}$ is a decreasing net and $u = \inf\{u_\alpha : \alpha \in \Lambda\}$, then we write $u_\alpha \downarrow u$ as $\alpha \in \Lambda$.

Definition 2.8. *i.* A subspace Θ of Υ is identified as a sublattice (or Riesz subspace, vector sublattice) of Θ if $u \vee v \in \Theta$ and $u \wedge v \in \Theta$, $\forall u, v \in \Theta$.

ii. A sublattice Θ of Υ is identified as order dense if $0 < u \in \Upsilon$, $\exists v \in \Theta$ with $0 < v \leq u$.

iii. A subset Λ of Υ is identified as a solid if $|u| \leq |v|$ for some $v \in \Lambda$ imply $u \in \Lambda$.

iv. Each solid subspace I of Υ is said to be an ideal (or an order ideal) in Υ.

v. An ideal B of Υ is said to be a band if $\sup(\Lambda) \in B$, $\forall$ subset $\Lambda \subset B$ which has a supremum in Υ.

Definition 2.9. A vector $e \in \Upsilon_+$ is said to be an order unit (or strong unit) if $\Upsilon_e = \Upsilon$ satisfies. If $B_e = \Upsilon$, then e is said to be a weak unit, that is, if $u \wedge ne \uparrow u$ satisfies $\forall u \in \Upsilon_+$.

If X is Archimedean, then a vector $e > 0$ is a weak unit $\Leftrightarrow |u| \wedge e = 0$ implies $u = 0$. Every order unit is a weak unit.

Definition 2.10. A linear topology (not necessarily Hausdorff) τ on a vector lattice Υ is called locally solid if τ has a base at "0" consisting of solid sets. (Υ, τ) is called locally solid vector lattice (or locally solid Riesz space) when Υ is a vector lattice equipped with a locally solid topology τ.

2.2 Baire Spaces and Some Topological Issues

Definition 2.11. A topological space Υ is said to be Hausdorff if for any pair u_1, u_2 of distinct points of Υ, there are neighborhoods V_1, and V_2 of u_1 and u_2, respectively, that are disjoint.

Definition 2.12 ([35])**.** A space Υ is called compact if every open covering Λ of Υ contains a finite subcollection that also covers Υ. A space Υ is called locally compact at u if there is some compact subspace Λ of Υ that contains a neighborhood of u. If Υ is locally compact at each of its points, Υ is called locally compact.

Example 2.2 (Example 3 in [35])**.** Every simply (totally) ordered set Υ having the least upper bound property is locally compact: Given a basis element for Υ, it is contained in a closed interval in Υ, which is compact.

Definition 2.13. Let Λ be a subset of a topological space. Then, Λ is said to be nowhere dense if $\text{Int}(\overline{\Lambda}) = \varnothing$.

Definition 2.14. Let Λ be a subset of a topological space Υ. Then,

i. Λ is called of the first category or meagre in Υ if Λ can be represented as a union of a sequence of nowhere dense sets.

ii. Λ is called of the second category or non-meager Υ if it is not meagre.

iii. Λ is called residual or co-meagre Υ if its complement is meagre Υ.

Definition 2.15. Υ is said to be a Baire space if it holds the Baire condition, that is, if every intersection of countably many dense open sets is dense. Equivalently, Υ is a Baire if every co-meagre set is dense.

Example 2.3. The space $\mathbb{Z}_+$ is a Baire space. Each singleton is open in $\mathbb{Z}_+$, so there does not exist any subset of $\mathbb{Z}_+$ such that having an empty interior except the empty set. Thus, this space satisfies closed Baire condition.

$(\mathbb{R}, |.|)$ is complete so it is a Baire space.

The space $\mathbb{Q}$ is not a Baire space because every singleton is closed and has empty interior in $\mathbb{Q}$.

Every locally compact Hausdorff space is Baire. Every nonempty open subspace of a Baire space is a Baire space. A space is Baire if and only if every point has a neighborhood which is also Baire.

Definition 2.16. Topological space Υ is called normal if it satisfies Axiom T_4: every disjoint closed sets of Υ have disjoint open neighborhoods, that is, for disjoint closed sets $C, D \subset \Upsilon$ there exist neighborhoods $C \subset U$, $D \subset V$ s. t. $U \cap V = \varnothing$.

There are several versions of Uryson's Lemma. We will give the most useful one for us below.

Theorem 2.2 (Uryson's Lemma)**.** *A topological space is normal if and only if any two disjoint closed subsets can be separated by a continuous function.*

Two disjoint closed subsets Λ and Θ are separated by a continuous function if there is a continuous function $f : \Upsilon \to [0, 1]$ such that $f(u) = 0, \forall u \in \Lambda$ and $f(v) = 1$, $\forall y \in \Theta$. It follows that if these two subsets are separated by a function, then so are their closures. Lastly, if Λ and Θ are separated by a continuous function, then they are separated by neighborhoods. If we fix Λ as a subset of Υ and choose $\Upsilon\backslash\Lambda$ instead of Θ, then the definition will be again true.

Definition 2.17. Let Υ be a topological space. Υ is said to be completely regular if the points can be seperated from closed sets via continuous real-valued functions. That is, for any closed set $\Lambda \subseteq \Upsilon$, any point $u \in \Upsilon\backslash\Lambda$, there exists a continuous real-valued function $f : \Upsilon \to \mathbb{R}$ such that

$$f(u) = 1$$

and

$$f|_{\Lambda} = 0.$$

Also, if Υ is a completely regular Hausdorff space, then Υ is said to be Tychonoff or completely T_3 space.

3
CONVERGENCE TYPES AND THEIR STRUCTURES

In this section, we will give some informations about convergence types which are defined on vector lattices such as net convergence structures, order convergence and unbounded order convergence.

3.1 Net Convergence Structure

The convergence structure theory was developed to handle convergences that are not topological. This theory uses the filters in order to identify convergence. In the paper [21], the equivalence of a filter convergence structure theory and a theory of net convergence structere was developed.

Definition 3.1. *i.* Assume that (u_α) is a net indexed by a directed set Λ. For $\alpha_0 \in \Lambda$ fixed, let $\Lambda_0 = \{\alpha \in \Lambda : \alpha \geq \alpha_0\}$. This is also a directed set which is under the pre-order relation that induced from Λ. The net which is created by restricting of the function u to Λ_0 is said to be a tail of (u_α). This tail is denoted by $(u_\alpha)_{\alpha \geq \alpha_0}$.

ii. This tail defines a new net which is named as a tail set of $(u_\alpha)_{\alpha\in\Lambda}$, and its range is shown as $\{u_\alpha\}_{\alpha\geq\alpha_0}$.

iii. The set which includes all tail sets of $(u_\alpha)_{\alpha\in\Lambda}$ is defined as follows:

$$\langle u_\alpha\rangle_{\alpha\in\Lambda} = \{\{u_\alpha\}_{\alpha\geq\alpha_0} : \alpha_0 \in \Lambda\}. \tag{3.1}$$

Definition 3.2. For nets $(u_\alpha)_{\alpha\in\Lambda}$ and $(v_\beta)_{\beta\in\Theta}$, $(v_\beta)_{\beta\in\Theta}$ is called a quasi-subnet of $(u_\alpha)_{\alpha\in\Lambda}$ if $\forall\alpha_0 \in \Lambda$, $\exists\beta_0 \in \Theta$ s. t. $\{v_\beta\}_{\beta\geq\beta_0} \subseteq \{u_\alpha\}_{\alpha\geq\alpha_0}$. That is, every tail of $(u_\alpha)_{\alpha\in\Lambda}$ includes a tail of $(v_\beta)_{\beta\in\Theta}$ as a subset. It is denoted by $(v_\beta)_{\beta\in\Theta} \preceq (u_\alpha)_{\alpha\in\Lambda}$.

Additionally, if $(v_\beta)_{\beta\in\Theta} \preceq (u_\alpha)_{\alpha\in\Lambda}$ and $(u_\alpha)_{\alpha\in\Lambda} \preceq (v_\beta)_{\beta\in\Theta}$, then this relation is called tail equivalence. That means, $(u_\alpha)_{\alpha\in\Lambda}$, $(v_\beta)_{\beta\in\Theta}$ are tail equivalent. This

relation is denoted by $(u_\alpha)_{\alpha\in\Lambda} \sim (v_\beta)_{\beta\in\Theta}$.

Definition 3.3. Suppose that $(u_\alpha)_{\alpha\in\Lambda}$ and $(v_\beta)_{\beta\in\Theta}$ are two nets in Υ. They are said to be strongly tail equivalent if these two nets satisfy the condition $\langle u_\alpha\rangle_{\alpha\in\Lambda} = \langle v_\beta\rangle_{\beta\in\Theta}$. This equivalence is written as $(u_\alpha)_{\alpha\in\Lambda} \approx (v_\beta)_{\beta\in\Theta}$.

Definition 3.4. Let $\mathcal{P}(\Upsilon)$ be a power set of Υ. Let us define

$$U_1(\Upsilon) = \Upsilon$$
$$U_{n+1}(\Upsilon) = U_n(\Upsilon) \cup \mathcal{P}(U_n(\Upsilon)),$$

$n \in \mathbb{N}$. It is called the superstructure over Υ.

Definition 3.5. Let $(u_\alpha)_{\alpha\in\Lambda}$ be a net in a vector lattice Υ. (u_α) is said to be admissible if $\Lambda \in \bigcup_{n=1}^{\infty} U_n(\Upsilon)$.

The collection of admissible nets forms a set and it is denoted by $\mathfrak{N}(\Upsilon)$. The restriction of the relation "$\preceq$" to the set $\mathfrak{N}(\Upsilon)$ states a pre-order relation. Additionally, restrictions of tail equivalence " $\sim$ " and strongly tail equivalence " $\approx$ " are equivalence relations [21].

Theorem 3.1 (Theorem 2.1, [21])**.** *All nets in Υ are strongly tail equivalent to an admissible net.*

Proof. Suppose that (u_α) is a net in Υ. By Proposition 5.1 in [21], there exists an admissible net such that its tail filter base is $\langle u_\alpha\rangle$. Therefore the latest net is strongly tail equivalent to (u_α). ■

A term "subnet" has too many different definitions in the literature. We will give two of them below and use Kelley's definition.

Definition 3.6 (Kelley, [36])**.** Let $u = (u_\alpha)$ and $v = (v_\beta)$ be two nets indexed by Λ and Θ, respectively. v is said to be a subnet of u if there is a function $f : \Theta \to \Lambda$ such that,

i. f maps u to v, that is, $\forall\beta \in \Theta, v_\beta = u_{f(\beta)}$;

ii. f is strongly co-final, that is, $\forall\alpha \in \Lambda$, $\exists\beta \in \Theta$ such that, $\forall\beta_1 \geq \beta \in \Theta$, $f(\beta_1) \geq \alpha$.

Definition 3.7 (Willard, [37])**.** Let $u = (u_\alpha)$ and $v = (v_\beta)$ be two nets indexed by Λ and Θ, respectively. v is said to be a subnet of u if there is a function $f : \Theta \to \Lambda$ such that,

i. f maps u to v, that is, $\forall \beta \in \Theta, v_\beta = u_{f(\beta)}$;

ii. f is monotone; that means, $\forall \beta_1 \geq \beta_2 \in \Theta, f(\beta_1) \geq f(\beta_2)$;

iii. f is co-final; that means, $\forall \alpha \in \Lambda$, $\exists \beta \in \Theta$ such that $f(\beta) \geq \alpha$.

Proposition 3.1 (Proposition 2.2, [21])**.** *All subnets are quasi-subnets. All quasi-subnets are tail equivalent to a subnet.*

Proof. Assume that $(v_\gamma)_{\gamma \in \Theta}$ is a quasi-subnet of a net $(u_\lambda)_{\lambda \in \Lambda}$. Take $\zeta = \{(\lambda, \gamma) \in \Lambda \times \Theta : u_\lambda = v_\gamma\}$. For $(\lambda, \gamma) \in \zeta$, fix $\psi(\lambda, \gamma) = \lambda$. Then, ζ is a directed set under the product order relation induced by $\Lambda \times \Theta$. In addition, $u \circ \psi$ is a subnet of (u_λ) by the means of not only Willard's definition but also Kelley's definition, that is tail equivalent to (v_γ). ■

Example 3.1. Let $(u_n) = (3, 4, 5, 6, 7, \dots)$ and $(v_n) = (8, 9, 10, 11, 12, \dots)$ be two sequences. Let choose $\beta = 1$ and $\alpha = 1$ such that $\{v_n\}_{1 \leq n} = \{8, 9, 10, 11, \dots\}$ and $\{u_n\}_{1 \leq n} = \{3, 4, 5, \dots\}$. Then $\{v_n\}_{1 \leq n} \subset \{u_n\}_{1 \leq n}$. For $\alpha = \beta = 2$, again we get $\{v_n\}_{2 \leq n} \subset \{u_n\}_{2 \leq n}$. If we continue this process, we can see that for all $\alpha \in \mathbb{N}$, $\exists\, \beta \in \mathbb{N}$ s.t. $\{v_n\}_{\beta \leq n} \subset \{u_n\}_{\alpha \leq n}$ This means that $(v_n) \preceq (u_n)$.

On the other hand, let choose $\gamma = 1$ and $\theta = 6$ such that $\{v_n\}_{1 \leq n} = \{8, 9, 10, 11, \dots\}$ and $\{u_n\}_{6 \leq n} = \{8, 9, 10, 11, \dots\}$. It can be clearly seen that $\{u_n\}_{\beta \leq n} \subset \{v_n\}_{\alpha \leq n}$. That is, $(u_n) \preceq (v_n)$.

From both consequences, (u_n) and (v_n) are tail equivalent. Indexes can be choosen differently but the result will be the same.

Definition 3.8. Assume that Υ is an arbitrary set. If the function

$$\eta : \Upsilon \to \mathcal{P}(\mathfrak{N}(\Upsilon)) \tag{3.2}$$

satisfies following axioms, it means a net convergence structure (or NCS) on Υ:

(N1) Constant nets converge: if $(u_\alpha) = u, \forall \alpha$, then $u_\alpha \to u$;

(N2) If a net is convergent to u then every quasi-subnet of that net is convergent to u;

(N3) Consider that $(u_\alpha)_{\alpha \in \Lambda} \to u$ and $(v_\alpha)_{\alpha \in \Lambda} \to u$. Assume that $(w_\alpha)_{\alpha \in \Lambda}$ is a net in Υ such that $w_\alpha \in \{u_\alpha, v_\alpha\}, \forall \alpha$. Then $w_\alpha \to u$.

In this circumstances, Υ equipped with this structure states a net convergence space. The convergence on Υ is denoted by $(\Upsilon, \xrightarrow{\eta})$ or $(\Upsilon, \underset{\eta}{\to})$. Instead of writing

$(u_\alpha)_{\alpha\in\Lambda} \in \eta(u)$, we use the notation $(u_\alpha)_{\alpha\in\Lambda} \underset{\eta}{\to} u$. In the present case, $(u_\alpha)_{\alpha\in\Lambda}$ is said to be $\eta-$ convergent to u. In this case, u is said to be a limit of $(u_\alpha)_{\alpha\in\Lambda}$.

As a consequence of 3.1, the axiom (N2) can be written as two extra provision:

(N2a) If a net is convergent to u, then all of its subnets is convergent to u, and

(N2b) if $(u_\alpha) \to u$ and $(v_\beta) \preceq (u_\alpha)$ then $(v_\beta) \to u$.

If $(u_\alpha) \to u$, (u_α) and (v_β) are tail equivalent to each other for some admissible nets (u_α), (v_β) then $(v_\beta) \to u$; this follows by (v_β) is a quasi-subnet of (u_α). Particularly, an admissible net is convergent to u if and only if some (and therefore every) tail of it converges to u.

Definition 3.9. Let η_1 and η_2 be two NCS on Υ. Then η_1 is called stronger than η_2 if $u_\alpha \underset{\eta_1}{\longrightarrow} u$ implies $u_\alpha \underset{\eta_2}{\longrightarrow} u$ for each (admissible) net (u_α) in Υ. In other words, η_2 is called weaker than η_1.

All convergences induced by some topologies ensure the NCS axioms. These convergences are called topological.

Definition 3.10. A space Υ with net convergence structure is called Hausdorff if all nets in Υ has at most one limit. The topology is Hausdorff, if the net convergence structure is induced by any topology.

Definition 3.11. Consider that the space Υ is a net convergence space and $\Lambda \subseteq \Upsilon$.

i. $\overline{\Lambda}$ is called closure or adherence of Λ. It denotes the set of the limits of each net in Λ.

ii. If the limit of all nets that are convergent in Λ belongs to Λ, in this case Λ is said to be closed. We denote it by $\overline{\Lambda} = \Lambda$.

iii. In the case that $\overline{\Lambda} = \Upsilon$, Λ is said to be dense.

iv. For an element $u \in \Lambda \subseteq \Upsilon$, Λ is called a neighborhood of u if for all convergent net in Υ that goes to u, a tail of the net is included in Λ.

v. If Λ is a neighborhood of all of its points, then Λ is called open.

Definition 3.12. A function $f : \Upsilon \to \Theta$ from a net convergence space to a net convergence space is called continuous at u if $u_\alpha \to u$ in Υ implies $f(u_\alpha) \to f(u)$ in Θ for each admissible net (u_α) in Υ. f is continuous if f is continuous at $\forall u \in \Upsilon$.

Definition 3.13. Υ is called compact if each net in Υ has a quasi-subnet which is convergent. A net convergence space is said to be locally compact if each convergent net owns a tail included in a compact set.

Proposition 3.2 (Proposition 3.3, [21])**.** *The assertions below hold in all net convergence structure:*

i. The composition of two continuous functions is continuous;

ii. Every set is included in its closure;

iii. A set is open $\Leftrightarrow$ its complement is closed;

iv. The intersection of any collection of closed sets is closed;

v. The union of any finite collection of closed sets is closed;

vi. The union of any collection of open sets is open;

vii. The intersection of any finite collection of open sets is open.

Proof. (i) and (ii) can be proven by using their definitions.

(iii) Firstly we assume that S is open. Suppose that $(u_\alpha)_{\alpha\in A}$ is a net in S^C such that $u_\alpha \to u$, but $u \in S$. In that case, a tail of that net belongs to S. This gives us a contradiction.

On the other hand, if S^C is closed, $(u_\alpha) \to u$ in S, and no tail of the net is not in the S, and so let $A_0 = \{\alpha \in A : u_\alpha \in S^C\}$. A_0 is again a directed set and the restriction of (u_α) to A_0 is a quasi-subnet of (u_α). This new net is included in S^C and still is convergent to u by N2. It is a contradiction because this implies $u \in S^C$.

(iv) It is enough to show that the union of any collection of open set is open; which is exactly the same with (vi).

Proof of *(vi)* is a consequence of *(iii)* and *(iv)*.

vii. Let U_1 and U_2 be two open sets. It is enough to prove that $U_1 \cap U_2$ is open. Let choose an element $u \in U_1 \cap U_2$ such that $u_\alpha \to u$. Since $u \in U_1$ and U_1 is open, then there is α_1 s.t. the tail set $\{u_\alpha\}_{\alpha\geq\alpha_1}$ belongs to U_1. In the same way, since $u \in U_2$ and U_2 is open, then there is α_2 such that the tail set $\{u_\alpha\}_{\alpha\geq\alpha_2}$ belongs to U_2. Let choose an element α_0 such that $\alpha_0 \geq \alpha_1, \alpha_2$ and so $\{u_\alpha\}_{\alpha\geq\alpha_0}$ is contained in $U_1 \cap U_2$.

v. Let $U_1, \dots U_n$ be closed subsets of U. It is enough to show that $U \backslash \bigcup_{i=1}^{n} U_i$ is open. By De Morgan's laws and (iii), we have

$$U \backslash \bigcup_{i=1}^{n} U_i = \bigcap_{i=1}^{n} U \backslash U_i,$$

which is open by (vii). ■

From now on, Υ will be a vector space over $\mathbb{K}(\mathbb{C}$ or $\mathbb{R})$, with the usual convergence. An NCS defined on Υ is called linear if the operations multiplication and addition are jointly continuous. In this case, Υ is called a net convergence vector space (or NCVS).

Definition 3.14. Let $(u_\alpha)_{\alpha \in \Lambda}$ be a net in Υ which is an NCVS is called Cauchy if the net $(u_\alpha - u_\lambda)_{(\alpha,\lambda) \in \Lambda \times \Lambda} \to 0$. By the way, the set $\Lambda \times \Lambda$ is again directed and coordinate-wise. Let choose $v_{(\alpha,\lambda)} = u_\alpha$ and $w_{(\alpha,\lambda)} = u_\lambda$. Then we have $(v_{(\alpha,\lambda)}) \approx (u_\alpha)$ and $(w_{(\alpha,\lambda)}) \approx (u_\alpha)$, that is why both are convergent to u. Followingly, $u_\alpha - u_\lambda = v_{(\alpha,\lambda)} - w_{(\alpha,\lambda)} \to 0$.

If every Cauchy net converges in an NCVS, then it is called complete.

3.2 Order Convergence

Order convergence has several definitions in the literature. We will give the most common used one below.

Definition 3.15 ([21])**.** Let $(u_\alpha)_{\alpha \in \Lambda}$ be a net in Υ. (u_α) is order convergent (to converge in order) to u if there is a net $(v_\beta)_{\beta \in \Theta} \in \Upsilon$ such that $v_\beta \downarrow 0$ and $\forall \beta \in \Theta$, $\exists \alpha_0 \in \Lambda$ satisfying $|u_\alpha - u| \leq v_\beta$, $\forall \alpha \geq \alpha_0$. It is written as $u_\alpha \underset{o}{\to} u$ or $u_\alpha \to u$ in order. A net (u_α) is said to be order Cauchy or shortly $o-$Cauchy if the double net $(u_\alpha - u_\beta)_{(\alpha,\beta)}$ converges to zero in order. Order convergent and $o-$Cauchy sequences are defined in the same way.

As a consequence of the definition, we can say that every net that converges in order has an order bounded tail. The element u in Definition 3.15 is the order limit of the net (u_α). The following lemma is about the uniqueness of the order limits.

Lemma 3.1 (Lemma 1.13, [16])**.** *In a vector lattice, a net has at most one o-limit.*

Proof. Consider that $u_\alpha \underset{o}{\to} u$ and $u_\alpha \underset{o}{\to} v$ hold in a vector lattice Υ. Let (s_α) and (t_α) be two nets satisfying $|u_\alpha - u| \leq s_\alpha \downarrow 0$ and $|u_\alpha - v| \leq t_\alpha \downarrow 0$. Then, for each

α we have

$$|u - v| \leq |u - u_\alpha| + |u_\alpha - v| \leq s_\alpha + t_\alpha.$$

Since $s_\alpha + t_\alpha \downarrow 0$, it follows that $|u - v| = 0$, that means $u = v$. ■

The following two propositions from [21] state other definitions of order convergence.

Proposition 3.3. *$u_\alpha \underset{o}{\to} u$ if there are two nets (v_β) and (w_β) such that $v_\beta \uparrow u$, $w_\beta \downarrow u$, and $\forall \beta$, a tail of (u_α) is included in the order interval $[v_\beta, w_\beta]$.*

Proposition 3.4. *For a net $(u_\alpha) \in \Upsilon$, $u_\alpha \underset{o}{\to} u$ if and only if there exists a nested decreasing net of order intervals s. t. the intersection of all these order intervals is a singleton $\{u\}$, each of these intervals includes a tail of (u_α).*

Theorem 3.2 (Theorem 1.14, [16])**.** *Consider that the nets $(u_\alpha)_{\alpha \in \Lambda}$ and $(v_\beta)_{\beta \in \Theta}$ in a vector lattice Υ order converge to u and v, respectively. Then the following conditions hold.*

i. $|u_\alpha| \underset{o}{\to} |u|$, ${u_\alpha}^+ \underset{o}{\to} u^+$, *and* ${u_\alpha}^- \underset{o}{\to} u^-$.

ii. $au_\alpha + bv_\beta \xrightarrow[o]{(\alpha,\beta)} au + bv$, $\forall a, b \in \mathbb{R}$.

iii. $u_\alpha \vee v_\beta \xrightarrow[o]{(\alpha,\beta)} u \vee v$ *and* $u_\alpha \wedge v_\beta \xrightarrow[o]{(\alpha,\beta)} u \wedge v$.

iv. If $u_\alpha \uparrow u$ *or* $u_\alpha \downarrow u$, *then* $u_\alpha \underset{o}{\to} u$.

v. If $u_\alpha \uparrow$ *and* $u_\alpha \underset{o}{\to} u$, *then* $u_\alpha \uparrow u$.

Proof. i. Let $(u_\alpha)_{\alpha \in \Lambda}$ be a net in a Riesz space Υ such that $u_\alpha \underset{o}{\to} u$. Then there is a net $(v_\beta)_{\beta \in \Theta}$ in Υ such that $v_\beta \downarrow 0$, $\forall \beta \in \Theta$, $\exists \alpha_0$ satisfying $|u_\alpha - u| \leq v_\beta$, $\forall \alpha \geq \alpha_0$. On the other hand, let fix $u_\alpha = u_\alpha - u + u$. If we use the Triangle inequality (see Theorem 2.1, v), we get

$$|u_\alpha| = |u_\alpha - u + u| \leq |u_\alpha - u| + |u|$$

and so $|u_\alpha| - |u| \leq |u_\alpha - u|$. It follows that $||u_\alpha| - |u|| \leq |u_\alpha - u| \leq v_\beta$.

For the rest of the assertion, we use the definition of $|.|$.

$$\begin{aligned}
|u_\alpha - u| \leq v_\beta \Rightarrow u_\alpha - u &\leq v_\beta \\
u_\alpha &\leq v_\beta + u \\
0 \vee u_\alpha &\leq 0 \vee (v_\beta + u) \\
0 \vee u_\alpha - (0 \vee u) &\leq 0 \vee (v_\beta + u) - (0 \vee u) \\
u_\alpha^+ - u^+ &\leq (v_\beta + u)^+ - u^+ \leq w_\beta \\
|u_\alpha^+ - u^+| &\leq w_\beta,
\end{aligned}$$

where $w_\beta \downarrow 0$. Hence, $u_\alpha^+ \underset{o}{\to} u^+$. $u_\alpha^- \underset{o}{\to} u^-$ can be shown in the similar way.

ii. Let a and b be two real numbers and (u_α) and (v_β) be two nets that are order convergent to u and v, respectively. So, we naturally have $t_\lambda \in C$ and $w_\gamma \in D$ in Υ such that $t_\lambda \downarrow 0$, $w_\gamma \downarrow 0$ and for any $\lambda \in C$, $\gamma \in D$, there exist $\alpha_0 \in \Lambda$ and $\beta_0 \in \Theta$ satisfying $|u_\alpha - u| \leq t_\lambda, \forall \alpha \geq \alpha_0$, $|v_\beta - v| \leq w_\gamma, \forall \beta \geq \beta_0$.

Now, we need to show that a tail of $au_\alpha + bv_\beta$ is dominated by another net that decreases to 0.

$$\begin{aligned}
|au_\alpha + bv_\beta - (au + bv)| &= |(au_\alpha - au) + (bv_\beta - bv)| \\
&\leq |au_\alpha - au| + |bv_\beta - bv| \ \textit{(by the Triangle Inequality)} \\
&= |a(u_\alpha - u)| + |b(v_\beta - v)| \\
&= |a||u_\alpha - u| + |b||v_\beta - v|, \textit{(for all } a, b \in \mathbb{R}\textit{, see Theorem 2.1, iii.)} \\
&\leq |a|t_\lambda + |b|w_\gamma.
\end{aligned}$$

iii. Let (u_α), (v_β), (t_λ) and (w_γ) define as the same in (ii).

$$u_\alpha \vee v_\beta = \frac{1}{2}(u_\alpha + v_\beta + |u_\alpha - v_\beta|) \tag{3.3}$$

and

$$u \vee v = \frac{1}{2}(u + v + |u - v|). \tag{3.4}$$

We take the difference of (3.3) and (3.4). Then we get,

$$\begin{aligned} u_\alpha \vee v_\beta - u \vee v &= \frac{1}{2}\left(u_\alpha + v_\beta + |u_\alpha - v_\beta| - u - v - |u - v|\right) \\ &= \frac{1}{2}\left[(u_\alpha - u) + (v_\beta - v) + (|u_\alpha - v_\beta| - |u - v|)\right] \\ &\leq |u_\alpha - u| + |v_\beta - v| + ||u_\alpha - v_\beta| - |u - v|| \ \textit{(by the Triangle Inequality)} \\ &\leq t_\lambda + w_\gamma + s_\sigma \ \textit{(by (i.) in this theorem)}, \end{aligned}$$

for some $s_\sigma \downarrow 0$. Hence, $u_\alpha \vee v_\beta \xrightarrow[o]{(\alpha,\beta)} u \vee v$.

On the other hand, if we apply the same operations to $u_\alpha \wedge v_\beta$, we get the following consequence:

$$\begin{aligned} u_\alpha \wedge v_\beta &= \frac{1}{2}\left(u_\alpha + v_\beta - |u_\alpha - v_\beta|\right) \\ u \wedge v &= \frac{1}{2}\left(u + v - |u - v|\right) \\ u_\alpha \wedge v_\beta - u \wedge v &= \frac{1}{2}\left(u_\alpha + v_\beta - |u_\alpha - v_\beta| - u - v + |u - v|\right)) \\ &= \frac{1}{2}\left[(u_\alpha - u) + (v_\beta - v) - (|u_\alpha - v_\beta| - |u - v|)\right] \\ &\leq |u_\alpha - u| + |v_\beta - v| + ||u_\alpha - v_\beta| - |u - v||, \end{aligned}$$

which gives the expecting result.

iv. and $v.$ immediately follows from Proposition 3.3. ■

Proposition 3.5 (Proposition 1.1.11, [34])**.** *Consider that $(u_n), (v_n)$ are sequences in Υ such that $u_n \underset{o}{\to} u$ and $v_n \underset{o}{\to} v$. Let (α_n) be a real sequence such that $\alpha_n \underset{o}{\to} \alpha$. Then;*

i. $(u_n + \alpha v_n) \underset{o}{\to} u + \alpha v$,

ii. $(u_n \vee v_n) \underset{o}{\to} u \vee v$,

iii. $(u_n \wedge v_n) \underset{o}{\to} u \wedge v$.

If Υ is Archimedean, then $\alpha_n u_n \underset{o}{\to} \alpha u$.

Proof. i. Let choose (u_n) and (v_n) as order convergent sequences and α as a real number. Then by the Triangle Inequality and by Theorem 3.2 (ii), we get the following consequence:

$$|u_n + av_n - (u + av)| \leq |u_n - u| + |av_n - av| \leq |u_n - u| + |a||v_n - v| \leq w_\beta + |a|t_\beta$$

where w_β and t_β are nets that decreases to zero, for some n_0 and for all $\beta \in B$, $n \geq n_0$.

ii. We know that $u_n \underset{o}{\to} u$ and $v_n \underset{o}{\to} v$ as $n \to \infty$. Then, there are sequences w_n and t_n in X such that

$$|u_n - u| \leq w_n \text{ when } w_n \downarrow 0 \text{ and,}$$

$$|v_n - v| \leq t_n \text{ when } t_n \downarrow 0$$

for all $n \to \infty$. Since $w_n \downarrow 0$ and $t_n \downarrow 0$, then $w_n + t_n \downarrow 0$. By the Birkhoff's Inequality,

$$\begin{aligned} |u_n \vee v_n - u \vee v| &\leq |u_n \vee v_n - u_n \vee v| + |u_n \vee v - u \vee v| \\ &\leq |u_n - u| + |v_n - v| \\ &\leq w_n + t_n. \end{aligned}$$

Thus, by the definition of order limit, we find a sequence $w_n + t_n \downarrow 0$ that dominates our sequence. So, we reach what we want; $u_n \vee v_n \underset{o}{\to} u \vee v$.

iii. We can prove third part in the same way:

Again let choose w_n and t_n with the same conditions in (ii). Then,

$$\begin{aligned} |u_n \wedge v_n - u \wedge v| &\leq |u_n \wedge v_n - u_n \wedge v| + |u_n \wedge v - u \wedge v| \\ &\leq |u_n - u| + |v_n - v| \\ &\leq w_n + t_n \end{aligned}$$

, $\forall n \to \infty$. From the definition of order limit, $u_n \wedge v_n \underset{o}{\to} u \wedge v$. ■

Definition 3.16. If each increasing net which is bounded above has a supremum, then a Riesz space Υ is called order complete (or o-complete or Dedekind complete).

Proposition 3.6 (Proposition 1.1.10, [34])**.** *Suppose that Υ is $\sigma-$ Dedekind complete. For each o-bounded sequence $(u_n) \subset \Upsilon$, $u \in \Upsilon$ TFAE:*

i. $u_n \underset{o}{\to} u$,

ii. $u = \sup_{n\in\mathbb{N}}\{\inf\{u_m : m \geq n\}\} = \inf_{n\in\mathbb{N}}\{\sup\{u_m : m \geq n\}\}$.

Proof. Let define $k_n = \sup\{u_m : m \geq n\}$, $l_n = \inf\{u_m : m \geq n\}$, $\forall n \in \mathbb{N}$.

$(i) \Rightarrow (ii)$. Suppose that $(v_n)_1{}^{\infty}$ is a sequence satisfies $v_n \downarrow 0$ when $n \to \infty$, $|u_n - u| \leq v_n, \forall n$. Then, $\forall m \geq n$ it yields that

$$-v_n \leq -v_m \leq u_m - u \leq v_m \leq v_n.$$

Therefore, $\forall n \in \mathbb{N}$ we get

$$-v_n \leq l_n - u \leq k_n - u \leq v_n.$$

Followingly,

$$\begin{aligned} u = \sup\{u - v_n : n \in \mathbb{N}\} &\leq \sup\{l_n : n \in \mathbb{N}\} \\ &\leq \inf\{k_n : n \in \mathbb{N}\} \\ &\leq \inf\{v_n + u : n \in \mathbb{N}\} = u. \end{aligned}$$

$(ii) \Rightarrow (i)$. $\forall n \in \mathbb{N}$, let choose $v_n = k_n - l_n$. By the definition, (v_n) is decreasing as $n \in [1, \infty)$ such that

$$0 \leq \inf\{v_n : n \in \mathbb{N}\} \leq \inf\{k_m - l_n : n \in \mathbb{N}\} = k_m - u$$

$\forall m \in \mathbb{N}$. This yields that

$$\inf\{v_n : n \in \mathbb{N}\} \leq \inf\{k_m - u : m \in \mathbb{N}\} = 0.$$

By the inequality $l_n \leq u \leq k_n, \forall n$, conclude that

$$\begin{aligned} |u_n - u| &= (u_n - u)_+ + (u - u_n)_- \\ &\leq (k_n - u)_+ + (u - l_n)_+ \\ &= k_n \vee u - l_n \wedge u = v_n. \end{aligned}$$

This completes the proof. ■

For a net $(u_\alpha) \subset \Upsilon$ and for $u \in \Upsilon$, we have $|u_\alpha - u| \underset{o}{\to} 0 \Leftrightarrow |u_\alpha| \underset{o}{\to} |u|$. That's why, without loss of generality, it is enough to use order-null nets in Υ_+.

Remark 3.1. For an o-bounded net (u_α) in a Riesz space which is o-complete,

$$u_\alpha \underset{o}{\to} u \Leftrightarrow \inf_\alpha \sup_{\beta \geq \alpha} |u_\beta - u| = 0 \Leftrightarrow u = \inf_\alpha \sup_{\beta \geq \alpha} u_\beta = \sup_\alpha \inf_{\beta \geq \alpha} u_\beta.$$

Definition 3.17 (Another definition of order boundedness)**.** Let Λ be a subset of a vector lattice Υ. Then Λ is called order bounded if it is included in an order interval.

Definition 3.18. Let Λ be a subset of a vector lattice Υ. Λ is called order closed if $u_\alpha \underset{o}{\to} u$, $u_\alpha \subseteq \Lambda$ imply $u \in \Lambda$.

Proposition 3.7 (Proposition 9.10, [21])**.** *Let Υ be an Archimedean Riesz space. Υ is o-complete if and only if every o-Cauchy net o-converges.*

Proof. Suppose that (u_λ) is an o-Cauchy net in an o-complete Riesz space. It will be shown that it o-converges. Since (u_λ) has an o-bounded tail, assume that (u_λ) is o-bounded. Fix $u = \sup \kappa_\lambda$, $\kappa_\lambda = \inf_{\beta \geq \lambda} u_\beta$, and $v = \inf \varpi_\lambda$, $\varpi_\lambda = \sup_{\beta \geq \lambda}$. Then $\kappa_\lambda \leq u \leq v \leq \varpi_\lambda$, $\forall \lambda$. By the previous argument, it is enough to show that $u = v$.

Consider that $(w_{\lambda,\beta})$ is a net satisfying $w_{\lambda,\beta} \downarrow 0$ and $|u_\lambda - u_\beta| \leq w_{\lambda,\beta}$. Put a pair (λ_0, β_0) such that $\lambda \geq \lambda_0$ and $\lambda \geq \beta_0$ for λ. $\forall \beta, \beta \geq \lambda$, we get $(\lambda, \beta) \geq (\lambda_0, \beta_0)$, and then $|u_\lambda - u_\beta| \leq w_{\lambda_0,\beta_0}$. Followingly,

$$u_\beta \in [u_\lambda - w_{\lambda_0,\beta_0}, u_\lambda + w_{\lambda_0,\beta_0}] \text{ which concludes } \kappa_\lambda, \varpi_\lambda \in [u_\lambda - w_{\lambda_0,\beta_0}, u_\lambda + w_{\lambda_0,\beta_0}].$$

Thus, we get $0 \leq \varpi_\lambda - \kappa_\lambda \leq 2w_{\lambda_0,\beta_0}$ and so $0 \leq v - u \leq 2w_{\lambda_0,\beta_0}$. Consequently, $u - v = 0$.

On the other hand, we assume that each o-Cauchy net in Υ converges in order. Assume that $0 \leq u_\lambda \uparrow \leq v$. Since each monotone o-bounded net in an Archimedean Riesz space is o-Cauchy (Lemma 9.9 in [21]), (u_λ) is o-Cauchy. Therefore, it converge in order, it yields that $\sup u_\lambda$ exists. ■

Definition 3.19. Assume that Θ is a sublattice of a Riesz space Υ. Then Θ is called

i. order dense if $\forall 0 < u \in \Upsilon$, $\exists 0 < v \in \Theta$ satisfying $0 < v \leq u$;

ii. dense w.r.t o-convergence if every vector in Υ is the o-limit of a net in Θ.

Definition 3.20. We define $f : \Upsilon \to \Theta$ as a function from a Riesz space to a Riesz space. f is called order continuous (o-continuous) if $u_\alpha \underset{o}{\to} u$ implies $f(u_\alpha) \underset{o}{\to} f(u)$.

Definition 3.21. A sublattice is regular if the inclusion map is o-continuous, that is, it conserves o-convergence of nets. Each o-dense sublattice is regular.

3.3 Unbounded Order Convergence

This section includes Theorem 3.3 and Corollary 3.1 which are taken from [20].

Definition 3.22. Let (u_α) be a net in a vector lattice Υ. (u_α) is called unbounded order convergent ($uo-$convergent) to u if $|u_\alpha - u| \wedge v \underset{o}{\to} 0$ for all $v \geq 0$. It is denoted by $u_\alpha \underset{uo}{\longrightarrow} u$.

Definition 3.23. A net $(u_\alpha) \subset \Upsilon$ is called unbounded order Cauchy ($uo-$Cauchy) if the double net $(u_\alpha - u_\beta)_{(\alpha,\beta)\in A\times A}$ $uo-$converges to zero.

Theorem 3.3. *Consider that Θ is a sublattice of a vector lattice Υ. TFAE:*

i. Θ is regular;

ii. for any net (v_α) in Θ, $v_\alpha \underset{uo}{\longrightarrow} 0$ in Θ implies that $v_\alpha \underset{uo}{\longrightarrow} 0$ in Υ;

iii. for any net (v_α) in Θ, $v_\alpha \underset{uo}{\longrightarrow} 0$ in Θ iff $v_\alpha \underset{uo}{\longrightarrow} 0$ in Υ.

Proof. $(iii) \Rightarrow (ii)$. Since our claim is that $v_\alpha \underset{uo}{\longrightarrow} 0$ in both Υ and Θ, it follows immediately (ii).

$(ii) \Rightarrow (i)$. Let suppose that $v_\alpha \downarrow 0$ in Θ. Let apply the rule that "unbounded order convergence agrees with o-convergence for o-bounded nets" to a tail of v_α, one can obtain $v_\alpha \downarrow 0$ in Υ by (ii). Followingly, Θ is regular.

$(i) \Rightarrow (iii)$. Given that Θ is regular in Υ, (v_α) is a net in Θ such that $v_\alpha \underset{uo}{\longrightarrow} 0$ in Θ. Since Υ is regular in Υ^δ, the completion of Υ, we get Θ is regular in Υ^δ. Suppose that I is the ideal generated by Θ in Υ^δ. Our claim is that $v_\alpha \underset{uo}{\longrightarrow} 0$ in I. For $\gamma \in I_+$, $\exists v \in \Theta_+$ such that $0 \leq \gamma \leq y$. By the assumption, $|v_\alpha| \wedge v \underset{o}{\to} 0$ in Y and since Θ is regular in Υ^δ, we get $|v_\alpha| \wedge v \underset{o}{\to} 0$ in Υ^δ. Additionally, since I is regular in Υ^δ, we get $|v_\alpha| \wedge v \underset{o}{\to} 0$ in I (Corollary 2.12 in [20]). By the inequality $0 \leq \gamma \leq v$, we get $|v_\alpha| \wedge \gamma \underset{o}{\to} 0$ in I. Hence, $v_\alpha \underset{uo}{\longrightarrow} 0$ in I. Lemma 3.4 in [19] says that for an order complete Riesz space, I -an ideal of Υ- and $(u_\alpha) \subset I$, we have $u_\alpha \underset{uo}{\longrightarrow} 0$ in $I \Leftrightarrow$ $u_\alpha \underset{uo}{\longrightarrow} 0$ in Υ. It follows from that lemma, $v_\alpha \underset{uo}{\longrightarrow} 0$ in Υ^δ. Lastly, for any $u \in \Upsilon_+$, $|v_\alpha| \wedge u \underset{o}{\to} 0$ in Υ^δ, so in Υ by Corollary 2.9 in [20]. Hence, $v_\alpha \underset{uo}{\longrightarrow} 0$ in Υ.

On the other hand, suppose that the net (v_α) belongs to Θ such that $v_\alpha \underset{uo}{\longrightarrow} 0$ in Υ. Let choose $\gamma \in \Theta_+$. Then, $|v_\alpha| \wedge \gamma \underset{o}{\to} 0$ in Υ. By Corollary 2.12 in [20], $|v_\alpha| \wedge \gamma \underset{o}{\to} 0$ in Θ. Consequently, $v_\alpha \underset{uo}{\longrightarrow} 0$ in Θ. ■

Corollary 3.1. *Let Υ be a Riesz space, $w > 0$ stand for a weak unit of it. Then, for a net (u_α) in Υ, $u_\alpha \underset{uo}{\longrightarrow} 0$ in Υ iff $|u_\alpha| \wedge w \underset{o}{\to} 0$.*

Proof. Consider that w is a weak unit of Υ^δ which is the o-completion of Υ. From Theorem 3.3 $u_\alpha \underset{uo}{\longrightarrow} 0$ in $\Upsilon \Leftrightarrow u_\alpha \underset{uo}{\longrightarrow} 0$ in Υ^δ, and so by the possibility that replacing

u (in the definition of uo-convergence) with a weak unit w by Theorem 2.2 in [6], $\Leftrightarrow |u_\alpha| \wedge w \underset{o}{\to} 0$ in Υ^δ. This is equivalent to $|u_\alpha| \wedge w \underset{o}{\to} 0$ in Υ. ■

4
ORDER AND UO-CONVERGENCE IN CONTINUOUS FUNCTION SPACE

In this chapter, characterization of unboundedly order convergence of nets in the continuous function space $C(X)$ will be given in local topological terms. This chapter based on the papers [33] and [38]. Recall that a net is o-convergent $\Leftrightarrow$ it is unbounded order convergent and it has an o-bounded tail [33]. In this view, characterizations of nets that unbounded order converges yield analogic characterizations of nets that are o-convergent.

Throughout the chapter, X is a Tychonoff space (completely regular Hausdorff topological space). It is the class of Hausdorff spaces where the consequence of Uryson's lemma is satisfied. It is a useful reminder that any locally compact Hausdorff space is completely regular. This also holds for all every normal spaces. $C(X)$ denotes all continuous real-valued function space where the functions defined on X, $\mathbb{1}$ is the constant-one function.

Proposition 4.1 (Proposition 4.13, [38]). *If $H \subset C(X)_+$, then $\inf H = 0 \Leftrightarrow \forall n \in \mathbb{N}$, every open U in X, $\exists h \in H$ and $u \in X$ such that $h(u) < \frac{1}{n}$.*

Proof. Assume that $g \in C(X)$ such that $0 \leq g \leq H$. For every open nonempty $U \subset X$ and $n \in \mathbb{N}$, $\exists h \in H$, $u \in X$ such that $g(u) \leq h(u) < \frac{1}{n}$. Then, $g = 0$.

Consider that there is $n \in \mathbb{N}$ and open nonempty U satisfies $h(u) \geq \frac{1}{n}$, $\forall h \in H$ and $u \in U$. Put $g \in (0, \mathbb{1}_U]$. Then we have $0 < \frac{1}{n}g \leq h$, for all $h \in H$, which is a contradiction. ■

Lemma 4.1 (Lemma 2.1, [33]). *For a net (u_α) in an Archimedean Riesz space X, $u_\alpha \underset{o}{\to} 0$ if and only if there is a set $H \subseteq X_+$ satisfies $\inf H = 0$, each element of H is a dominator for a tail of (u_α), that is, $\forall h \in H$, $\exists \alpha_0$ such that $|u_\alpha| \leq h$, $\forall \alpha \geq \alpha_0$.*

4.1 Unbounded Order Convergence in $C(X)$

The next lemma is actually Proposition 4.1; the proof is ensured for completeness:

Lemma 4.2 (Lemma 3.1, [33]). *Assume that X is Tychonoff and $H \subseteq C(X)_+$. The following are equivalent:*

i. $\inf H = 0$;

ii. $\forall U \neq \varnothing$, $\forall \varepsilon > 0$, $\exists s \in U$, $h \in H$ *such that* $h(s) < \varepsilon$;

iii. $\forall U \neq \varnothing$, $\forall \varepsilon > 0$, $\exists$ *an open set* $\varnothing \neq V \subseteq U$, $h \in H$ *with* $h(s) < \varepsilon$, $\forall s \in V$.

Proof. (i) $\Rightarrow$ (ii) Assume that $\inf H = 0$, but (ii) fails, i.e., there is an open set $U \neq \varnothing$, $\varepsilon > 0$ s.t. $\forall h \in H$, $h \geq \varepsilon \mathbb{1}$ on U. Since X is Tychonoff, there is a function $0 \neq g \in C(X)_+$ s. t. $g \leq \varepsilon \mathbb{1}$, g vanishes outside of U. Therefore, $g \leq H$, this gives a contradiction because our assumption is $\inf H = 0$.

(ii) $\Rightarrow$ (i) Assume that $\inf H \neq 0$. Then there is $g \in C(X)$ with $0 < g \leq H$. It is able to find an open set $U \neq \varnothing$, $\varepsilon > 0$ s. t. $g \geq \varepsilon \mathbb{1}$ on U. Followingly, each $h \in H$ is greater than ε on U, it gives a contradiction to (ii).

(iii) $\Rightarrow$ (ii) For all $U \neq \varnothing$ and ε let choose an open subset $V \subseteq U$, $h \in H$ s. t. $h(s) < \varepsilon$, $\forall s \in V$. Since $s \in V \subseteq U$, so $s \in U$, which yields (ii).

(ii) $\Rightarrow$ (iii) For all $U \neq \varnothing$ and ε let choose an element $s \in U$ and $h \in H$ such that $h(s) < \varepsilon$. We can find an open subset V of U that covers all s satisfies the assertion. This gives the remaining consequence. ■

The following theorem characterizes unbounded order convergence in the space $C(X)$. Since $f_\alpha \underset{uo}{\longrightarrow} f$ if and only if $|f_\alpha - f| \underset{uo}{\longrightarrow} 0$, it is enough to characterize that positive nets are o-null.

Theorem 4.1 (Theorem 3.2, [[33]). *] Suppose that X is Tychonoff and (f_α) is a net in $C(X)_+$. Then $f_\alpha \underset{uo}{\longrightarrow} 0 \Leftrightarrow \forall$ open $U \neq \varnothing$, $\forall \varepsilon > 0$, $\exists$ open $\varnothing \neq V \subseteq U$, an index α_0 s. t. for $\alpha \geq \alpha_0$, f_α is less than ε on V.*

Proof. Consider that $f_\alpha \underset{uo}{\longrightarrow} 0$. In that case, $f_\alpha \wedge \mathbb{1} \underset{o}{\to} 0$. There is a set $H \subseteq C(X)_+$ s. t. $\inf H = 0$ by Lemma 4.1, each element of H dominates a tail of $(f_\alpha \wedge \mathbb{1})$. Take an open set $U \neq \varnothing$, $\varepsilon \in (0, 1)$. Let choose V and h as it were in Lemma 4.2(iii). As h dominates a tail of $(f_\alpha \wedge \mathbb{1})$, $\exists \alpha_0$ satisfies $f_\alpha \wedge \mathbb{1} \leq h$, $\forall \alpha \geq \alpha_0$. Particularly, $f_\alpha(t) \wedge 1 \leq h(t) < \varepsilon$, so $f_\alpha(t) < \varepsilon$, $\forall t \in V$.

Conversely, consider that the statement in the theorem is provided. As we can use $\mathbb{1}$ as a weak unit, it is enough to show that $f_\alpha \wedge \mathbb{1}$ is order-null. We use Lemma 4.1 again. Take an open set $U \neq \varnothing$, $\varepsilon > 0$. We choose V and α_0 as in the hypothesis. Take any $s \in V$. As X is Tychonoff, there is a $g \in C(X)_+$ such that $g(s) = 0$, $g = \mathbb{1}$ outside of V. Again take $k = g \vee \varepsilon\mathbb{1}$. Thus, $k(s) = \varepsilon$. Our claim is that $f_\alpha \wedge \mathbb{1} \leq k$, $\forall \alpha \geq \alpha_0$. So indeed, if $t \in V$ then we have $f_\alpha(t) < \varepsilon \leq k(t)$, and if $t \notin V$, so $(f_\alpha \wedge \mathbb{1})(t) \leq 1 = g(t) \leq k(t)$.

Recap this operations $\forall(U, \varepsilon)$, where $\varnothing \neq U$ is open and $\varepsilon > 0$. Suppose that H is the set of the resulting functions h. Every such h is dominator for a tail of $(f_\alpha \wedge \mathbb{1})$. Lemma 4.2 yields $\inf H = 0$. ■

Corollary 4.1 (Corollary 3.3, [33]). *Consider that X is Tychonoff space, (f_α) is a net in $C(X)$. Then (f_α) is unbounded order-Cauchy if and only if $\forall$ open $U \neq \varnothing$, $\varepsilon > 0$, $\exists$ open $\varnothing \neq V \subseteq U$, α_0 s. t. $|f_\alpha(s) - f_\beta(s)| < \varepsilon$, $\forall s \in V$, $\alpha, \beta \geq \alpha_0$.*

Proof. $(\Rightarrow)$ Let $(f_\alpha)_{\alpha \in A}$ be an $uo - Cauchy$ net. That is, for $\alpha, \beta \in A$, the net $(f_\alpha - f_\beta)_{A \times A}$ is $uo-$null. By the Theorem 4.1 we get the result directly.
$(\Leftarrow)$ It is enough to use Theorem 4.1 for the proof. ■

Lemma 4.3 (Lemma 3.6, [33]). *Suppose that X is a Tychonoff Baire space. For $H \subseteq C(X)_+$, the following are equivalent:*

i. $\inf H = 0$*;*

ii. $\exists$ *dense V satisfies* $\inf_{h \in H} h(s) = 0$ *for all $s \in V$;*

iii. $\exists$ *residual V s. t.* $\inf_{h \in H} h(s) = 0$ *for all $s \in V$.*

Proof. $(ii) \Rightarrow (i)$ It is a consequence of Proposition 4.1.

$(iii) \Rightarrow (ii)$ Since every residual set in a Baire space is dense, this implication holds.

$(i) \Rightarrow (iii)$ Consider that $\inf H = 0$. For $n \in \mathbb{N}$, take $G_n = \bigcup_{h \in H} \{h < \frac{1}{n}\}$. Then G_n is open. $\forall$ open $U \neq \varnothing$, Lemma 4.2 yields a point $s \in U$, $h \in H$ such that $h(s) < \frac{1}{n}$. Hence, $s \in G_n$, and so G_n is dense. Take $V := \bigcap_{n=1}^{\infty} G_n$, then V is residual. Let $s \in V$, then for all $n \in \mathbb{N}$ we get $s \in G_n$. Thus, $\inf_{h \in H} h(s) \leq \frac{1}{n}$. Hence $\inf_{h \in H} h(s) = 0$. ■

Theorem 4.2 (Theorem 3.7, [33]). *Suppose that X is a Tychonoff Baire space, $(f_\alpha) \subset C(X)$. If $f_\alpha \underset{uo}{\longrightarrow} f$ then f_α is pointwise convergent to f on a residual*

set. The opposite part of the theorem holds for countable nets (particularly, for sequences).

Proof. Without loss of generality, let choose $f = 0$. Consider that $f_\alpha \underset{uo}{\longrightarrow} 0$. So $|f_\alpha| \wedge \mathbb{1} \underset{o}{\to} 0$. We can find a net (h_λ) satisfying $h_\lambda \downarrow 0$, $\forall \lambda$, $\exists \alpha_0$ s. t. $|f_\alpha| \wedge \mathbb{1} \leq h_\lambda$ implies $\alpha \geq \alpha_0$. Fix $H = \{h_\lambda\}$. Thus $\inf H = \inf h_\lambda = 0$. From the previous lemma, we can find a residual set V s. t. $\forall s \in V$, we get $0 = \inf_{h \in H} h(s) = \inf_\lambda g_\lambda(s)$. Hence, $\lim_\alpha f_\alpha(s) = 0$.

It is easier using sequences than using nets. Assume that $(f_n)_{n\in\mathbb{N}}$ is convergent to zero on a residual set V. By using Theorem 4.1, we will demonstrate $f_n \underset{uo}{\longrightarrow} 0$. Without loss of generality, $f_n \geq 0$, $\forall n$. Take an open set $U \neq \varnothing$, $\varepsilon > 0$. $\forall k$, fix $G_k = \bigcup_{n\geq k}\{f_n > \varepsilon\}$. So, G_k is open. If $s \in \bigcap_k G_k$ then $\forall k$, $\exists n \geq k$ such that $f_n(s) > \varepsilon$, and therefore $s \notin V$. This provides that $\bigcap_k G_k$ is contained in $X\backslash V$. That's why, it is of the first category. Since G_k is open, ∂G_k is nowhere dense for all k. We accomplish that $\bigcup_k \partial G_k$ is of the first category. It follows from

$$\bigcap_k \overline{G_k} \subset \left(\bigcap_k G_k\right) \cup \left(\bigcup_k \partial G_k\right)$$

that $\bigcap_k \overline{G_k}$ is of the first category, and so its complement $\bigcup_k (X\backslash \overline{G_k})$ is residual, hence dense. Thus, it meets U. Consequently, $U \cap (X\backslash \overline{G_k}) \neq \varnothing$ for some k. Therefore, the statement in Theorem 4.1 is provided. ■

Example 4.1 (Example 3.9, [33])**.** This example shows that an o-bounded net which is pointwise convergent to zero at every point, but it fails to be o-convergent. That means the converse condition in Theorem 4.2 usually fails for nets.

Assume that $X = C([0,1])$, ζ is the set of all finite subsets of $[0,1]$ that contains the numbers 0 and 1. Also, ζ is ordered by inclusion. $\forall \varpi \in A$, $\varpi = \{m_1, m_2, m_3, \ldots, m_n\}$ where $0 = m_0 < m_1 < m_2 < m_3 < \cdots < m_n = 1$, let us define f_ϖ such that for every t, $f_\varpi(m_t) = 0$. Let us put $f_\varpi = 1$ at the midpoints, that is, $f_\varpi\left(\frac{m_{t-1}+m_t}{2}\right) = 1$ for every $t \in \{1, 2, 3, \ldots, n\}$ and define f_ϖ to be linear in between. Then, f_ϖ is pointwise convergent to zero on $[0,1]$.

Conversely, let the supremum of each tail of the present net in $C([0,1])$ be $\mathbb{1}$. So the nets are not order convergent to zero. Furthermore, let choose $\varpi \in A$, and assume that $g \geq f_\lambda$ for every $\varpi \leq \lambda$. Let $t \notin \varpi$. Then, there exists $\lambda \supseteq \varpi$ such that t is a midpoint for λ. Therefore, $g(t) \geq f_\lambda(t) = 1$. Hence, $1 \leq g(t)$ for every $t \notin \varpi$. As ϖ is finite, g is continuous, we get $\mathbb{1} \leq g$.

5
UO-CONVERGENCE OF OPERATOR NETS

In this chapter, we characterize uo-convergence of operator nets which are defined on continuous functions. In the paper [33], order and unbounded order convergence on $C(\Omega)$ were characterized where Ω is a Baire space and $C(\Omega)$ is the space of all real valued functions. For a short look, please see the Chapter 4.

Based on that work, we study the uo-convergence of operator nets defined on the space of continuous functions satisfying the same conditions in [33]. We give a definition of uo-convergence and uo-Cauchy nets for operator nets. These results are published in the paper [39].

5.1 Preliminary Notes

For a net $(u_\alpha) \in Y$, which is a regular sublattice of X, $u_\alpha \underset{uo}{\longrightarrow} 0$ in $X \Leftrightarrow u_\alpha \underset{uo}{\longrightarrow} 0$ in Y.

Throughout the section, X stands for a Hausdorff locally solid vector lattice. Additionally, X is a Tychonoff. X' be a topological dual of X. $C(X)$ denotes the space of all continuous real-valued functions on X, $\mathbb{1}$ denotes the constant one function. Let $B(X')_+$ be a family of all continuous positive operators defined as follows:

$$B(X')_+ = \{T : X' \to X' \mid Tf \geq 0 \text{ for } f \in X'\}. \tag{5.1}$$

5.2 Unbounded Order Convergence in $B(X')_+$

The next lemma is actually Lemma 4.2, we rewrite it for our main goal:

Lemma 5.1. *Assume that X is a Hausdorff locally solid vector lattice (and Tychonoff), X' be the topological dual of it and $H \subseteq B(X')_+$. The following are equivalent:*

i. $\inf H = 0$,

ii. $\forall$ *open* $U \neq \varnothing$, $g \in X'$, $\forall \varepsilon > 0$, $\exists s \in U$, $T \in H$ *such that* $(Th)(s) < \varepsilon$,

iii. $\forall$ *open* $U \neq \varnothing$, $g \in X'$, $\forall \varepsilon > 0$, $\exists$ *open* $\varnothing \neq V \subseteq U$, $T \in H$ *satisfies* $(Th)(s) < \varepsilon$, $\forall s \in V$.

Proof. (i) $\Rightarrow$ (ii) Consider that $\inf H = 0$, but (ii) does not hold, that means, there is an open set $U \neq \varnothing$, $h \in X'$, $\varepsilon > 0$ satisfying for all $T \in H$, $(Th)(s) \geq \varepsilon$ on U for all s. Since X is Tychonoff, there is a nonzero $f \in C(X)_+$ and an operator $S \in B(X')_+$ such that $Sf \leq \varepsilon \mathbb{1}$ and Sf vanishes outside of U. So, $S \leq H$, it contradicts $\inf H = 0$.

(ii) $\Rightarrow$ (i) Suppose that (ii) holds and $\inf H \neq 0$. Then there is $S \in B(X')_+$ such that $0 < S \leq H$. Next, there is an open set $U \neq \varnothing$, $f \in C(X)$, $\varepsilon > 0$ satisfying $Sf > \varepsilon$ on U. Followingly, $\forall T \in H, h \in X'$ we obtain that Th is greater than ε on U, it contradicts with (ii).

$(ii) \Leftrightarrow (iii)$ follows from Lemma 4.2. ■

Now, we will define uo-convergence and characterize it on $B(X')_+$ by using operator nets motivated from [33].

Definition 5.1. Let T_λ be a net in $B(X')_+$. $T_\lambda \underset{uo}{\longrightarrow} T$ if and only if $|T_\lambda - T| \wedge S \underset{o}{\to} 0$ for all $S \in B(X')_+$.

Since $T_\lambda \underset{uo}{\longrightarrow} T$ if and only if $|T_\lambda - T| \underset{uo}{\longrightarrow} 0$, it is enough to characterize $T_\lambda \underset{uo}{\longrightarrow} 0$.

Theorem 5.1. *Let* X' *be the dual of a Hausdorff locally solid vector lattice (and Tychonoff)* X *and* (T_λ) *be a net in* $B(X')_+$. *Then* $T_\lambda \underset{uo}{\longrightarrow} 0$ *if and only if for each open set* $U \neq \varnothing$, $f \in X'_+$, *each* $\varepsilon > 0$, *we have an open set* $\varnothing \neq V \subseteq U$, *an index* λ_0 *s. t.* $T_\lambda f$ *is less than* ε *on* V, $\lambda \geq \lambda_0$.

Proof. Let $T_\lambda \underset{uo}{\longrightarrow} 0$. So we have $T_\lambda \wedge \mathbb{1} \underset{o}{\to} 0$. From the Lemma 4.1, we can find a set $H \subseteq B(X')_+$ satisfies $\inf H = 0$, each element of H dominates a tail of $(T_\lambda \wedge \mathbb{1})$. Take an open set $U \neq \varnothing$, $\varepsilon \in (0, 1)$. Choose such V,T that Lemma 4.2 (iii) holds. Since $T \in H$ then T is a dominator of a tail of $(T_\lambda \wedge \mathbb{1})$, followingly, there is λ_0 satisfies $T_\lambda \wedge \mathbb{1} \leq T$ for all $\lambda \geq \lambda_0$. In particular,

$$(T_\lambda f)(s) \wedge \mathbb{1}(s) = (T_\lambda f)(s) \wedge 1 \leq (Tf)(s) < \varepsilon,$$

and so $(T_\lambda f)(s) < \varepsilon$, $\forall s \in V$. That part of the theorem is completed.

Conversely, consider that the assumption in the theorem holds. Since $\mathbb{1}$ is a weak unit, so it is enough to prove that $T_\lambda \wedge \mathbb{1} \underset{o}{\to} 0$. Let use Lemma 4.1 for the proof of it. Choose an open set $U \neq \varnothing$, $\varepsilon > 0$. We assume that $\varnothing \neq V \subseteq U$ is an open set, λ_0 is an index satisfying $T_\lambda f$ is less than ε on V for all $f \in {X'}_+$ whenever $\lambda \geq \lambda_0$. Take any $s \in V$. Since X is Tychonoff, there exists $g \in C(X)_+$ such that $g(s) = 0$, g is equal to 1 outside of V. Define $k = g \vee \varepsilon\mathbb{1}$. Then

$$k(s) = g(s) \vee \varepsilon \cdot \mathbb{1}(s) = \varepsilon.$$

Our assumption is that $(T_\lambda \wedge \mathbb{1})(f) \leq k$ for all $\lambda \geq \lambda_0$. Indeed, if $m \in V$, so we get

$$(T_\lambda f)(m) < \varepsilon \leq k(s).$$

If $m \notin V$, so we have

$$[(T_\lambda \wedge \mathbb{1})(f)](m) \leq 1 = g(s) \leq k(s).$$

Recap this $\forall(U, \varepsilon)$ where $U \neq \varnothing$ is open set, $\varepsilon > 0$. Assume that H is the set of the receiving functions k. Every such k is a dominator for a tail of $(T_\lambda \wedge \mathbb{1})$ Lemma 4.2 implies $\inf H = 0$. That means the proof is completed. ■

Corollary 5.1. *Suppose that X is a Hausdorff locally solid vector lattice (and Tychonoff), X' is a topological dual of it and $(T_\lambda)_{\lambda\in\Lambda}$ is a net in $B(X')_+$. Then (T_λ) is unbounded order Cauchy if and only if for each open set $U \neq \varnothing$, $f \in {X'}_+$, each $\varepsilon > 0$, there is an open set $\varnothing \neq V \subseteq U$, λ_0 satisfying*

$$|(T_\lambda f)(s) - (T_\beta f)(s)| < \varepsilon$$

for every $s \in V$ and $\lambda, \beta \geq \lambda_0$.

Proof. ($\Rightarrow$) Let (T_λ) be an $uo - Cauchy$ net. That is, for $\lambda, \beta \in \Lambda$, the net $(T_\lambda - T_\beta)_{\Lambda\times\Lambda}$ is $uo-$null. By the Theorem 5.1 we get the result directly.
($\Leftarrow$) It can be easily seen by using Theorem 5.1 again. ■

6
CONCLUSION

In this thesis, we introduced basic lattice theory. We gave historical view of order and unbounded order convergence theory.

We gave some preliminary notes about vector lattices and their properties. We introduced basic definitions for reader to understand the rest of this thesis. Since our main concept is to examine convergence types, we stated three different convergence types and relationships between them.

Then, we gave order and unbounded order convergence in the space of all real valued continuous functions which are defined on a Baire space (and also it is Tychonoff). Choosing spaces like this enables us to generalize order and uo-convergence. By using this argument, we defined and characterized unbounded order convergence in the class of all positive continuous operators defined on the topological dual of a completely regular Hausdorff space. By stating unbounded order convergence for operator nets, we rebuilt the structure from real valued functions to positive continuous operators. We used functionals to carry structure to operator nets and they became compatible. Consequently, we studied a different class of operators than real valued continuous functions. There is no similar research with our work.

There exist some operator classes which haven't been studied yet. Unbounded convergence can be extended more general class. Our class can be generalized with the same convergence type or can be characterized with another convergence type. Also, this study can be used in economical researches.

REFERENCES

[1] H. Nakano, "Ergodic theorems in semi-ordered linear spaces," *Annals of Mathematics*, vol. 49, pp. 538–556, 1948.

[2] R. DeMarr, "Partially ordered linear spaces and locally convex linear topological spaces," *Illinois Journal of Mathematics*, vol. 8, no. 4, pp. 601–606, 1964.

[3] J. van der Walt, "Order convergence in sets of hausdorff continuous functions," *Honors Essay, University of Pretoria*, 2004.

[4] R. Anguelov, J. H. van der Walt, "Order convergence structure on $C(X)$," *Quaestiones Mathematicae*, vol. 28, no. 4, pp. 425–457, 2005.

[5] J. H. van der Walt, "Order convergence on archimedean vector lattices and applications," Ph.D. dissertation, University of Pretoria, 2007.

[6] S. Kaplan, "On unbounded order convergence," *Real Analysis Exchange*, vol. 23(1), pp. 175–184, 1997.

[7] A. Wickstead, "Weak and unbounded order convergence in banach lattices," *Journal of the Australian Mathematical Society*, vol. 24, no. 3, pp. 312–319, 1977.

[8] W. A. Luxemburg, A. C. Zaanen, *Riesz spaces I*. North-Holland, 1971.

[9] A. C. Zaanen, *Riesz spaces II*. North-Holland, 1983.

[10] D. H. Fremlin, *Topological Riesz Spaces and Measure Theory*. Cambridge University Press, 1974.

[11] H. H. Schaefer, *Banach Lattices and Positive Operators*. Springer Berlin, Heidelberg, 1975.

[12] H.-U. Schwarz, *Banach Lattices and Operators*. Springer Berlin, Heidelberg, 1984.

[13] A. C. Zaanen, *Introduction to operator theory in Riesz spaces*. Springer Berlin, Heidelberg, 1997.

[14] S. S. Kutateladze, *Nonstandard Analysis and Vector Lattices*. Springer Dordrecht, 2000, vol. 525.

[15] C. D. Aliprantis, K. C. Border, W. A. Luxemburg, *Positive Operators, Riesz Spaces, and Economics: Proceedings of a Conference at Caltech, Pasadena, California, April 16–20, 1990*. Springer Science & Business Media, 2012, vol. 2.

[16] C. D. Aliprantis, O. Burkinshaw, *Locally solid Riesz spaces with applications to economics*. American Mathematical Society, 2003.

[17] O. Burkinshaw, C. D. Aliprantis, *Locally Solid Riesz Spaces*. Springer Berlin, Heidelberg, 1997.

[18] N. Gao, "Unbounded order convergence in dual spaces," *Journal of Mathematical Analysis and Applications*, vol. 419, no. 1, pp. 347–354, 2014.

[19] N. Gao, F. Xanthos, "Unbounded order convergence and application to martingales without probability," *Journal of Mathematical Analysis and Applications*, vol. 415, no. 2, pp. 931–947, 2014.

[20] N. Gao, V. G. Troitsky, F. Xanthos, "Uo-convergence and its applications to cesaro means in banach lattices," *Israel Journal of Mathematics*, vol. 220, pp. 649–689, 2017.

[21] M. O'Brien, V. Troitsky, J. van der Walt, "Net convergence structures with applications to vector lattices," *Quaestiones Mathematicae*, vol. 46, no. 2, pp. 243–280, 2023.

[22] M. Kandic, M. A. Marabeh, V. G. Troitsky, "Unbounded norm topology in banach lattices," *Journal of Mathematical Analysis and Applications*, vol. 451, no. 1, pp. 259–279, 2017.

[23] O. Zabeti, "Some results about unbounded convergences in banach lattices," *Mathematical Researches*, vol. 8, no. 2, pp. 129–137, 2022.

[24] M. A. Taylor, "Unbounded convergences in vector lattices," Master's thesis, University of Alberta, 2018.

[25] M. A. Marabeh, "Unbounded p-convergence in lattice-normed vector lattices," Ph.D. dissertation, 2017.

[26] N. E. Özcan, N. A. Gezer, O. Zabeti, "Unbounded absolute weak dunford-pettis and unbounded absolute weak compact operators," *arXiv preprint, arXiv:1708.03970*, 2017.

[27] N. E. Özcan, N. A. Gezer, Ş. E. Özdemir, İ. M. Geyikçi, "Order compact and unbounded order compact operators," *Turkish Journal of Mathematics*, vol. 45, no. 2, pp. 634–646, 2021.

[28] S. Alpay, E. Emelyanov, S. Gorokhova, "Bibounded uo-convergence and b-property in vector lattices," *Positivity*, vol. 25, no. 5, pp. 1677–1684, 2021.

[29] A. Aydın, E. Y. Emelyanov, N. E. Özcan, M. A. Marabeh, "Compact-like operators in lattice-normed spaces," *Indagationes Mathematicae*, vol. 29, no. 2, pp. 633–656, 2018.

[30] A. Aydın, "Compact-like operators in lattice-normed spaces," Ph.D. dissertation, 2017.

[31] A. Aydın, "Statistical unbounded order convergence in riesz spaces," *Facta Universitatis, Series: Mathematics and Informatics*, pp. 585–593, 2022.

[32] Ö. Gök, E. Demir, "On unbounded order weakly demicompact operators," in *International Mathematical Forum*, vol. 17, 2022, pp. 61–66.

[33] E. Bilokopytov, V. G. Troitsky, "Order and uo-convergence in spaces of continuous functions," *Topology and its Applications*, vol. 308, p. 107 999, 2022.

[34] P. Meyer-Nieberg, *Banach lattices*. Springer Verlag, 1991.

[35] J. Munkres, *Topology, Pearson New International Edition*, 2nd ed. Pearson Education Limited, 2014.

[36] J. L. Kelley, *General topology*. D. Van Nostrand Company, Inc., Toronto/New York/London, 1955.

[37] S. Willard, *General topology*. Addison-Wesley Publishing Co., 1970.

[38] E. Bilokopytov, "Order continuity and regularity on vector lattices and on lattices of continuous functions," *arXiv preprint, arXiv:2103.08776*, 2021.

[39] E. Aydogan, E. Demir, "On unbounded order convergence of operator nets," in *International Mathematical Forum*, vol. 18, 2023, pp. 83–88.

PUBLICATIONS FROM THE THESIS

Papers

1. E. Aydogan, E. Demir, "On Unbounded Order Convergence of Operator Nets", *International Mathematical Forum*, vol. 18(2), pp. 83 - 88, 2023.

Conference Papers

1. E. Demir, E. Aydogan, "Unbounded Convergence Structure Properties in Riesz Spaces", *5th International Online Conference on Mathematics,* 2021.

Projects

1. E. Demir, E. Aydogan, "Unbounded Convergence Structure Properties in Riesz Spaces", *BAP (Scientific Research Projects) YTU*, Project No. FYL-2022-5224, 2022-2023.