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A Second Regularized Trace Formula for a Fourth Order Differential Operator

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Abstract: In applications, many states given for a system can be expressed by orthonormal elements, called "state elements," taken in a separable Hilbert space (called "state space"). The exact nature of the Hilbert space depends on the system; for example, the state space for position and momentum states is the space of square-integrable functions. The symmetries of a quantum system can be represented by a class of unitary operators that act in the Hilbert space. The operators called ladder operators have the effect of lowering or raising the energy of the state. In this paper, we study the spectral properties of a self-adjoint, fourth-order differential operator with a bounded operator coefficient and establish a second regularized trace formula for this operator.

Keywords: self-adjoint operator; trace-class operator; spectrum; regularized trace

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1. Introduction

The trace formulae of a differential operator may be seen as a generalization of the traces of matrices or trace-class operators. These formulae are, in general, referred as regularized trace formulae for operators, and they can be used to solve inverse problems [1] and can be applied to index theory [2]. The regularized trace formula of a scalar differential operator was first introduced by I. M. Gelfand and B. M. Levitan [3]. Then, several works on the regularized traces of scalar differential operators appeared (see [4–9]). The trace formulae for differential operators with operator coefficients were studied in many works [10–18]. Recently a second regularized trace formula was obtained in [19] for the Sturm–Liouville operator with the antiperiodic boundary conditions.

To explain our motivation here, let $\mathcal{H}$ be a separable Hilbert space. On Hilbert space $\mathcal{H}_1 = L^2(\mathcal{H};[0,\pi])$, we consider two differential operators L_0 and L given by the differential statements:

$$l_0(u) = u^{iv}(t) \qquad \text{and} \qquad l(u) = u^{iv}(t) + Q(t)u(t)$$

with the identical symmetric boundary conditions $u'(0) = u'(\pi) = u'''(0) = u'''(\pi) = 0$. Our aim was to find a trace formula called the second regularized trace for the operator L by taking advantage from spectral properties of the unperturbed operator L_0. Here we refer to [20] for the first regularized trace formula of the same operator.

Here $Q(t)$ is an operator function with the properties:

(i) $Q(t)$ has a weak fourth-order derivative in interval $[0,\pi]$ and for every $t \in [0,\pi]$, $Q^{(i)}(t) \quad (i = \overline{0,4})$ are self-adjoint trace-class operators on $\mathcal{H}$.
(ii) $\|Q\| < \frac{1}{2}$.
(iii) $\mathcal{H}$ has an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \|Q(t)\varphi_n\| < \infty$.
(iv) $\|Q^{(i)}(t)\|_{\sigma_1(\mathcal{H})} (i = \overline{0,4})$ is bounded and measurable in $[0,\pi]$.

Let $\sigma_1(\mathcal{H})$ denote the space of trace-class operators from $\mathcal{H}$ to $\mathcal{H}$ [21]. Moreover, the norms in $\mathcal{H}$ and $\mathcal{H}_1$ are denoted by $\|.\|_{\mathcal{H}}$ and $\|.\|$ and the inner products are denoted

by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle$, respectively. Denote by trA the sum of the eigenvalues of a trace-class operator A [22] and the notation "$\cdot$" stands for the product.

Each point in the spectrum $\{m^4\}_{m=0}^{\infty}$ of L_0 gives an eigenvalue of L_0 with its infinite multiplicity. We can easily check that the orthonormal eigenvectors corresponding to these eigenvalues are given by the system

$$\psi_{mn}(t) = d_m \cos mt \cdot \varphi_n \quad (m = 0, 1, 2, \dots; n = 1, 2, \dots), \tag{1}$$

where

$$d_m = \begin{cases} \frac{1}{\sqrt{\pi}} & \text{if } m = 0 \\ \frac{2}{\sqrt{\pi}} & \text{if } m = 1, 2, \dots . \end{cases}$$

This system constitutes an orthonormal basis of the Hilbert space $\mathcal{H}_1$, and we will often refer to this fact through the paper.

At the end, we will obtain a formula for the sum of

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\left(\lambda_{mn}^2 - m^8\right) - \frac{2m^4}{\pi}\int_0^{\pi} \mathrm{tr}Q(t)dt - \frac{m^2}{2\pi}\left(\mathrm{tr}Q'(\pi) - \mathrm{tr}Q'(0)\right) + 2C\right]$$

where the sequences $\{\lambda_{mn}\}_{n=1}^{\infty}$ represent the eigenvalues of L for $m = 0, 1, 2, \dots$ belonging to the interval $[m^4 - 1/2, m^4 + 1/2]$, and C is a constant depending on $Q(t)$. This formula is said to be a second regularized trace formula of the operator L.

2. Some Relations about Eigenvalues and Resolvents

Let R_λ^0 and R_λ be resolvents of L_0 and L, respectively. Since the operator function $Q(t)$ satisfies condition (iii) and the system (1) is an orthonormal basis of $\mathcal{H}_1$, the operator $QR_\lambda^0 : \mathcal{H}_1 \to \mathcal{H}_1$ is a trace-class operator for every $\lambda \notin \{m^4\}_{m=0}^{\infty}$ [20]. Moreover, since $Q(t)$ satisfies also conditions (ii) and (iii), then the spectrum of L is a subset of the union of pairwise disjoint intervals $I_m = \left[m^4 - \|Q\|, m^4 + \|Q\|\right]$ $(m = 0, 1, 2, \dots)$ on the real line. Furthermore, we have:

(a) Each point of the spectrum of L which is not the same as m^4 in I_m is an isolated eigenvalue of finite multiplicity.
(b) m^4 is the possible eigenvalue of L of any multiplicity.
(c) $\lim_{n\to\infty} \lambda_{mn} = m^4$ such that $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of L in I_m.

Let $\rho(L)$ be the resolvent set of L. Since $QR_\lambda^0 \in \sigma_1(\mathcal{H}_1)$ for every $\lambda \in \rho(L)$, the equation

$$R_\lambda - R_\lambda^0 = -R_\lambda Q R_\lambda^0 \tag{2}$$

gives $R_\lambda - R_\lambda^0 \in \sigma_1(\mathcal{H}_1)$. On the other hand, since the series

$$\sum_{n=1}^{\infty}\left(\lambda_{mn} - m^4\right), \quad m = 0, 1, 2, \dots$$

are absolutely convergent, we have:

$$\mathrm{tr}\left(R_\lambda - R_\lambda^0\right) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{m^4 - \lambda}\right)$$

for every $\lambda \in \rho(L)$. Multiply by $\lambda^2/2\pi i$ both sides of this equality and integrate it over the circle $|\lambda| = b_p = (p^2 + p + 1)^2 - 1/2$ $(p = 1, 2, \dots)$. We obtain

$$\frac{1}{2\pi i}\int_{b_p} \lambda^2 \mathrm{tr}\left(R_\lambda - R_\lambda^0\right)d\lambda = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(m^8 - \lambda_{mn}^2\right). \tag{3}$$

By using the relation (2) we find, for any positive integer N:

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^{N} (-1)^j R_\lambda^0 (QR_\lambda^0)^j + (-1)^{N+1} R_\lambda (QR_\lambda^0)^{N+1}.$$

If we use this expression in (3), we obtain

$$\sum_{m=0}^{p} \sum_{n=1}^{\infty} \left(\lambda_{mn}^2 - m^8\right) = \sum_{j=0}^{N} D_{pj} + D_p^{(N)} \tag{4}$$

where

$$D_{pj} = \frac{(-1)^{j+1}}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 \operatorname{tr}\left[R_\lambda^0 (QR_\lambda^0)^j\right] d\lambda\,, \tag{5}$$

$$D_p^{(N)} = \frac{(-1)^N}{2\pi i} \int_{|\lambda|=b_p} \lambda^2 \operatorname{tr}\left[R_\lambda (QR_\lambda^0)^{N+1}\right] d\lambda. \tag{6}$$

The fact that $Q(t)$ satisfies the condition (iii) implies $QR_\lambda^0 \in \sigma_1(\mathcal{H}_1)$ for every $\lambda \neq m^4$ $(m = 0, 1, \dots)$, and the operator function QR_λ^0 in the domain $\mathbb{C} - \{m^4\}_{m=0}^{\infty}$ is analytic with respect to the norm in $\sigma_1(\mathcal{H}_1)$. By [8], we have:

$$D_{pj} = \frac{(-1)^j}{\pi i j} \int_{|\lambda|=b_p} \lambda \operatorname{tr}\left[(QR_\lambda^0)^j\right] d\lambda. \tag{7}$$

Theorem 1. *If $\|Q(t)\|_{\sigma_1(\mathcal{H})}$ is integrable in the interval $[0, \pi]$ and $Q'(0), Q'(\pi), Q'''(0), Q'''(\pi) \in \sigma_1(\mathcal{H})$ then we have:*

$$\begin{aligned} D_{p1} = {} & \frac{2}{\pi} \sum_{m=1}^{p} m^4 \int_0^\pi \operatorname{tr} Q(t) dt + \frac{1}{12\pi} p(p+1)(2p+1)\left(\operatorname{tr} Q'(\pi) - \operatorname{tr} Q'(0)\right) \\ & + \frac{1}{8\pi} p\left(\operatorname{tr} Q'''(0) - \operatorname{tr} Q'''(\pi)\right) + \frac{1}{8\pi} \sum_{m=1}^{p} \int_0^\pi \operatorname{tr} Q^{iv}(t) \cos 2mt\, dt. \end{aligned} \tag{8}$$

Proof. According to (7) we get

$$D_{p1} = \frac{-1}{\pi i} \int_{|\lambda|=b_p} \lambda \operatorname{tr}(QR_\lambda^0) d\lambda. \tag{9}$$

Since the system in (1) is an orthonormal basis of $\mathcal{H}_1$ and QR_λ^0 is a trace-class operator for every $\lambda \in \rho(L_0)$, we have

$$\operatorname{tr}(QR_\lambda^0) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \langle QR_\lambda^0 \psi_{mn}, \psi_{mn} \rangle.$$

Replacing this expression in (9), we get:

$$D_{p1} = -\frac{1}{\pi i} \int_{|\lambda|=b_p} \sum_{m=0}^{p} \sum_{n=1}^{\infty} \langle QR_\lambda^0 \psi_{mn}, \psi_{mn} \rangle d\lambda. \tag{10}$$

By (1) and the fact that

$$R_\lambda^0 \psi_{mn} = \left(L_0 - \lambda L\right)^{-1} \psi_{mn} = \left(m^4 - \lambda\right)^{-1} \psi_{mn}$$

we obtain

$$\left|\left(QR_\lambda^0 \psi_{mn}, \psi_{mn}\right)\right| \leq \sqrt{\pi} |m^4 - \lambda|^{-1} \|Q(t)\varphi_n\|.$$

Together with the condition (iii) on $Q(t)$ and last inequality, the series

$$\sum_{n=1}^{\infty}\left\langle QR_{\lambda}^{0}\psi_{mn},\psi_{mn}\right\rangle \quad (m=0,1,2,\dots) \quad \text{and} \quad \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\langle QR_{\lambda}^{0}\psi_{mn},\psi_{mn}\right\rangle$$

are absolutely and uniformly convergent on the circle $|\lambda|=b_p$. Therefore, using Cauchy integral formula and the system (1), the equality (10) becomes

$$D_{p1}=\frac{2}{\pi}\sum_{m=1}^{p}m^{4}\int_{0}^{\pi}\mathrm{tr}Q(t)\,dt+\frac{2}{\pi}\sum_{m=1}^{p}m^{4}\int_{0}^{\pi}\mathrm{tr}Q(t)\cos 2mt\,dt. \tag{11}$$

By applying partial integration four times successively to the second integral in (11), we get (8). □

Theorem 2. *With the same hypothesis as in Theorem 1, we have the following equality:*

$$\begin{aligned}D_{p2}=&\frac{6p+7}{8\pi}\int_{0}^{\pi}\mathrm{tr}Q^{2}(t)\,dt+\frac{1}{\pi}\sum_{m=1}^{p}\int_{0}^{\pi}\mathrm{tr}Q^{2}(t)\,\cos 2mt\,dt\\&+\frac{2p+1}{8\pi^{2}}\mathrm{tr}\left(\int_{0}^{\pi}Q(t)dt\right)^{2}+\mathcal{O}(p^{-1}).\end{aligned} \tag{12}$$

Proof. Using (7) we have:

$$D_{p2}=\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\mathrm{tr}\left[(QR_{\lambda}^{0})^{2}\right]d\lambda=\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\langle(QR_{\lambda}^{0})^{2}\psi_{mn},\psi_{mn}\right\rangle d\lambda. \tag{13}$$

Since $QR_{\lambda}^{0}\psi_{mn}=(m^{4}-\lambda)^{-1}Q\psi_{mn}$ we obtain

$$(QR_{\lambda}^{0})^{2}\psi_{mn}=\frac{QR_{\lambda}^{0}}{m^{4}-\lambda}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\langle Q\psi_{mn},\psi_{rq}\rangle\psi_{rq}=\frac{1}{m^{4}-\lambda}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\frac{\langle Q\psi_{mn},\psi_{rq}\rangle\psi_{rq}}{r^{4}-\lambda}.$$

Inserting this expression into (13), we get:

$$D_{p2}=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left|\langle Q\psi_{mn},\psi_{rq}\rangle\right|^{2}\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{(\lambda-m^{4})(\lambda-r^{4})}d\lambda.$$

By separating the series according to m and r into four series and applying the Cauchy integral formula, we obtain:

$$D_{p2}=\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left|\langle Q\psi_{mn},\psi_{rq}\rangle\right|^{2}-\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left(1+\frac{2m^{4}}{r^{4}-m^{4}}\right)\left|\langle Q\psi_{mn},\psi_{rq}\rangle\right|^{2}.$$

Let

$$\alpha_{p}:=\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{r^{4}+m^{4}}{r^{4}-m^{4}}\left|\langle Q\psi_{mn},\psi_{rq}\rangle\right|^{2}. \tag{14}$$

Hence, we get:

$$D_{p2}=\sum_{m=0}^{p}\sum_{n=1}^{\infty}\|Q\psi_{mn}\|^{2}-\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p}\,. \tag{15}$$

For $m = \overline{0,p}$ and $r = \overline{p+1,\infty}$ we have:

$$\begin{aligned}\left|\langle Q\psi_{mn},\psi_{rq}\rangle\right|^2 &= \frac{d_m^2}{2\pi}\Big|\int_0^\pi \langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(m-r)t\,dt\Big|^2 \\ &+ \frac{d_m^2}{\pi}Re\Big[\int_0^\pi \langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(m-r)t\,dt.\int_0^\pi \overline{\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}}\cos(m+r)t\,dt\Big]^2 \\ &+ \frac{d_m^2}{2\pi}\Big|\int_0^\pi \langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(m+r)t\,dt\Big|^2.\end{aligned}$$

Hence, (14) becomes

$$\alpha_p = \alpha_{p1} + \alpha_{p2} + \alpha_{p3} \tag{16}$$

with

$$\alpha_{p1} = \frac{1}{2\pi}\sum_{m=0}^{p} d_m^2 \sum_{r=p+1}^{\infty}\frac{r^4+m^4}{r^4-m^4}\Big|\int_0^\pi \langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(m-r)t\,dt\Big|^2, \tag{17}$$

$$\begin{aligned}\alpha_{p2} = \frac{1}{\pi}\sum_{m=0}^{p} d_m^2 \sum_{r=p+1}^{\infty}\frac{r^4+m^4}{r^4-m^4}Re\Big[\int_0^\pi \langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(m-r)t\,dt \\ \times\int_0^\pi \overline{\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}}\cos(m+r)t\,dt\Big]^2,\end{aligned} \tag{18}$$

$$\alpha_{p3} = \frac{1}{2\pi}\sum_{m=0}^{p} d_m^2 \sum_{r=p+1}^{\infty}\frac{r^4+m^4}{r^4-m^4}\Big|\int_0^\pi \langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(m+r)t\,dt\Big|^2. \tag{19}$$

For any positive integers p and i, let $E = \{(r,m) : r,m \in \mathbb{N}; r-m = i, r > p\}$. Therefore, we rewrite (17) as

$$\begin{aligned}\alpha_{p1} &= \frac{1}{2\pi^2}\sum_{r=p+1}^{\infty}\Big|\int_0^\pi \langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos rt\,dt\Big|^2 \\ &+ \frac{1}{\pi^2}\sum_{i=1}^{\infty}\Big|\int_0^\pi \langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos it\,dt\Big|^2\sum_{m,r\in E}\frac{r^4+m^4}{r^4-m^4}.\end{aligned} \tag{20}$$

Consider the case $i \le p$ $(m \ge 1)$. First we get:

$$\begin{aligned}\sum_{m,r\in E}\frac{r^4+m^4}{r^4-m^4} &= \sum_{\substack{r-m=i\\ m\le p, r>p}}\Big(1+\frac{2m^4}{r^4-m^4}\Big) = i + 2\sum_{m,r\in E}\frac{m^4}{r^4-m^4} \\ &= i + 2\Big[\sum_{j=0}^{i-1}\frac{(p-1)^4}{(p-j+i)^4-(p-j)^4}\Big] \\ &= i + 2\Big[\sum_{j=0}^{i-1}\Big[\frac{p}{4i}-\frac{j}{4i}-\frac{3}{8}+\frac{i}{8[2(p-j)+i]}+\frac{i[2(p-j)+i]}{4[2(p-j)^2+2i(p-j)+i^2]}\Big]\Big] \\ &= i + 2\Big[\frac{p}{4}-\frac{1-4i}{8}+\frac{i}{8}\sum_{j=0}^{i-1}\frac{1}{2p-2j+i}+\frac{i}{4}\sum_{j=0}^{i-1}\frac{2(p-j)^2+i]}{2(p-j)^2+2i(p-j)+i^2}\Big] \\ &= \frac{p}{2}+\frac{1}{4}+\frac{i}{4}\sum_{j=0}^{i-1}\frac{1}{2p-j+i}+\frac{i}{2}\sum_{j=0}^{i-1}\frac{2(p-j)^2+i]}{2(p-j)^2+2i(p-j)+i^2}.\end{aligned}$$

Since

$$\frac{i}{4}\sum_{j=0}^{i-1}\frac{1}{2p-j+i} < \frac{i}{4}\sum_{j=0}^{i-1}\frac{1}{p+(p-j)+(i-j)} < \frac{i}{4}\sum_{j=0}^{i-1}\frac{1}{p} < \frac{i^2}{p}$$

and

$$\sum_{j=0}^{i-1} \frac{2(p-j)^2+i]}{2(p-j)^2+2i(p-j)+i^2} < \frac{i^2}{p}$$

we obtain

$$\sum_{m,r\in E} \frac{r^4+m^4}{r^4-m^4} = \frac{p}{2} + \frac{1}{4} + i^2\mathcal{O}(\frac{1}{p}), \tag{21}$$

where $0 < \mathcal{O}(\frac{1}{p}) < const\cdot \frac{1}{p}$.

Similarly, for $i > p$ with $m \geq 1$ we get:

$$\begin{aligned}\sum_{m,r\in E} \frac{r^4+m^4}{r^4-m^4} &= \sum_{\substack{r-m=i \\ m\leq p, r>p}} (1+\frac{2m^4}{r^4-m^4}) = i + 2\sum_{m,r\in E} \frac{m^4}{r^4-m^4} \\ &= p + 2\sum_{j=1}^{p} \frac{j^4}{(i+j)^4-j^4} \\ &= p + \frac{p(p+1)}{4i} - \frac{3p}{4} + \frac{i}{4}\sum_{j=1}^{p}\frac{1}{1+2j} + \frac{i}{2}\sum_{j=1}^{p}\frac{i+2j}{2j^2+2ij+i^2} \\ &< \frac{p}{2} + \frac{1}{4} + \frac{i}{4}\sum_{j=1}^{p}\frac{1}{i} + i\sum_{j=1}^{p}\frac{1}{i+j}\end{aligned}$$

or

$$\sum_{m,r\in E} \frac{r^4+m^4}{r^4-m^4} = \mathcal{O}(p), \tag{22}$$

where $\mathcal{O}(p)$ depends on p and i and

$$0 < \mathcal{O}(p) < const\cdot p\,. \tag{23}$$

From (20)–(22) we get

$$\alpha_{p1} = \alpha_{p1}^{(1)} + \alpha_{p1}^{(2)} + \alpha_{p1}^{(3)} \tag{24}$$

where

$$\begin{aligned}\alpha_{p1}^{(1)} &= \frac{1}{\pi^2}(\frac{p}{2}+\frac{1}{4})\sum_{i=1}^{\infty}\Big|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos it\,dt\Big|^2 \\ \alpha_{p1}^{(2)} &= \sum_{i=1}^{p} i^2\mathcal{O}(p^{-1})\Big|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos it\,dt\Big|^2 \\ \alpha_{p1}^{(3)} &= \sum_{i=p+1}^{\infty}\mathcal{O}(p)\Big|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos it\,dt\Big|^2.\end{aligned}$$

Clearly we get:

$$\begin{aligned}&\alpha_{p1}^{(1)} = \frac{1}{8\pi}(2p+1)\int_0^{\pi}\big|\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\big|^2dt - \frac{1}{8\pi^2}(2p+1)\Big|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}dt\Big|^2, \\ &0 < \alpha_{p1}^{(2)} < \frac{1}{\pi}\int_0^{\pi}\big|\langle Q'(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\big|^2dt\,, \\ &|\alpha_{p1}^{(3)}| < \frac{1}{\pi}\int_0^{\pi}\big|\langle Q'(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\big|^2dt\,.\end{aligned}$$

These relations give:

$$\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p1}=\frac{2p+1}{8\pi}\int_0^{\pi}\mathrm{tr}Q^2(t)-\frac{2p+1}{8\pi^2}\mathrm{tr}\Big[\int_0^{\pi}Q(t)\Big]^2+\mathcal{O}(p^{-1}). \tag{25}$$

Since

$$\Big|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(m-r)t\,dt\Big|^2\le\frac{1}{(m-r)^2}\Big[|\langle Q'(0)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|+|\langle Q'(\pi)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|+\int_0^{\pi}|\langle Q''(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|dt\Big]$$

we obtain:

$$\Big|\int_0^{\pi}\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}\cos(m-r)t\,dt\int_0^{\pi}\overline{\langle Q(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}}\cos(m+r)t\,dt\Big|\le$$
$$\frac{3}{(m-r)^2(m+r)^2}\Big[|\langle Q'(0)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|^2+|\langle Q'(\pi)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|^2+\pi\int_0^{\pi}|\langle Q''(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|^2dt\Big].$$

By (18) and the last inequality, we find:

$$|\alpha_{p2}|\le\frac{3}{\pi}\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{r^4+m^4}{(r+m)^2(r-m)^2(r^4-m^4)}\Big[|\langle Q'(0)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|^2+|\langle Q'(\pi)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|^2+\pi\int_0^{\pi}|\langle Q''(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|^2dt\Big].$$

Using (21)–(23) we get $\sum_{m,r\in E}\frac{r^4+m^4}{r^4-m^4}<\mathcal{O}(p)$. Therefore, we can show that:

$$\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{r^4+m^4}{(r+m)^2(r-m)^2(r^4-m^4)}\le\frac{1}{p^2}\sum_{i=1}^{\infty}\frac{1}{i^2}\sum_{m,r\in E}\frac{r^4+m^4}{r^4-m^4}<const\cdot p^{-1}.$$

By the last two inequalities we find:

$$\begin{aligned}\Big|\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p2}\Big|&\le\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\frac{const}{p}\Big[|\langle Q'(0)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|^2+|\langle Q'(\pi)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|^2\\&\quad+\int_0^{\pi}|\langle Q''(t)\varphi_n,\varphi_q\rangle_{\mathcal{H}}|^2dt\Big]\\&<\frac{const}{p}\Big[\sum_{n=1}^{\infty}\|Q'(0)\varphi_n\|_{\mathcal{H}}^2+\sum_{n=1}^{\infty}\|Q'(\pi)\varphi_n\|_{\mathcal{H}}^2+\int_0^{\pi}\sum_{n=1}^{\infty}\|Q''(t)\varphi_n\|_{\mathcal{H}}^2dt\Big]\\&=\frac{const}{p}\Big[\mathrm{tr}Q'(0)^2+\mathrm{tr}Q'(\pi)^2+\int_0^{\pi}\mathrm{tr}Q''(t)^2dt\Big]<\frac{const}{p}.\end{aligned} \tag{26}$$

The inequality

$$\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{r^4+m^4}{(r+m)^4(r^4-m^4)}<\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{r^4+m^4}{(r+m)^2(r-m)^2(r^4-m^4)}$$

gives

$$0\le\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p3}\le\frac{const}{p}. \tag{27}$$

Moreover, the first part of (15) becomes:

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty}\|Q\psi_{mn}\|^2=\frac{p+1}{\pi}\int_0^{\pi}\mathrm{tr}Q^2(t)\,dt+\frac{1}{\pi}\sum_{m=1}^{p}\int_0^{\pi}\mathrm{tr}Q^2(t)\cos 2mt\,dt$$

Thus, using (15), (16), (25)–(27) and this last equation, we obtain (12). □

3. The Second Regularized Trace Formula

In this section, we first compute D_{p3} and we show

$$\lim_{p\to\infty} D_{pj} = 0 \ (j \geq 4) \quad \text{and} \quad \lim_{p\to\infty} D_p^N = 0 \ (N \geq 4)$$

for $|\lambda| = (p^2+p+1)^2 - \frac{1}{2}$. Then we give the second regularized trace formula. Let us first recall that we have:

$$D_{pj} = \frac{(-1)^j}{\pi i j} \sum_{m_1=0}^{\infty}\sum_{n_1=0}^{\infty} \cdots \sum_{m_j=0}^{\infty}\sum_{n_j=0}^{\infty} * \Big[\int_{|\lambda|=b_p} \lambda \prod_{s=1}^{j} (m_s^4 - \lambda)^{-1} d\lambda \Big] . \prod_{s=1}^{j} \langle Q\psi_{m_s n_s}, \psi_{m_{g(s)} n_{g(s)}} \rangle$$

where the sign $*$indicates the existence of the numbers greater or less than b_p between $m_1^4, m_2^4, \ldots, m_j^4$, and

$$g(s) = \begin{cases} s+1 & \text{if } s < j \\ s & \text{if } s = j \end{cases}$$

Hence, we get:

$$D_{p3} = -\frac{1}{3\pi i} \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\sum_{s=0}^{\infty}\sum_{k=1}^{\infty} * \Big[\int_{|\lambda|=b_p} \frac{\lambda}{(m^4-\lambda)(r^4-\lambda)(s^4-\lambda)} d\lambda \Big] \times \langle Q\psi_{mn}, \psi_{rq} \rangle \langle Q\psi_{rq}, \psi_{sk} \rangle \langle Q\psi_{sk}, \psi_{mn} \rangle$$

which is absolutely convergent. Taking

$$F(m,r,s) := \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{k=1}^{\infty} \langle Q\psi_{mn}, \psi_{rq} \rangle \langle Q\psi_{rq}, \psi_{sk} \rangle \langle Q\psi_{sk}, \psi_{mn} \rangle,$$

$$G(m,r,s) := \int_{|\lambda|=b_p} \frac{\lambda}{(\lambda - m^4)(\lambda - r^4)(\lambda - s^4)} d\lambda$$

we rewrite D_{p3} as:

$$D_{p3} = -\frac{1}{3\pi i} \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} * G(m,r,s) F(m,r,s). \tag{28}$$

Moreover, the inner product in $\mathcal{H}_1$ says that $F(m,r,s)$ is given by:

$$\begin{aligned} F(m,r,s) = d_m^2 d_r^2 d_s^2 \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{k=1}^{\infty} \int_0^{\pi} \langle Q(t)\varphi_n, \varphi_q \rangle_{\mathcal{H}} \cos mt \ \cos rt \, dt \\ \times \int_0^{\pi} \langle Q(t)\varphi_n, \varphi_q \rangle_{\mathcal{H}} \cos rt \ \cos st \, dt \int_0^{\pi} \langle Q(t)\varphi_n, \varphi_q \rangle_{\mathcal{H}} \cos st \ \cos mt \, dt. \end{aligned} \tag{29}$$

We have:

$$F(m,r,s) = F(m,s,r) = F(r,s,m) = F(r,m,s) = F(s,m,r) = F(s,r,m).$$

Using the Cauchy integral formulae, (28) becomes:

$$\begin{aligned} D_{p3} = 4 \sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} F(m,r,s) - 2 \sum_{m=0}^{p}\sum_{s=p+1}^{\infty} \frac{s^4}{(s^4-m^4)^2} F(m,m,s) \\ + 2 \sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\sum_{s=p+1}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} F(m,r,s). \end{aligned}$$

Let us denote by

$$F_1(m,r,s) := \frac{d_m^2 d_r^2 d_s^2}{8} \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{k=1}^{\infty} \int_0^{\pi} \left\langle Q(t)\varphi_n, \varphi_q \right\rangle_{\mathcal{H}} \cos(m-r)t\, dt \times \int_0^{\pi} \left\langle Q(t)\varphi_n, \varphi_q \right\rangle_{\mathcal{H}} \cos(r-s)t\, dt \int_0^{\pi} \left\langle Q(t)\varphi_n, \varphi_q \right\rangle_{\mathcal{H}} \cos(s-m)t\, dt, \tag{30}$$

$$F_2(m,r,s) := F(m,r,s) - F_1(m,r,s) \tag{31}$$

and

$$A_i := \sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\sum_{s=p+1}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} F_i(m,r,s), \qquad (i=1,2) \tag{32}$$

$$B_i := \sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} F_i(m,r,s), \qquad (i=1,2) \tag{33}$$

$$C_i := \sum_{m=0}^{p}\sum_{s=p+1}^{\infty} \frac{s^4}{(s^4-m^4)^2} F_i(m,m,s). \qquad (i=1,2). \tag{34}$$

Then we get $D_{p3} = 2A_1 + 4B_1 - 2C_1 + 2A_2 + 4B_2 - 2C_2$. The relations

$$F_1(m,r,s) = F_1(m,s,r) = F_1(r,s,m) = F_1(r,m,s) = F_1(s,m,r) = F_1(s,r,m),$$
$$F_1(m,s,s) = \frac{d_s^2}{d_m^2} F_1(m,m,s) = F_1(m,m,s) \quad (m \geq 1)$$

replaced in A_1 give:

$$A_1 = 2\sum_{m=0}^{p}\sum_{\substack{r=p+1 \\ r>s}}^{\infty}\sum_{s=p+1}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} F_1(m,r,s) + \sum_{m=0}^{p}\sum_{\substack{s=p+1 \\ r=s}}^{\infty} \frac{m^4}{(m^4-s^4)^2} F_1(m,m,s). \tag{35}$$

Similarly, by (33) we first get

$$\begin{aligned} B_i &= \sum_{m=0}^{p}\sum_{\substack{r=0 \\ m<r}}^{p}\sum_{s=p+1}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} F_i(m,r,s) \\ &+ \sum_{m=0}^{p}\sum_{\substack{r=0 \\ m>r}}^{p}\sum_{s=p+1}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} F_i(m,r,s) \\ &= \sum_{m=0}^{p}\sum_{\substack{r=0 \\ m<r}}^{p}\sum_{s=p+1}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} F_i(m,r,s) \\ &+ \sum_{m=0}^{p}\sum_{\substack{r=0 \\ m<r}}^{p}\sum_{s=p+1}^{\infty} \frac{r^4}{(r^4-m^4)(r^4-s^4)} F_i(r,m,s) \end{aligned}$$

and we obtain:

$$B_i = -\sum_{m=0}^{p}\sum_{\substack{r=0 \\ m<r}}^{p}\sum_{s=p+1}^{\infty} \frac{s^4}{(s^4-m^4)(s^4-r^4)} F_i(m,r,s). \tag{36}$$

Thus, D_{p3} takes the form

$$D_{p3} = 4A' - 4B' - 2C' + 2A_2 + 4B_2 - 2C_2 \tag{37}$$

where

$$A' = \sum_{m=0}^{p} \sum_{\substack{r=p+1 \\ r>s}}^{\infty} \sum_{s=p+1}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} F_1(m,r,s), \tag{38}$$

$$B' = \sum_{m=0}^{p} \sum_{\substack{r=0 \\ m<r}}^{p} \sum_{s=p+1}^{\infty} \frac{s^4}{(s^4-m^4)(s^4-r^4)} F_1(m,r,s), \tag{39}$$

$$C' = \sum_{m=0}^{p} \sum_{s=p+1}^{\infty} \frac{1}{(s^4-m^4)} F_1(m,m,s). \tag{40}$$

For any integers p, i and j such that $i > j$ and $p \geq j$, let

$$E_1 = \{(m,r,s) : m,r,s \in \mathbb{N}\,; r-m = i\,; s-m = j\,; m \leq p\,; r,s > p\}.$$

Let us consider

$$\sum_{m=0}^{p} \sum_{\substack{\mathrm{r=p+1} \\ r>s}}^{\infty} \sum_{\mathrm{s=p+1}}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} |F_1(m,r,s)| < \infty$$

and

$$\begin{aligned} \gamma_{ij} = & \frac{1}{\pi^3} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} \sum_{k=1}^{\infty} \int_0^{\pi} \langle Q(t)\varphi_n, \varphi_q \rangle_{\mathcal{H}} \cos it\, dt \int_0^{\pi} \langle Q(t)\varphi_q, \varphi_k \rangle_{\mathcal{H}} \cos(i-j)t\, dt \\ & \times \int_0^{\pi} \langle Q(t)\varphi_k, \varphi_n \rangle_{\mathcal{H}} \cos jt\, dt\,. \end{aligned} \tag{41}$$

Then we rewrite A' as the following:

$$\begin{aligned} A' &= \sum_{i=2}^{\infty} \sum_{j=1}^{p} \Big[\Big(\sum_{m,r,s \in E_1} \frac{m^4}{(m^4-r^4)(m^4-s^4)} \Big) \gamma_{ij} \Big] \\ &\quad + \sum_{m=0}^{p} \sum_{\substack{r=p+1 \\ r>s,\, s-m>p}}^{\infty} \sum_{s=p+1}^{\infty} \frac{m^4}{(m^4-r^4)(m^4-s^4)} F_1(m,r,s) \\ &= A'^1 + A'^2. \end{aligned} \tag{42}$$

It is clear that

$$\sum_{m,r,s \in E_1} \frac{m^4}{(m^4-r^4)(m^4-s^4)} = \frac{1}{i\pi} \mathcal{O}(1)$$

where $|\mathcal{O}(1)| < const$ and, $\mathcal{O}(1)$ depends on p,i and j. Hence,

$$A'^1 = \sum_{i=2}^{\infty} \sum_{j=1}^{p} \left(\frac{1}{i\pi} \mathcal{O}(1) \right) \gamma_{ij}. \tag{43}$$

Put:

$$B'^1 = \sum_{m=0}^{p} \sum_{\substack{r=0 \\ m<s,\, s-m \leq p}}^{p} \sum_{s=p+1}^{\infty} \frac{s^4}{(s^4-m^4)(s^4-r^4)} F_1(m,r,s), \tag{44}$$

$$B'^2 = \sum_{m=0}^{p} \sum_{\substack{r=0 \\ m<s,\, s-m>p}}^{p} \sum_{s=p+1}^{\infty} \frac{s^4}{(s^4-m^4)(s^4-r^4)} F_1(m,r,s); \tag{45}$$

then (39) can be written shortly as

$$B' = B'^1 + B'^2. \tag{46}$$

By the fact

$$\sum_{m=0}^{p}\sum_{r=0}^{\infty}\sum_{s=p+1}^{\infty}\frac{s^4}{(s^4-m^4)(s^4-r^4)}|F_1(m,r,s)| < \infty$$

and by (30) and (41), we write (44) as:

$$B'^1 = \sum_{j=2}^{p}\sum_{i=1}^{p-1}\Big(\sum_{m,r,s\in E_2}\frac{s^4}{(s^4-m^4)(s^4-r^4)}\Big)\gamma_{ij}. \tag{47}$$

Here E_2 expresses a set for any integers p, i and j, providing $i < j \leq p$, that is,

$$E_2 = \{(m,r,s) : m,r,s \in \mathbb{N}\,; r-m = i\,; s-m = j\,; m,r \leq p\,; s > p\}.$$

This gives:

$$\sum_{m,r,s\in E_2}\frac{s^4}{(s^4-m^4)(s^4-r^4)} = \frac{1}{ij\pi}\mathcal{O}(1) \tag{48}$$

By (47) and (48), we get:

$$B'^1 = \sum_{\substack{\mathrm{j=1}\\ j>i}}^{\mathrm{p}}\sum_{\mathrm{i=1}}^{\mathrm{p\text{-}1}}\Big(\frac{1}{ij\pi}\mathcal{O}(1)\Big)\gamma_{ij}. \tag{49}$$

Since $\gamma_{ij} = \gamma_{ji}$ and, by (37), (40), (42), (43), (46) and (49), we obtain:

$$D_{p3} = 4\sum_{\substack{\mathrm{i=2}\\ i>j}}^{\infty}\sum_{\mathrm{j=1}}^{\mathrm{p}}\Big(\frac{\mathcal{O}(1)}{i\pi}\Big)\gamma_{ij} - 4\sum_{\substack{\mathrm{i=2}\\ i>j}}^{\mathrm{p}}\sum_{\mathrm{j=1}}^{\mathrm{p\text{-}1}}\Big(\frac{\mathcal{O}(1)}{ij\pi}\Big)\gamma_{ij} + 4A'^2 - 4B'^2 - 2C' + 2A_2 + 4B_2 - 2C_2.$$

Moreover, if $Q(t)$ has a continuous derivative of second order with respect to the norm in $\sigma_1(\mathcal{H})$ on the interval $[0,\pi]$, then $|\gamma_{ij}| \leq \frac{const}{i^2j^2}$. This implies that

$$\Big|\sum_{i=2}^{\infty}\sum_{j=1}^{p}\Big(\frac{\mathcal{O}(1)}{i\pi}\Big)\gamma_{ij}\Big| \leq \frac{1}{p}\sum_{i=2}^{\infty}\sum_{j=1}^{p}\Big(\frac{1}{i}\Big)|\gamma_{ij}| < const\frac{1}{p}\Big(\sum_{i=2}^{\infty}\frac{1}{i^3}\Big)\Big(\sum_{j=1}^{p}\frac{1}{j^2}\Big) < \mathcal{O}(\frac{1}{p}),$$

$$\Big|\sum_{i=2}^{p}\sum_{j=1}^{p-1}\Big(\frac{\mathcal{O}(p^{-1})}{ij}\Big)\gamma_{ij}\Big| \leq \frac{2}{p}\sum_{i=2}^{p}\sum_{j=1}^{p-1}\Big(\frac{1}{ij}\Big)|\gamma_{ij}| < const\frac{1}{p}\Big(\sum_{i=2}^{p}\frac{1}{i^3}\Big)\Big(\sum_{j=1}^{p-1}\frac{1}{j^3}\Big) < \mathcal{O}(\frac{1}{p}).$$

Hence, D_{p3} becomes

$$D_{p3} = 4A'^2 - 4B'^2 - 2C' + 2A_2 + 4B_2 - 2C_2 + \mathcal{O}(p^{-1}). \tag{50}$$

Now, we will show that

$$\lim_{p\to\infty}A'^2 = \lim_{p\to\infty}B'^2 = \lim_{p\to\infty}C' = \lim_{p\to\infty}A_2 = \lim_{p\to\infty}B_2 = \lim_{p\to\infty}C_2 = 0.$$

By (29)–(31), and using the partial integration we can show that for $m \neq r \neq s$, we have:

$$|F_2(m,r,s) \leq const\Big[\frac{1}{(m-r)^2(r+s)^2(s-m)^2} + \frac{1}{(m+r)^2(r-s)^2(s-m)^2} + \frac{1}{(m-r)^2(r-s)^2(s+m)^2}\Big], \tag{51}$$

or

$$|F_2(m,r,s) \leq const\Big[\frac{1}{(m-r)^2(r+s)^2(s-m)^2} + \frac{1}{(m+r)^2(s-m)^2} + \frac{1}{(m-r)^2(s+m)^2}\Big], \tag{52}$$

or

$$|F_2(m,r,s) \leq const\Big[\frac{1}{(s+m)^2(s-m)^2} + \frac{1}{(s-m)^4}\Big]. \tag{53}$$

Moreover, from (30), (35) and (41), we obtain:

$$|F_1(m,r,s)| \leq const\frac{1}{(r-m)^2(s-m)^2} \qquad \text{for } m \neq r \text{ and } m \neq s\,, \tag{54}$$

$$|F_1(m,r,s)| \leq const\frac{1}{(s-m)^2(s-r)^2} \qquad \text{for } m \neq s \text{ and } r \neq s. \tag{55}$$

From (42) and (54) we get

$$|A'^2| \leq const\cdot p^{-5/2}\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s^4-m^4)} < const\cdot p^{-4}.$$

Additionally, the formulae (45) and (55) give:

$$|B'^2| \leq const\frac{(p+1)^2}{p^5}\sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-p)^3)} < const\cdot p^{-3}.$$

By using (32) and (52), we get the restricted version for A_2 as:

$$|A_2| < 4p^{-3}\sum_{r=1}^{\infty}\frac{1}{r^3}\sum_{s=1}^{\infty}\frac{1}{s^3} + 2p^{-2}\sum_{r=p+1}^{\infty}\frac{1}{(r^2-p^2)}\sum_{s=1}^{\infty}\frac{1}{s^3} < const(p^{-3}+p^{-5/2}).$$

The last three inequalities give:

$$\lim_{p\to\infty}A'^2 = 0\,; \quad \lim_{p\to\infty}B'^2 = 0\,; \quad \lim_{p\to\infty}A_2 = 0. \tag{56}$$

Using (36) and (51), we write $|B_2| \leq const(B_{21}+B_{22}+B_{23})$ where

$$B_{21} = \sum_{m=0}^{p}\sum_{\substack{r=0\\m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{s^4}{(s^4-m^4)(s^4-r^4)(m-r)^2(r+s)^2(s-m)^2},$$

$$B_{22} = \sum_{m=0}^{p}\sum_{\substack{r=0\\m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{s^4}{(s^4-m^4)(s^4-r^4)(m+r)^2(r-s)^2(s-m)^2},$$

$$B_{23} = \sum_{m=0}^{p}\sum_{\substack{r=0\\m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{s^4}{(s^4-m^4)(s^4-r^4)(m-r)^2(r-s)^2(s+m)^2}$$

with their respective restrictions:

$$B_{21} < \sum_{m=0}^{p}\sum_{\substack{r=0\\m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{1}{s^4(s-r)^2(s-m)^2} < \Big[\sum_{m=0}^{p}\frac{1}{(p+1-m)^2}\Big]^2\sum_{s=p+1}^{\infty}\frac{1}{s^4} < \frac{const}{p^3},$$

$$\begin{aligned}B_{22} &< \frac{1}{4p^2}\sum_{m=0}^{2p}\sum_{\substack{r=0\\m<r}}^{2p}\sum_{s=p+1}^{\infty}\frac{1}{(m+r)^2(s-r)^3(s-m)^3}\\ &\le \frac{1}{4p^2}\Big[\sum_{m=0}^{p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-m)^3}\sum_{r=0}^{2p}\frac{1}{(s-r)^3}+\sum_{m=p+1}^{2p}\sum_{r=0}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-r)^6(2m)^2}\Big]\\ &\le \frac{1}{4p^2}\Big[\frac{2(p+1)^2}{p^3}\sum_{s=1}^{\infty}\frac{1}{s^3}+\frac{1}{4p}\Big(\sum_{r=0}^{2p}\frac{1}{(2p+1-r)^6}+\sum_{r=0}^{2p}\int_{2p+1}^{\infty}\frac{1}{(t-r)^6}dt\Big)\Big]\\ &< \frac{const}{p^3}\end{aligned}$$

and

$$\begin{aligned}B_{23} &\le \sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)(s-r)^3(s+m)^4}\\ &< \sum_{m=0}^{p}\frac{1}{(p+1-m)}\sum_{r=0}^{p}\frac{1}{(p+1-r)^3}\sum_{s=p+1}^{\infty}\frac{1}{s^4}\\ &< const\Big(1+\int_1^{p+1}\frac{1}{t}dt\Big)\Big(\int_p^{\infty}\frac{1}{t^4}dt\Big)\\ &= const\frac{1+ln(p+1)}{p^3}.\end{aligned}$$

Hence, we get:

$$\lim_{p\to\infty} B_2 = 0. \tag{57}$$

Now, by using (34) and (53) we restrict C_2 as

$$|C_2| \le const(C_{21}+C_{22})$$

where

$$\begin{aligned}C_{21} &= \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{s^4}{(s^4-m^4)^2(s+m)^2(s-m)^2},\\ C_{22} &= \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{s^4}{(s^4-m^4)^2(s-m)^2}.\end{aligned}$$

Since

$$\begin{aligned}C_{21} &\le \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^4(s+m)^4} < \Big(\sum_{m=0}^{p}\frac{1}{(p+1-m)^4}\Big)\Big(\sum_{s=p+1}^{\infty}\frac{1}{s^4}\Big) < \frac{const}{p^3},\\ C_{22} &\le \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^6(s+m)^2} < \frac{1}{p^2}\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-p)^6} < \frac{const}{p}\end{aligned}$$

we obtain

$$\lim_{p\to\infty} C_2 = 0. \tag{58}$$

By (40) and (55) we get:

$$\begin{aligned}|C'| &\le const \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s^4-m^4)(s-m)^4}\\ &< const\Big(\sum_{m=0}^{p}\frac{1}{(p+1-m)^4}\Big)\Big(\sum_{s=p+1}^{\infty}\frac{1}{(s^4-p^4)}\Big)\\ &< \frac{const}{p^{5/2}}.\end{aligned}$$

This implies

$$\lim_{p\to\infty} C' = 0. \tag{59}$$

The conclusion from (50) and (56)–(59) is $\lim_{p\to\infty} D_{p3} = 0$.

Now we claim that $\lim_{p\to\infty} D_{pj} = 0$ for $j \ge 4$: It is easy to show that, for $|\lambda| = b_p = (p^2+p+1)^2 - 1/2$ $(p = 1, 2, \dots)$, there exists a constant c satisfying the following inequalities:

$$\|QR_\lambda^0\|_{\sigma_1(\mathcal{H}_1)} < c, \quad \|R_\lambda^0\| < p^{-3}, \quad \|R_\lambda\| < cp^{-3}. \tag{60}$$

Note that

$$|D_{pj}| = \frac{1}{\pi j}\Big|\int_{|\lambda|=b_p}\lambda \mathrm{tr}[(QR_\lambda^0)^j]d\lambda\Big| \le \frac{b_p}{\pi j}\int_{|\lambda|=b_p}\|QR_\lambda^0\|_{\sigma_1(\mathcal{H}_1)}\|(QR_\lambda^0)^{j-1}\|\,|d\lambda|.$$

Therefore, by the fact that $Q(t)$ satisfies the condition (ii), we obtain

$$|D_{pj}| \le \frac{cb_p}{\pi j}\int_{|\lambda|=b_p}\|Q\|^{j-1}\|R_\lambda^0\|^{j-1}|d\lambda| < \frac{cb_p}{\pi j}\int_{|\lambda|=b_p}(\frac{1}{2})^{j-1}p^{3(1-j)}|d\lambda| < \frac{c}{p^{3j-11}}.$$

This proves our claim. Thus, we have:

$$\lim_{p\to\infty} D_{pj} = 0 \quad (j \ge 3). \tag{61}$$

Finally, by (6) and by the inequalities in (60), we get

$$\begin{aligned}|D_p^{(N)}| &= \frac{1}{2\pi}\Big|\int_{|\lambda|=b_p}\lambda^2 \mathrm{tr}\big[R_\lambda(QR_\lambda^0)^{N+1}d\lambda\big]\Big|\\ &\le \frac{{b_p}^2}{2\pi}\int_{|\lambda|=b_p}\|R_\lambda(QR_\lambda^0)^{N+1}\|_{\sigma_1(\mathcal{H}_1)}|d\lambda|\\ &\le \frac{c{b_p}^2}{2\pi p^3}\int_{|\lambda|=b_p}\|QR_\lambda^0\|^N\|QR_\lambda^0\|_{\sigma_1(\mathcal{H}_1)}|d\lambda|\\ &\le \frac{c}{p^{3N-9}}.\end{aligned}$$

Hence, we have:

$$\lim_{p\to\infty} D_p^{(N)} = 0 \quad (N \ge 4). \tag{62}$$

Now, we are ready to announce the second regularized trace formula of L.

Theorem 3. *If operator function $Q(t)$ satisfies the conditions (i)–(iv), then the following formula is satisfied:*

$$\sum_{m=0}^{\infty}\Big(\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^8)-\frac{2m^4}{\pi}\int_0^{\pi}\mathrm{tr}Q(t)\,dt-\frac{m^2}{2\pi}\big(\mathrm{tr}Q'(\pi)-\mathrm{tr}Q'(0)\big)+2C\Big)$$
$$=C+\frac{1}{32}\big[\mathrm{tr}Q^{iv}(0)+\mathrm{tr}Q^{iv}(\pi)\big]+\frac{1}{4}\big[\mathrm{tr}Q^2(0)+\mathrm{tr}Q^2(\pi)\big], \tag{63}$$

where

$$C=\frac{1}{16\pi}\big(\mathrm{tr}Q'''(\pi)-\mathrm{tr}Q'''(0)\big)-\frac{3}{8\pi}\int_0^{\pi}\mathrm{tr}Q^2(t)\,dt-\frac{1}{8\pi^2}\mathrm{tr}\Big[\int_0^{\pi}Q(t)\,dt\Big]^2.$$

Proof. By (4), (8), (12), (61) and (62), and by taking $h(t)=\mathrm{tr}Q^{iv}(t)+8\mathrm{tr}Q^2(t)$ we obtain

$$\sum_{m=0}^{\infty}\Big(\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^8)-\frac{2m^4}{\pi}\int_0^{\pi}\mathrm{tr}Q(t)\,dt-\frac{m^2}{2\pi}\big(\mathrm{tr}Q'(\pi)-\mathrm{tr}Q'(0)\big)$$
$$-\frac{1}{8\pi}\big(\mathrm{tr}Q'''(0)-\mathrm{tr}Q'''(\pi)\big)-\frac{3}{4\pi}\int_0^{\pi}\mathrm{tr}Q^2(t)\,dt-\frac{1}{4\pi^2}\mathrm{tr}\Big[\int_0^{\pi}Q(t)\,dt\Big]^2\Big)$$
$$=\frac{1}{8\pi}\int_0^{\pi}\mathrm{tr}Q^2(t)\,dt+\frac{1}{8\pi}\sum_{m=1}^{\infty}\int_0^{\pi}h(t)\cos 2mt\,dt-\frac{1}{8\pi^2}\mathrm{tr}\Big[\int_0^{\pi}Q(t)\,dt\Big]^2$$
$$-\frac{1}{8\pi}\big(\mathrm{tr}Q'''(0)-\mathrm{tr}Q'''(\pi)\big) \tag{64}$$

when $p\to\infty$. Note that

$$\frac{1}{8\pi}\sum_{m=1}^{\infty}\int_0^{\pi}h(t)\cos 2mt\,dt=\frac{1}{16\pi}\sum_{m=1}^{\infty}\Big[\int_0^{\pi}h(t)\cos 2mt\,dt+(-1)^m\int_0^{\pi}h(t)\cos 2mt\,dt\Big]$$
$$=\frac{1}{32}\Big(\sum_{m=1}^{\infty}\Big[\frac{2}{\pi}\int_0^{\pi}h(t)\cos mt\,dt\Big]\cos m0+\frac{1}{\pi}\int_0^{\pi}h(t)\,dt\,\cos 0\Big)$$
$$+\frac{1}{32}\Big(\sum_{m=1}^{\infty}\Big[\frac{2}{\pi}\int_0^{\pi}h(t)\cos mt\,dt\Big]cosm\pi+\frac{1}{\pi}\int_0^{\pi}h(t)\,dt\,\cos 0\pi\Big)-\frac{1}{16\pi}\int_0^{\pi}h(t)\,dt$$
$$=\frac{1}{32}[h(0)+h(\pi)]-\frac{1}{16\pi}\big(\mathrm{tr}Q'''(\pi)-\mathrm{tr}Q'''(0)\big)-\frac{1}{2\pi}\int_0^{\pi}\mathrm{tr}Q^2(t)\,dt. \tag{65}$$

Thus, from (64) and (65) we find:

$$\sum_{m=0}^{\infty}\Big(\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^8)-\frac{2m^4}{\pi}\int_0^{\pi}\mathrm{tr}Q(t)\,dt-\frac{m^2}{2\pi}\big(\mathrm{tr}Q'(\pi)-\mathrm{tr}Q'(0)\big)$$
$$+\frac{1}{8\pi}\big(\mathrm{tr}Q'''(\pi)-\mathrm{tr}Q'''(0)\big)-\frac{3}{4\pi}\int_0^{\pi}\mathrm{tr}Q^2(t)\,dt-\frac{1}{4\pi^2}\mathrm{tr}\Big[\int_0^{\pi}Q(t)\,dt\Big]^2\Big)$$
$$=\frac{1}{16\pi}\big(\mathrm{tr}Q'''(\pi)-\mathrm{tr}Q'''(0)\big)-\frac{3}{8\pi}\int_0^{\pi}\mathrm{tr}Q^2\,dt-\frac{1}{8\pi^2}\mathrm{tr}\Big[\int_0^{\pi}Q(t)\,dt\Big]^2$$
$$+\frac{1}{32}\big[\mathrm{tr}Q^{iv}(0)+\mathrm{tr}Q^{iv}(\pi)\big]+\frac{1}{4}\big[\mathrm{tr}Q^2(0)+\mathrm{tr}Q^2(\pi)\big].$$

Denoting by

$$C=\frac{1}{16\pi}\big(\mathrm{tr}Q'''(\pi)-\mathrm{tr}Q'''(0)\big)-\frac{3}{8\pi}\int_0^{\pi}\mathrm{tr}Q^2(t)\,dt-\frac{1}{8\pi^2}\mathrm{tr}\Big[\int_0^{\pi}\mathrm{tr}Q(t)\,dt\Big]^2$$

we obtain (63), which is the formula for the second regularized trace of the operator L. □

Example

Take $\mathcal{H}_1 = L^2(\mathcal{H};[0,\pi])$ where $\mathcal{H}$ is a separable Hilbert space. Consider the operator function $Q(t) = (2\pi)^{-1}tT$ ($t \in [0,\pi]$) where, for every $u \in \mathcal{H}$, $T : \mathcal{H} \longrightarrow \mathcal{H}$ is given by

$$Tu = \sum_{i=1}^{\infty} i^{-2}\langle u, \phi_i\rangle_{\mathcal{H}}\phi_i$$

with the orthonormal basis $\{\phi_i\}_{i\geq 1}$ in $\mathcal{H}$. Here $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ is the inner product on $\mathcal{H}$. We first show that, for every $t \in [0,\pi]$ the operator function $Q(t)$ is a trace-class (kernel) operator on $\mathcal{H}$. To understand this, it is enough to see that T is a trace-class operator: For every $u, v \in \mathcal{H}$ we have

$$\begin{aligned}\langle Tu, v\rangle_{\mathcal{H}} &= \Big\langle \sum_{i=1}^{\infty} i^{-2}\langle u, \phi_i\rangle_{\mathcal{H}}\phi_i, v\Big\rangle_{\mathcal{H}} = \sum_{i=1}^{\infty} i^{-2}\langle u, \phi_i\rangle_{\mathcal{H}}\langle \phi_i, v\rangle_{\mathcal{H}} \\ &= \sum_{i=1}^{\infty}\langle u, i^{-2}, \langle v, \phi_i\rangle_{\mathcal{H}}\phi_i\rangle_{\mathcal{H}} = \Big\langle u, \sum_{i=1}^{\infty} i^{-2}\langle v, \phi_i\rangle_{\mathcal{H}}\phi_i\Big\rangle_{\mathcal{H}} \\ &= \langle u, Tv\rangle_{\mathcal{H}}.\end{aligned}$$

Since T has eigenvalues $\lambda_i = s_i = i^{-2}$ ($i = 1,2,3,\cdots$), called s-numbers, T is also a trace-class operator. It is also easy to see that $Q'(t) = (2\pi)^{-1}T$ and $Q^{(i)}(t) = 0$ for $i \geq 2$ with respect to the norm of $\sigma_1(\mathcal{H})$. This implies the self-adjointness of $Q^{(i)}(t)$ for $i = 0,1,2,\cdots$; that is, we have

$$[Q^{(i)}(t)]^* = Q^{(i)}(t)\ ; \quad (i = 0,1,2,\cdots).$$

Here, we also notice that Q is a self-adjoint, trace-class operator from $\mathcal{H}_1$ to $\mathcal{H}_1$.

On the other hand, since $\|T\|_{\mathcal{H}} = 1$ we find $\|Q\| < \frac{1}{2}$; and since $\|Q(t)\phi_n\| \leq \sqrt{\pi}n^{-2}$ we get

$$\sum_{i=1}^{\infty}\|Q(t)\phi_n\| < \infty.$$

4. Conclusions

We introduced and computed a new second regularized trace formula for a fourth-order differential operator with a bounded operator coefficient defined on a separable Hilbert space. This formula can be generalized to an even-order differential operator through the techniques we used here. On the other hand, the regularized trace can be also computed on a separable Banach space, which is a continuous dense embedding in a separable Hilbert space [23]. The trace formulae of these operators are used in many branches of mathematics, mathematical physics and quantum mechanics. For example, the resonant frequencies of the rotating turbine blade can be determined using fourth-order differential operators with the operator coefficients. The quantum mechanics of particles in the wave mechanical formulation cannot be completely represented by a wave-like structure. For example, electron spin degrees of freedom do not imply the action of a gradient operator. Therefore, it is useful to reformulate quantum mechanics in a framework that only includes differential operators.

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