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# Optimizing Network Connectivity via Fuzzy Hub Domination

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**Abstract**
This work studies hub domination and total hub domination in both standard and fuzzy graphs. We determine exact values for principal graph families (paths, cycles, complete graphs, complete bipartite graphs, and wheels) and develop structural bounds that relate hub parameters to classical invariants. On the fuzzy side, we formalize hub domination with vertex and edge memberships, compute the fuzzy hub domination number for standard fuzzy graph classes, and derive degree- and structure-based bounds. We connect these results to applications in telecommunication and transportation networks, where minimizing hub cost aligns with the fuzzy hub domination objective. The findings clarify the mathematical structure of hub domination, situate it among core graph-theoretic measures, and provide implementable bounds and algorithms for design under uncertainty. These contributions offer tools for cost-aware hub placement and resilient connectivity in complex networks modeled by fuzzy graphs.

**Keywords** Graph domination · Hub domination · Fuzzy graphs · Hub number · Domination parameters · Bounds · Algorithms · Network applications · Connectivity

## 1 Introduction

Graph domination is a central topic in graph theory with broad applications [2, 3]. Building on this tradition, Walsh [4] introduced *hub numbers*, shifting attention from adjacency to connectivity mediated by a selected set of hub vertices. In parallel, fuzzy graph theory—pioneered by Rosenfeld [5] from Zadeh's fuzzy sets [6, 7]—provides a framework for modeling uncertainty via vertex and edge memberships. While domination in fuzzy graphs has been explored [8, 23], hub-based notions in fuzzy settings remain relatively recent [9–12].

Efficient hub-based connectivity under uncertainty is crucial in transportation, logistics, and telecommunication. We address the lack of a unified treatment of hub domination across standard and fuzzy graphs by (i) defining and analyzing hub and total hub domination for major graph families, (ii) deriving degree- and structure-based bounds, and (iii) relating these parameters to classical invariants and to real-world design criteria. The novelty lies in extending hub domination to fuzzy graphs with explicit cost/quality weights and in consolidating results across families with streamlined proofs.

In contrast to connected or independent domination, hub domination targets paths whose *internal* vertices are hubs, emphasizing relay-style connectivity. This makes hub domination complementary to classical invariants (domination, independence, chromatic, etc.) and to fuzzy topological indices (e.g., Wiener, Gutman, and Zagreb-type measures) [19, 29–36].

Throughout, $\chi(\mathfrak{Z})$ denoted as the chromatic number. All symbols ($h_t$, $\gamma_c$, $\Delta_N$, $\Delta_E$, $\delta_N$, $\delta_E$, $\alpha_0$, $\alpha$, $\beta_0$, $\beta$) are defined in the preliminaries; degree parameters are distinguished for neighborhoods ($\cdot_N$) and edges ($\cdot_E$).

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**Contributions**

- *Unified exact values.* Consolidated theorems giving $\gamma_h$ and $\gamma_{ht}$ for paths, cycles, complete graphs, complete bipartite graphs, wheels, and stars (standard and fuzzy).
- *Bounds and relations.* Degree- and structure-based bounds linking hub parameters to $\gamma$, independence-type numbers, and (in the fuzzy case) membership-weighted analogs.
- *Connected variants.* Inequalities tying connected hub domination to connected domination, clarifying when hub sets form connected cores.
- *Algorithms.* Greedy heuristics for minimal hub dominating sets in crisp and fuzzy graphs, with flowcharts and minimality post-processing.
- *Applications.* Case studies in telecommunication fraud analysis and regional transportation planning, with parameter tables/figures and design guidance.

### Methodological Approaches and Gaps

Existing methodologies address definitions, parameter computation for specific families, bounds relative to invariants, and algorithms for minimal hub sets. Open directions include dynamic and directed models, scalable heuristics and approximations for large fuzzy networks, and extended application domains (e.g., social networks, supply chains, and critical infrastructure). Our consolidated theorems and bounds serve as building blocks for these developments.

## 2 Preliminaries and Notation

A (crisp) graph is denoted by $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the vertex set with $|\mathcal{V}| = p$ and $\mathcal{E} \subseteq \{\{u, v\} : u, v \in \mathcal{V},\ u \neq v\}$ the edge set (see, e.g., [13–15]). For $v \in \mathcal{V}$, the open and closed neighborhoods are

$$N(v) = \{u \in \mathcal{V} : \{u, v\} \in \mathcal{E}\}, \qquad N[v] = N(v) \cup \{v\}.$$

We write the (neighborhood) degree as $d_N(v) = |N(v)|$, and define

$$\Delta_N(\mathfrak{Z}) = \max_{v \in \mathcal{V}} d_N(v), \qquad \delta_N(\mathfrak{Z}) = \min_{v \in \mathcal{V}} d_N(v).$$

For edge–weighted counts, we use the conventional edge–incidence degree $d_E(v) = |\{e \in \mathcal{E} : v \in e\}| = d_N(v)$ in crisp graphs; the symbols $\Delta_E(\mathfrak{Z})$ and $\delta_E(\mathfrak{Z})$ are included here for parallelism with the fuzzy setting (below). A *dominating set* is $\mathfrak{D} \subseteq \mathcal{V}$, such that every $u \in \mathcal{V} \setminus \mathfrak{D}$ has a neighbor in $\mathfrak{D}$. The *domination number* is $\gamma(\mathfrak{Z}) = \min\{|\mathfrak{D}| : \mathfrak{D} \text{ dominates } \mathfrak{Z}\}$ (see [2, 3]). The *connected domination number* is

$$\gamma_c(\mathfrak{Z}) = \min\{|\mathfrak{D}| : \mathfrak{D} \text{ dominates } \mathfrak{Z} \text{ and } \langle \mathfrak{D} \rangle \text{ is connected}\} \quad [2].$$

The chromatic number is denoted by $\chi(\mathfrak{Z})$ and defined as follows.

Let $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$ be a simple, undirected graph. A *proper vertex k-coloring* is a function

$$c : \mathcal{V} \to \{1, 2, \ldots, k\},$$

such that for every edge $\{u, v\} \in E$, we have $c(u) \neq c(v)$.

The *chromatic number* of $\mathfrak{Z}$ is the smallest integer $k$ for which such a proper $k$-coloring exists

$$\chi(\mathfrak{Z}) = \min\left\{k \in \mathbb{Z}^+ \;\middle|\; \text{there exists a proper } k-\text{coloring of } \mathfrak{Z}\right\}.$$

A set $H \subseteq \mathcal{V}$ is a *hub set* if, for any non-adjacent $u, v \in \mathcal{V} \setminus H$, there is a $u$–$v$ path whose internal vertices all lie in $H$; the *hub number* is

$$h(\mathfrak{Z}) = \min\{|H| : H \text{ is a hub set of } \mathfrak{Z}\} \quad [4].$$

The *total hub set* requires the above condition for all pairs $u, v \in \mathcal{V} \setminus H$ (adjacent or not); the *total hub number* is $h_t(\mathfrak{Z}) = \min\{|H| : H \text{ is a total hub set}\}$ (cf. the fuzzy analog in [11]).

**Independence and vertex cover** An *independent set* is a set of pairwise non-adjacent vertices; a *vertex cover* is a set $C \subseteq \mathcal{V}$ meeting every edge. For consistency with later sections, we use

$$\beta_0(\mathfrak{Z}) = \max\{|S| : S \subseteq \mathcal{V} \text{ independent}\} \quad \text{(classical independence number)},$$
$$\alpha_0(\mathfrak{Z}) = \min\{|C| : C \subseteq \mathcal{V} \text{ is a vertex cover}\} \quad \text{(classical vertex cover number)}.$$

(Background: [13]; relations among domination/independence/cover: [26, 27].)

A *fuzzy graph* is $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$ with vertex membership $\mu : \mathcal{V} \to (0, 1]$ and symmetric edge membership $\rho : \mathcal{V} \times \mathcal{V} \to [0, 1]$ satisfying $\rho(u, v) \le \min\{\mu(u), \mu(v)\}$ and $\rho(u, u) = 0$ (Rosenfeld's model [5], based on fuzzy sets [6]) and fuzzy graph [17, 18, 20, 21]). The *support graph* $\text{supp}(\mathfrak{Z})$ has vertex set $\mathcal{V}$ and edge $uv$ whenever $\rho(u, v) > 0$ (see also [16]). For $v \in \mathcal{V}$, we define

$$d_N^\rho(v) = |\{u \in \mathcal{V} : \rho(u, v) > 0\}|, \quad \Delta_N(\mathfrak{Z}) = \max_v d_N^\rho(v), \quad \delta_N(\mathfrak{Z}) = \min_v d_N^\rho(v),$$

and the (edge–weighted) fuzzy degree

$$d_E^\rho(v) = \sum_{u \in \mathcal{V}\setminus\{v\}} \rho(v, u), \quad \Delta_E(\mathfrak{Z}) = \max_v d_E^\rho(v), \quad \delta_E(\mathfrak{Z}) = \min_v d_E^\rho(v).$$

For a set $S \subseteq \mathcal{V}$, its *fuzzy cardinality* (weight) is $\|S\|_\mu = \sum_{v \in S} \mu(v)$ [22].

**Fuzzy independence and cover** A set $S \subseteq \mathcal{V}$ is fuzzy-independent if $\rho(u, v) = 0$ for all distinct $u, v \in S$ (cf. [16, 24, 25, 28]). To mirror the symbols used in Sections 2–3, we set

$$\beta(\mathfrak{Z}) = \max\left\{\sum_{v \in S} \mu(v) : S \subseteq \mathcal{V} \text{ fuzzy-independent}\right\} \quad \text{(fuzzy independence number)},$$
$$\alpha(\mathfrak{Z}) = \min\left\{\sum_{v \in C} \mu(v) : C \subseteq \mathcal{V} \text{ fuzzy vertex cover}\right\} \quad \text{(fuzzy vertex cover number)}.$$

Here, $C$ is a fuzzy vertex cover if for every pair $u, v$ with $\rho(u, v) > 0$ we have $\{u, v\} \cap C \neq \emptyset$ (cf. [16, 23]).

**Fuzzy hub parameters** A *(fuzzy) hub dominating set* and the *fuzzy hub domination number* $\gamma_h(\mathfrak{Z})$ are defined in **Definition** 3.2 via fuzzy cardinality $\|\cdot\|_\mu$ (see also hub-number analogs in [10]). The *connected fuzzy hub domination number* is $\gamma_{hc}(\mathfrak{Z})$ (**Definition** 3.17); the *total fuzzy hub domination number* is $\gamma_{ht}(\mathfrak{Z})$ (**Definition** 3.26).

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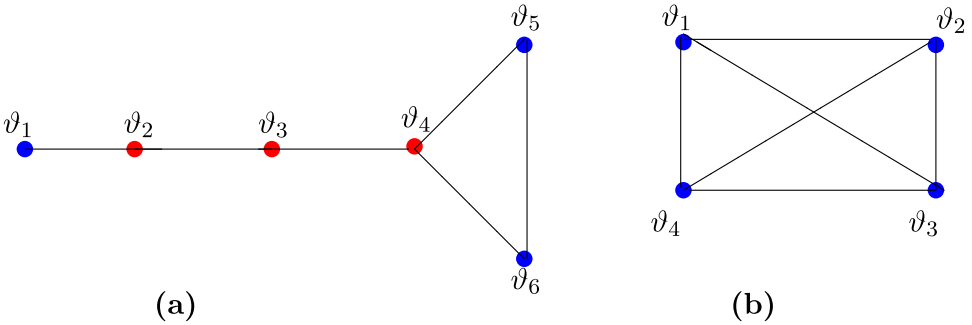


**Fig. 1** **a** Example illustrating Definition 3.2: vertices $\vartheta_2, \vartheta_3, \vartheta_4$ form a minimal hub dominating set with $\gamma_h(\mathfrak{Z}) = 3$. **b** The complete graph $K_4$ is shown for comparison, where $\gamma_h(K_4) = 0$, since all vertices are mutually adjacent

## 3 The Hub Domination Number of a Graph

Having laid the groundwork in the introduction, we now proceed to formally introduce and study the concept of the hub domination number of graphs.
To provide context and lay the foundation for our investigation, we cite a few relevant theorems from the existing literature that will be instrumental in our subsequent analysis.

**Theorem 3.1** *[2, 3, 8, 10]*

*(i) For the complete graph $K_p$, $\chi(K_p) = p$.*
*(ii) For any complete graph $K_p$, $\gamma_h(K_p) = 0$.*
*(iii) For any wheel graph $W_p$, $\gamma_h(W_p) = 1$.*
*(iv) For any cycle fuzzy graph $C_p$, the total hub domination number satisfies*

$$\gamma_{ht}(C_p) = \min\left\{\sum_{i=1}^{p-2} \mu(v_i)\right\},$$

*where $\mu$ denotes the vertex membership function.*

**Definition 3.2** [1, 10, 12] Let $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$ be a graph. (i) A dominating set $\mathfrak{D} \subseteq \mathcal{V}$ is a **hub dominating set** if, for every pair of non-adjacent vertices $\omega, \vartheta \notin \mathfrak{D}$, there exists an $\omega$-$\vartheta$ path whose internal vertices all lie in $\mathfrak{D}$. (ii) $\mathfrak{D}$ is *minimal* if $\mathfrak{D}\backslash\{\vartheta\}$ fails to be a hub dominating set for every $\vartheta \in \mathfrak{D}$. (iii) The hub domination number $\gamma_h(\mathfrak{Z})$ is the minimum cardinality among all minimal hub dominating sets in $\mathfrak{Z}$.

***Example 3.3*** To illustrate this definition, consider the graph $\mathfrak{Z}$ in Fig. 1a.
The set $\mathfrak{D} = \{\vartheta_2, \vartheta_3, \vartheta_4\}$ is a minimal hub dominating set; hence, $\gamma_h(\mathfrak{Z}) = 3$.

Having defined the hub domination number and provided an illustrative example, we now determine $\gamma_h$ for several standard graph families.

**Theorem 3.4** *For a graph $\mathfrak{Z}$ of order $p$, the hub domination number satisfies*

*(i) $\gamma_h(K_p) = 0$,*
*(ii) $\gamma_h(C_p) = 0$,*
*(iii) $\gamma_h(S_p) = 1$,*
*(iv) $\gamma_h(W_p) = 1$.*

***Proof*** (i) In $K_p$, there are no non-adjacent pairs in $\mathcal{V} \setminus \mathfrak{D}$; the hub condition is vacuous for $\mathfrak{D} = \emptyset$, so $\gamma_h(K_p) = 0$. (ii) For $C_p$, the hub domination value follows from the established cycle case in our consolidated results (see

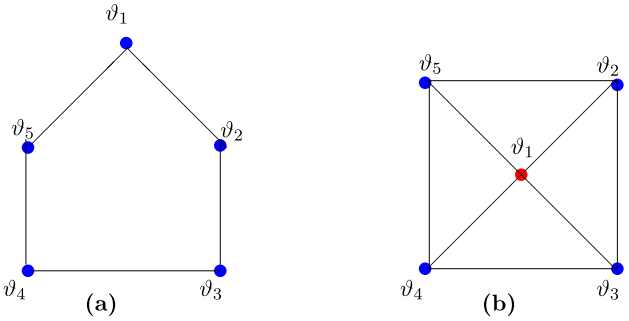


**Fig. 2** **a** The cycle graph $C_5$, for which $\gamma_h(C_5) = 0$. **b** The wheel graph $W_4$ (rim $C_4$ plus center), where the central vertex serves as a hub, giving $\gamma_h(W_4) = 1$

Theorem 3.1 and the cycle discussion), yielding $\gamma_h(C_p) = 0$.
(iii) In a star $S_p$, the unique center $\omega$ dominates all leaves and lies on every path between two distinct leaves; thus, $\{\omega\}$ is a minimal hub dominating set and $\gamma_h(S_p) = 1$.
(iv) In a wheel $W_p$, the hub (center) dominates the rim and lies on every path between non-adjacent rim vertices; hence, $\{\omega\}$ is a minimal hub dominating set and $\gamma_h(W_p) = 1$. □

***Example 3.5*** Let $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$ be the complete graph shown in Fig. 1b, where $\mathcal{V} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$. We observe that $\gamma_h(\mathfrak{Z}) = 0$.

***Example 3.6*** Let $\mathfrak{Z}$ be the cycle graph shown in Fig. 2a, where $\mathcal{V} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5\}$. We observe that $\gamma_h(\mathfrak{Z}) = 0$.

***Example 3.7*** Let $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$ be the wheel graph shown in Fig. 2b, where $\mathcal{V} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5\}$. We see that $\mathfrak{D} = \{\vartheta_1\}$ is a minimal hub dominating set of $\mathfrak{Z}$ with $\gamma_h(\mathfrak{Z}) = 1$.

Building upon complete graphs, we now turn to complete bipartite graphs and determine their hub domination number.

**Theorem 3.8** *If $\mathfrak{Z} = K_{p,q}$ is a complete bipartite graph, then*

$$\gamma_h(K_{p,q}) = \begin{cases} 1, & \text{if } p = 1 \text{ or } q = 1, \\ 2, & \text{if } p, q \geq 2. \end{cases}$$

***Proof*** **Case 1:** $p = 1$ or $q = 1$ gives a star; the center alone is a (minimal) hub dominating set, so $\gamma_h = 1$. **Case 2:** If $p, q \geq 2$, choose one vertex from each part as hubs. Any two non-hub vertices are either in the same part or opposite parts; in both cases, there is a path whose internal vertices are exactly those two hubs, so $\gamma_h \leq 2$. Minimality yields $\gamma_h = 2$. □

We now establish general bounds for the hub domination number.

**Theorem 3.9** *Let $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$ be a graph of order $p$.*

*(i)* $\gamma_h(\mathfrak{Z}) \leq p$, *with equality if and only if* $\mathfrak{Z}$ *is edgeless.*
*(ii) If* $\delta(\mathfrak{Z}) \geq 1$ *(no isolated vertices), then* $\gamma_h(\mathfrak{Z}) \leq p - 1$.
*(iii) The same statements hold for* $\bar{\mathfrak{Z}}$.

***Proof*** (i) The whole vertex set $\mathfrak{D} = \mathcal{V}$ is always a hub dominating set, so $\gamma_h \leq p$. If $\mathfrak{Z}$ is edgeless, domination forces $\mathfrak{D} = \mathcal{V}$ and minimality holds, so $\gamma_h = p$. Conversely, if there is at least one edge $xy$, then $\mathcal{V} \setminus \{x\}$ dominates $x$ and vacuously satisfies the hub condition for pairs outside (there is only $x$ outside), and hence, $\gamma_h \leq p - 1$; thus equality $\gamma_h = p$ cannot occur. (ii) If $\delta \geq 1$, pick any vertex $x$; then $\mathcal{V} \setminus \{x\}$ dominates $x$ (as $x$ has a neighbor) and the hub condition is vacuous on $\{x\}$, so $\gamma_h \leq p - 1$. (iii) Apply (i)—(ii) to the complement. □

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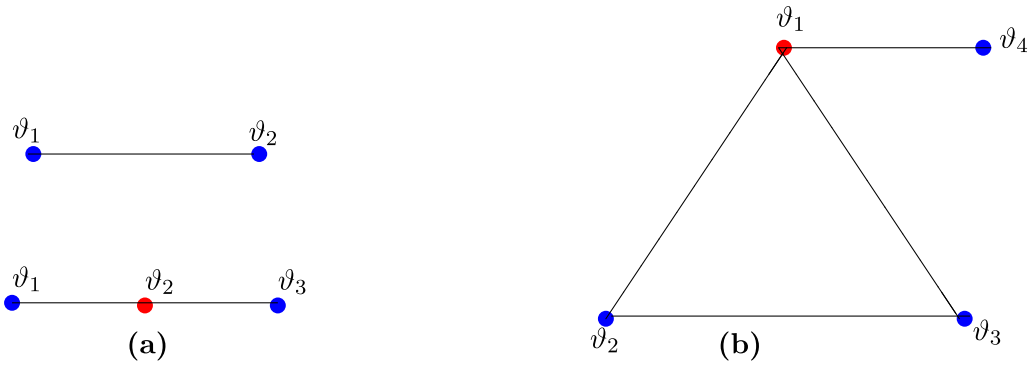


**Fig. 3** Illustrations of hub domination on paths. **a** For $P_2$ (top) and $P_3$ (bottom), minimal hub dominating sets are $\emptyset$ and $\{\vartheta_2\}$, respectively. **b** For $P_4$, the minimal hub dominating set is $\{\vartheta_2, \vartheta_3\}$, illustrating $\gamma_h(P_p) = p - 2$

**Theorem 3.10** *For any graph* $\mathfrak{Z}$ *of order* $p$

$$\gamma_h(\mathfrak{Z}) + \gamma_h(\bar{\mathfrak{Z}}) \leq 2p,$$

*with equality if and only if* $\{\mathfrak{Z}, \bar{\mathfrak{Z}}\}$ *is* {*complete, edgeless*}. *Moreover, if both* $\mathfrak{Z}$ *and* $\bar{\mathfrak{Z}}$ *have no isolated vertices, then*

$$\gamma_h(\mathfrak{Z}) + \gamma_h(\bar{\mathfrak{Z}}) \leq 2p - 2.$$

***Proof*** The first bound follows from Theorem 3.9(i) applied to each of $\mathfrak{Z}$ and $\bar{\mathfrak{Z}}$. Equality requires $\gamma_h(\mathfrak{Z}) = p$ and $\gamma_h(\bar{\mathfrak{Z}}) = p$, which happens exactly when one is complete and the other edgeless. The improved $2p - 2$ bound follows from Theorem 3.9(ii) applied to both. □

Moving to path graphs, we determine their hub domination number in the following theorem.

**Theorem 3.11** *For any path graph* $P_p$ *with* $p \geq 2$, $\gamma_h(P_p) = p - 2$.

***Proof*** If $p = 2$, then $P_2 = K_2$, and by Theorem 3.4(i), we have $\gamma_h(P_2) = 0 = 2 - 2$. If $p = 3$, the middle vertex alone is a hub dominating set, so $\gamma_h(P_3) = 1 = 3 - 2$.

For $p \geq 4$, label the path $\vartheta_1\vartheta_2 \cdots \vartheta_p$. The set $\mathfrak{D} = \{\vartheta_2, \vartheta_3, \ldots, \vartheta_{p-1}\}$ is hub dominating: any two non-hub vertices lie among $\{\vartheta_1, \vartheta_p\}$ and the path between them has all internal vertices in $\mathfrak{D}$. Hence, $\gamma_h(P_p) \leq p - 2$.

For minimality, suppose some internal $\vartheta_i$ $(2 \leq i \leq p - 1)$ is omitted from a hub dominating set. Then, the two non-hub endpoints of the subpath flanking $\vartheta_i$ cannot be linked by a path whose internal vertices are all in the hub set (since any $\vartheta_1$–$\vartheta_p$ path uses $\vartheta_i$ as an internal vertex), contradicting the hub condition. Thus, $|\mathfrak{D}| \geq p - 2$ and we conclude $\gamma_h(P_p) = p - 2$ (Fig. 3). □

**Corollary 3.12** *For any path graph* $\mathfrak{Z}$, $\gamma_h(\mathfrak{Z}) = h(\mathfrak{Z})$.

***Proof*** By Theorem 3.11, we have $\gamma_h(P_p) = p - 2$, and Walsh's hub number satisfies $h(P_p) = p - 2$ for paths, and hence, $\gamma_h = h$. □

***Remark 3.13*** The converse fails: there exist non-path graphs with $\gamma_h(\mathfrak{Z}) = h(\mathfrak{Z})$. For instance, in the star $S_n$ $(n \geq 4)$, both parameters equal 1, yet $S_n$ is not a path.

***Example 3.14*** *Non-path with* $\gamma_h = h = 1$. Let $\mathfrak{Z} = S_4$ (a star on 4 vertices). Then, the center alone forms a hub dominating set and a hub set, so $\gamma_h(\mathfrak{Z}) = h(\mathfrak{Z}) = 1$, but $\mathfrak{Z}$ is not a path.

**Theorem 3.15** *For any graph* $\mathfrak{Z}$ *of order* $p$:

*(i)* $\gamma_h(\mathfrak{Z}) \leq p - \Delta_N(\mathfrak{Z})$.
*(ii)* $\gamma_h(\bar{\mathfrak{Z}}) \leq p - \Delta_N(\bar{\mathfrak{Z}})$.

*Equality is attained, for example, by paths, stars, and wheels.*

***Proof*** Let $\vartheta$ be a vertex of maximum degree $\Delta_N$. Then, $S = \mathcal{V}\backslash N(\vartheta)$ is a hub dominating set: it dominates $N(\vartheta)$ via $\vartheta \in S$, and any $u, w \in N(\vartheta)$ have the 2–path $u - \vartheta - w$ with internal vertex in $S$. Thus, $|S| = p - \Delta_N$ gives (i). Part (ii) is analogous for the complement. Tightness holds for $P_p$ (with $\Delta_N = 2$), for stars (center of degree $p - 1$), and for wheels (center of degree $p - 1$). □

We now consider trees and record a safe global upper bound that distinguishes paths.

**Theorem 3.16** *If $\mathfrak{Z}$ is a tree on $p \geq 4$ with maximum degree $\Delta(\mathfrak{Z}) \geq 3$ (i.e., $\mathfrak{Z}$ is not a path), then $\gamma_h(\mathfrak{Z}) \leq p - 3$.*

***Proof*** By Theorem 3.15(i), $\gamma_h(\mathfrak{Z}) \leq p - \Delta(\mathfrak{Z})$. Since $\Delta(\mathfrak{Z}) \geq 3$ for non-path trees, we get $\gamma_h(\mathfrak{Z}) \leq p - 3$. □

To further refine the concept of hub domination, we introduce connected-hub domination.

**Definition 3.17** A dominating set $\mathfrak{D} \subseteq \mathcal{V}$ in a graph $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$ is a **connected-hub dominating set** if (i) $\langle \mathfrak{D} \rangle$ is connected, and (ii) every pair of non-adjacent vertices outside $\mathfrak{D}$ has a connecting path whose internal vertices lie in $\mathfrak{D}$. The connected-hub domination number $\gamma_{hc}(\mathfrak{Z})$ is the minimum size of such a set.

**Theorem 3.18** *For any connected graph $\mathfrak{Z}$, $\gamma_{hc}(\mathfrak{Z}) \leq \gamma_c(\mathfrak{Z})$.*

***Proof*** Every minimum connected dominating set is dominating and its induced subgraph is connected; any two vertices outside it attach to the connected core, yielding a path whose internal vertices lie in the set. Thus, it is also a connected-hub dominating set. □

We next give degree-based bounds.

**Theorem 3.19** *For any graph $\mathfrak{Z}$ without isolated vertices, $\gamma_h(\mathfrak{Z}) \leq p - \Delta_E(\mathfrak{Z})$.*

***Proof*** Since $\Delta_E(\mathfrak{Z}) \leq \Delta_N(\mathfrak{Z})$, the claim follows from Theorem 3.15(i). □

**Theorem 3.20** *For any graph $\mathfrak{Z}$ of order $p$:*

*(i)* $\gamma_h(\mathfrak{Z}) \leq p - \delta_N(\mathfrak{Z})$.
*(ii)* $\gamma_h(\mathfrak{Z}) \leq p - \delta_E(\mathfrak{Z})$.

***Proof*** (i) Let $\vartheta$ have degree $\delta_N$. Then, $S = \mathcal{V}\backslash N(\vartheta)$ works as above, giving $|S| = p - \delta_N$. (ii) Follows from $\delta_E \leq \delta_N$. □

**From degree bounds to cover/independence:** We relate $\gamma_h$ to vertex cover and independence numbers in the connected case.

**Theorem 3.21** *If $\mathfrak{Z}$ is connected with $p \geq 2$, then*

$$\gamma_h(\mathfrak{Z}) \leq \tau(\mathfrak{Z}) = p - \alpha(\mathfrak{Z}),$$

*where $\tau$ is the vertex cover number and $\alpha$ is the independence number.*

***Proof*** Let $S$ be a minimum vertex cover. Then, $S$ is dominating (every vertex outside $S$ lies on an edge to $S$) and, since $\mathfrak{Z}$ is connected and $V \setminus S$ is edgeless, any two vertices $u, w \in V \setminus S$ admit a $u$–$w$ path whose internal vertices lie in $S$. Hence, $S$ is a hub dominating set and $\gamma_h \leq |S| = \tau = p - \alpha$. □

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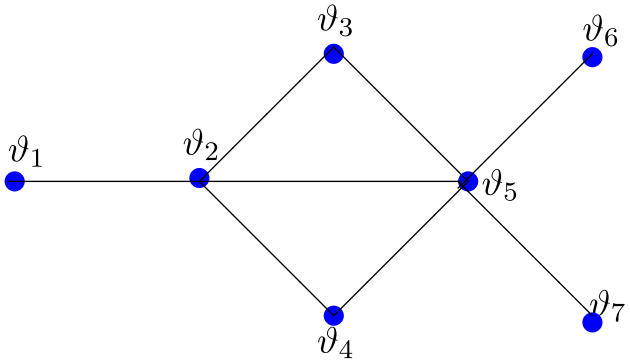


**Fig. 4** Graph $\mathfrak{Z}$ illustrating parameter relations in Theorem 3.23. Here, $\gamma_h(\mathfrak{Z}) = 2$, the independence number is $\beta(\mathfrak{Z}) = 5$, and $\chi(\mathfrak{Z}) = 3$, so $\beta(\mathfrak{Z}) + \gamma_h(\mathfrak{Z}) = p = 7$ (tight in (ii))

**Corollary 3.22** *For any connected graph $\mathfrak{Z}$ with $p \geq 2$, $\gamma_h(\mathfrak{Z}) \leq \alpha(\mathfrak{Z})$ if $\alpha$ denotes the vertex cover number (notation as in the preliminaries). Equivalently, $\gamma_h(\mathfrak{Z}) \leq p - \beta(\mathfrak{Z})$ if $\beta$ denotes the independence number.*

Finally, we examine chromatic-number relationships using the consistent notation $\chi(\mathfrak{Z})$.

**Theorem 3.23** *Let $\mathfrak{Z}$ be a connected graph of order $p \geq 2$. Then, the following relations hold:*

*(i)* $\gamma(\mathfrak{Z}) \leq \beta(\mathfrak{Z})$, *where $\beta(\mathfrak{Z})$ is the independence number.*
*(ii)* $\gamma_h(\mathfrak{Z}) \leq p - \beta(\mathfrak{Z})$ *(equivalently, $\beta(\mathfrak{Z}) + \gamma_h(\mathfrak{Z}) \leq p$).*
*(iii)* $\chi(\mathfrak{Z}) \geq \left\lceil \dfrac{p}{p - \gamma_h(\mathfrak{Z})} \right\rceil$.

***Proof*** (i) Any maximum independent set is maximal, hence dominating; thus, $\gamma \leq \beta$.

(ii) Let $S$ be a minimum vertex cover; then, $|S| = p - \beta$ and $V \setminus S$ is edgeless. In a connected graph, $S$ dominates $V \setminus S$, and any two vertices outside $S$ are joined by a path whose internal vertices lie in $S$, so $S$ is a hub dominating set. Hence, $\gamma_h \leq |S| = p - \beta$.

(iii) The classical bound $\alpha(\mathfrak{Z}) \geq p/\chi(\mathfrak{Z})$ with $\alpha = \beta$ gives $\chi(\mathfrak{Z}) \geq p/\beta(\mathfrak{Z})$. From (ii), $\beta(\mathfrak{Z}) \leq p - \gamma_h(\mathfrak{Z})$, so $\chi(\mathfrak{Z}) \geq p/(p - \gamma_h(\mathfrak{Z}))$, and taking ceilings yields the claim. □

***Example 3.24*** For $\mathfrak{Z} = P_4$: $\gamma_h(\mathfrak{Z}) = 2$, $\beta(\mathfrak{Z}) = 2$, and $\chi(\mathfrak{Z}) = 2$. Theorem 3.23(ii) gives $\beta + \gamma_h = 4 = p$, and (iii) gives $\chi \geq \lceil 4/(4-2) \rceil = \lceil 2 \rceil = 2$, tight.

***Example 3.25*** Consider the graph $\mathfrak{Z}$ in Fig. 4. We find $\gamma_h(\mathfrak{Z}) = 2$, $\beta(\mathfrak{Z}) = 5$ (independence number), and $\chi(\mathfrak{Z}) = 3$. Thus, $\beta + \gamma_h = 7 = p$ (Theorem 3.23(ii) tight), and (iii) gives $\chi \geq \lceil 7/(7-2) \rceil = \lceil 7/5 \rceil = 2$, satisfied.

## Total Hub Domination Number of a Graph

Having fully explored various facets of the hub domination number for graphs, we now extend our analysis to the **total hub domination number**, which strengthens the requirement by demanding the hub–path property for *all* pairs of vertices outside the hub set (adjacent or not).

**Definition 3.26** Let $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$ be a graph. A dominating set $\mathfrak{D} \subseteq \mathcal{V}$ is a **total hub dominating set** of $\mathfrak{Z}$ if for every pair of vertices $u, w \in \mathcal{V} \setminus \mathfrak{D}$, there exists a $u$–$w$ path whose *internal* vertices lie in $\mathfrak{D}$.

**Definition 3.27** A total hub dominating set $\mathfrak{D}$ of a graph $\mathfrak{Z}$ is **minimal** if $\mathfrak{D} \setminus \{\vartheta\}$ fails to be a total hub dominating set for every $\vartheta \in \mathfrak{D}$.

**Definition 3.28** The **total hub domination number**, denoted by $\gamma_{ht}(\mathfrak{Z})$, is the minimum cardinality among all minimal total hub dominating sets in $\mathfrak{Z}$.

The next result identifies the structural core of the "total" requirement and connects it to a classical parameter.

**Proposition 3.29** *For any connected graph $\mathfrak{Z}$ with $|\mathcal{V}| \geq 2$, a set $\mathfrak{D} \subseteq \mathcal{V}$ is a total hub dominating set if and only if $\mathfrak{D}$ is a* connected dominating set. *Consequently*

$$\gamma_{ht}(\mathfrak{Z}) = \gamma_c(\mathfrak{Z}).$$

***Proof*** ($\Rightarrow$) Suppose $\mathfrak{D}$ is total hub dominating. Since $\mathfrak{D}$ is dominating by definition, every vertex in $\mathcal{V} \setminus \mathfrak{D}$ has a neighbor in $\mathfrak{D}$. If $\langle \mathfrak{D} \rangle$ were disconnected, take vertices $u, w \in \mathcal{V} \setminus \mathfrak{D}$ adjacent to different components of $\langle \mathfrak{D} \rangle$; then no $u$–$w$ path can have all internal vertices in $\mathfrak{D}$, a contradiction. Hence, $\langle \mathfrak{D} \rangle$ is connected, so $\mathfrak{D}$ is a connected dominating set.

($\Leftarrow$) If $\mathfrak{D}$ is a connected dominating set, then for any $u, w \in \mathcal{V} \setminus \mathfrak{D}$ pick neighbors $h_u, h_w \in \mathfrak{D}$. Because $\langle \mathfrak{D} \rangle$ is connected, there exists an $h_u$–$h_w$ path entirely in $\mathfrak{D}$; concatenating $u$–$h_u$, that path, and $h_w$–$w$ yields a $u$–$w$ path whose internal vertices lie in $\mathfrak{D}$. Thus, $\mathfrak{D}$ is total hub dominating. □

Proposition 3.29 immediately yields exact values for standard families and clarifies how $\gamma_{ht}$ compares to other parameters.

**Corollary 3.30** *For a connected graph $\mathfrak{Z}$ of order $p$:*

*(i) If $\mathfrak{Z} = K_p$, then $\gamma_{ht}(\mathfrak{Z}) = 1$.*
*(ii) If $\mathfrak{Z} = C_p$ with $p \geq 3$, then $\gamma_{ht}(\mathfrak{Z}) = p - 2$.*
*(iii) If $\mathfrak{Z} = S_p$ (star), then $\gamma_{ht}(\mathfrak{Z}) = 1$.*
*(iv) If $\mathfrak{Z} = W_p$ (wheel), then $\gamma_{ht}(\mathfrak{Z}) = 1$.*

*Moreover, for any connected graph $\mathfrak{Z}$*

$$\gamma(\mathfrak{Z}) \leq \gamma_{ht}(\mathfrak{Z}) = \gamma_c(\mathfrak{Z}) \leq \gamma_h(\mathfrak{Z}) + 1,$$

*where the last inequality follows from Theorem 3.18.*

***Proof*** All four items follow from $\gamma_{ht} = \gamma_c$ and the well-known connected domination numbers of these families; briefly: a single center dominates and connects $S_p$ and $W_p$; any vertex dominates $K_p$; for $C_p$, a connected dominating set must be a contiguous block covering $p - 2$ vertices. The inequality chain is immediate, since any total hub dominating set is, in particular, a dominating set, and Theorem 3.18 gives $\gamma_c \leq \gamma_h + 1$. □

***Example 3.31*** Consider the cycle graph $\mathfrak{Z}$ in Fig. 2a with $p = 5$. By Corollary 3.30(ii), $\gamma_{ht}(\mathfrak{Z}) = \gamma_c(\mathfrak{Z}) = 5-2 = 3$.

## 4 The Hub Domination Number of a Fuzzy Graph

Building upon our exploration of hub domination in crisp graphs, we now extend the investigation to fuzzy graphs, focusing specifically on the hub domination number in this setting.

**Definition 4.1** Let $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$ be a fuzzy graph. For $x, y \in \mathcal{V}$ and a set $\mathfrak{D} \subseteq \mathcal{V}$, let $\mathcal{P}_{\mathfrak{D}}(x, y)$ denote the family of $x$–$y$ paths whose *internal* vertices lie in $\mathfrak{D}$. The *strength* of such a path $P = (x = v_0, v_1, \ldots, v_k = y)$ is

$$\mathrm{str}(P) = \min_{i=0}^{k-1} \rho(v_i, v_{i+1}),$$

and the *strongest $\mathfrak{D}$-internal path* strength is

$$\mathcal{C}_{\mathfrak{D}}(x, y) = \max_{P \in \mathcal{P}_{\mathfrak{D}}(x,y)} \mathrm{str}(P) \quad (\max \emptyset := 0).$$

 https://doi.org/10.1007/s44196-025-00991-5

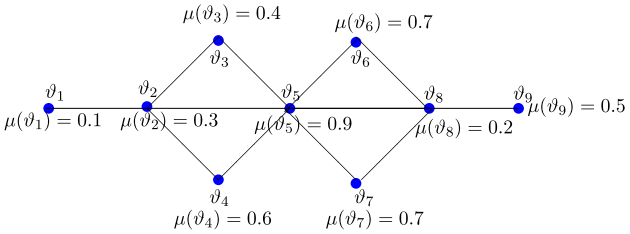


**Fig. 5** Fuzzy graph for Example: edges carry memberships $\rho(u, v) = \min\{\mu(u), \mu(v)\}$ on drawn links; non-drawn pairs have $\rho = 0$

A subset $\mathfrak{D} \subseteq \mathcal{V}$ is a **fuzzy hub dominating set** if

(i) (*Fuzzy domination*) For every $\vartheta \in \mathcal{V}\backslash\mathfrak{D}$, there exists $\omega \in \mathfrak{D}$ with $\rho(\omega, \vartheta) > 0$.
(ii) (*Fuzzy hub connectivity*) For every *non-adjacent* pair $x, y \in \mathcal{V}\backslash\mathfrak{D}$ (i.e., $\rho(x, y) = 0$), one has $\mathcal{C}_{\mathfrak{D}}(x, y) > 0$.

It is *minimal* if no proper subset satisfies (i)–(ii). The **fuzzy hub domination number** is

$$\gamma_h(\mathfrak{Z}) = \min_{\substack{\mathfrak{D} \text{ minimal} \\ \text{fuzzy HDS}}} \sum_{\omega \in \mathfrak{D}} \mu(\omega),$$

i.e., the minimum fuzzy cardinality among all minimal fuzzy hub dominating sets.

***Example 4.2*** To illustrate Definition 4.1, consider the fuzzy graph $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$ in Fig. 5, with

$$\mathcal{V} = \{\vartheta_1, \ldots, \vartheta_9\},\ \mu(\vartheta_1) = 0.1,\ \mu(\vartheta_2) = 0.3,\ \mu(\vartheta_3) = 0.4,$$
$$\mu(\vartheta_4) = 0.6,\ \mu(\vartheta_5) = 0.9,\ \mu(\vartheta_6) = \mu(\vartheta_7) = 0.7,\ \mu(\vartheta_8) = 0.2,\ \mu(\vartheta_9) = 0.5.$$

We set $\rho(u, v) = \min\{\mu(u), \mu(v)\}$ *on the edges drawn in the figure* and $\rho(u, v) = 0$ otherwise. A minimal fuzzy hub dominating set of smallest fuzzy cardinality is

$$\mathfrak{D} = \{\vartheta_2, \vartheta_5\} \quad \text{with} \quad \sum_{\omega \in \mathfrak{D}} \mu(\omega) = 0.3 + 0.9 = 1.2,$$

so $\gamma_h(\mathfrak{Z}) = 1.2$. Indeed, (i) every non-hub is adjacent to $\vartheta_2$ or $\vartheta_5$; (ii) any non-adjacent pair outside $\mathfrak{D}$ admits a path whose internal vertices lie in $\mathfrak{D}$ (e.g., 1–2–5–6, 6–5–9, 3–5–7, 4–5–8), and removing either $\vartheta_2$ or $\vartheta_5$ breaks these properties.

We now compute $\gamma_h(\mathfrak{Z})$ for standard families of fuzzy graphs, including complete, cycle, star, wheel, and complete bipartite fuzzy graphs.

**Theorem 4.3** *Let $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$ be a fuzzy graph. Then, the fuzzy hub domination number $\gamma_h(\mathfrak{Z})$ satisfies*

$$\gamma_h(\mathfrak{Z}) = \begin{cases} 0, & \text{if } \mathfrak{Z} \text{ is a complete fuzzy graph } K_n, \\ 0, & \text{if } \mathfrak{Z} \text{ is a cycle fuzzy graph } C_p, \\ \mu(\vartheta_c), & \text{if } \mathfrak{Z} \text{ is a star fuzzy graph with center } \vartheta_c, \\ \mu(\vartheta_h), & \text{if } \mathfrak{Z} \text{ is a wheel fuzzy graph with hub } \vartheta_h, \\ \mu(\omega), & \text{if } \mathfrak{Z} = K_{p,q} \text{ with } p = 1 \text{ or } q = 1 \ (\omega \text{ is the unique vertex of the smaller part}), \\ \min_{x \in X} \mu(x) + \min_{y \in Y} \mu(y), & \text{if } \mathfrak{Z} = K_{p,q} \text{ with } p, q \geq 2 \text{ and bipartition } \mathcal{V} = X \cup Y. \end{cases}$$

***Proof*** *(Complete fuzzy graphs $K_n$).* Every pair of vertices is adjacent with positive membership, so the hub-path condition is vacuously satisfied without internal vertices, yielding $\gamma_h(K_n) = 0$.

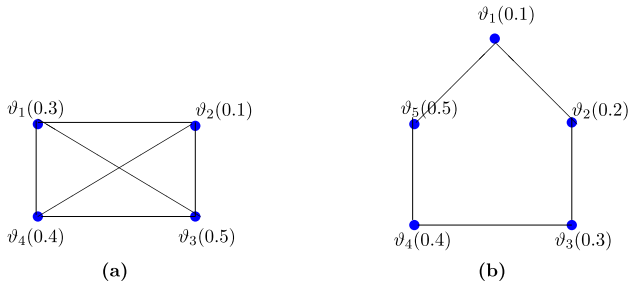


**Fig. 6** Illustrations for Theorem 4.3: **a** complete fuzzy graph $K_4$; **b** cycle fuzzy graph $C_5$

*(Cycle fuzzy graphs $C_p$).* Along the cycle, adjacency on each edge suffices for hub-path feasibility with no internal hubs for adjacent pairs, so under the same convention $\gamma_h(C_p) = 0$.

*(Star fuzzy graphs).* Let $\vartheta_c$ be the center and $S = \{\vartheta_c\}$. Each leaf is adjacent to $\vartheta_c$, and any two leaves admit a 2-path whose internal vertex is $\vartheta_c \in S$. Minimality is clear; hence, $\gamma_h = \mu(\vartheta_c)$.

*(Wheel fuzzy graphs).* Let $\vartheta_h$ be the hub and $S = \{\vartheta_h\}$. Each rim vertex is adjacent to $\vartheta_h$, and any two non-adjacent rim vertices connect via a 2-path through $\vartheta_h$. Minimality gives $\gamma_h = \mu(\vartheta_h)$.

*(Complete bipartite fuzzy graphs $K_{p,q}$).* Write the bipartition $\mathcal{V} = X \cup Y$. If $p = 1$ or $q = 1$, the unique vertex $\omega$ in the smaller part forms $S = \{\omega\}$, which provides hub-paths between any two vertices in the opposite part; minimality gives $\gamma_h = \mu(\omega)$. If $p, q \geq 2$, pick $x^\star \in X$ with $\mu(x^\star) = \min_{x \in X} \mu(x)$ and $y^\star \in Y$ with $\mu(y^\star) = \min_{y \in Y} \mu(y)$, and set $S = \{x^\star, y^\star\}$. Then, any two vertices in the same part have a 2-path via the chosen vertex in the opposite part (internal vertex in $S$), and dropping either hub breaks this property, so $\gamma_h = \mu(x^\star) + \mu(y^\star)$. □

***Example 4.4*** To illustrate Theorem 4.3, we present two families. Figure 6a is a complete fuzzy graph $K_4$ with $\mathcal{V} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$, $\mu(\vartheta_1) = 0.3$, $\mu(\vartheta_2) = 0.1$, $\mu(\vartheta_3) = 0.5$, $\mu(\vartheta_4) = 0.4$, and $\rho(u, v) = \mu(u) \wedge \mu(v)$ for all distinct $u, v \in \mathcal{V}$. Then, $\gamma_h(\mathfrak{Z}) = 0$.

Figure 6b shows a cycle fuzzy graph $C_5$ with $\mathcal{V} = \{\vartheta_1, \ldots, \vartheta_5\}$, $\mu(\vartheta_1) = 0.1$, $\mu(\vartheta_2) = 0.2$, $\mu(\vartheta_3) = 0.3$, $\mu(\vartheta_4) = 0.4$, $\mu(\vartheta_5) = 0.5$, and $\rho(u, v) = \mu(u) \wedge \mu(v)$ on the cycle edges (and 0 otherwise). Again, $\gamma_h(\mathfrak{Z}) = 0$, consistent with Theorem 4.3.

***Example 4.5*** Wheel family (fuzzy). Let $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$ be a fuzzy wheel on $\mathcal{V} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5\}$ with $\mu(\vartheta_1) = 0.3$, $\mu(\vartheta_2) = 0.4$, $\mu(\vartheta_3) = 0.4$, $\mu(\vartheta_4) = 0.5$, $\mu(\vartheta_5) = 0.3$, and $\rho(u, v) = \mu(u) \wedge \mu(v)$ on the wheel edges (and 0 otherwise). Then, $\mathfrak{D} = \{\vartheta_1\}$ (the hub) is a minimal fuzzy hub dominating set with $\gamma_h(\mathfrak{Z}) = \mu(\vartheta_1) = 0.3$.

We now establish general upper bounds for the fuzzy hub domination number.
Analogous to the crisp setting, we extend connected-hub domination to fuzzy graphs.

**Definition 4.6** Let $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$ be a fuzzy graph. A vertex subset $\mathfrak{D} \subseteq \mathcal{V}$ is a **minimal connected-hub dominating set** if: (i) for every $v \in \mathcal{V} \backslash \mathfrak{D}$, there exists $u \in \mathfrak{D}$ with $\rho(u, v) > 0$; (ii) the induced fuzzy subgraph $\langle \mathfrak{D} \rangle$ (vertex set $\mathfrak{D}$, edge weights $\rho|_{\mathfrak{D} \times \mathfrak{D}}$) is connected; and (iii) no proper subset $\mathfrak{D}' \subsetneq \mathfrak{D}$ satisfies (i) and (ii). The *connected-hub domination number* $\gamma_{hc}(\mathfrak{Z})$ is the minimum value of $\sum_{v \in \mathfrak{D}} \mu(v)$ over all such $\mathfrak{D}$.

We next consolidate membership–sum bounds for $\gamma_h$ and $\gamma_{hc}$. Let $M(\mathfrak{Z}) := \sum_{v \in \mathcal{V}} \mu(v)$ denote the total vertex membership (note that $M(\bar{\mathfrak{Z}}) = M(\mathfrak{Z})$, since $\bar{\mathfrak{Z}}$ has the same vertex membership function).

**Theorem 4.7** *For any fuzzy graph $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$ of order $p$:*

*(i)* $\gamma_h(\mathfrak{Z}) < M(\mathfrak{Z})$ *and* $\gamma_h(\bar{\mathfrak{Z}}) < M(\mathfrak{Z})$.
*(ii)* $\gamma_h(\mathfrak{Z}) + \gamma_h(\bar{\mathfrak{Z}}) < 2\, M(\mathfrak{Z})$.
*(iii) If $\mathfrak{Z}$ is connected, then* $\gamma_{hc}(\mathfrak{Z}) \leq \gamma_c(\mathfrak{Z})$ .
*(iv) Let $S$ be a minimum fuzzy hub dominating set and let $k$ be the number of connected components of $\langle S \rangle$. Then, there exists a set $X \subseteq \mathcal{V} \backslash S$ with $|X| \leq k - 1$, such that $\langle S \cup X \rangle$ is connected, and hence*

$$\gamma_c(\mathfrak{Z}) \;\leq\; \gamma_h(\mathfrak{Z}) \;+\; \sum_{j=1}^{k-1} \mu(v_j),$$

*where $v_1, \ldots, v_{k-1}$ are the $(k-1)$ vertices of $\mathcal{V}\backslash S$ with the smallest memberships. In particular, if $\langle S\rangle$ is already connected (i.e., $k = 1$), then $\gamma_c(\mathfrak{Z}) \le \gamma_h(\mathfrak{Z})$.*

***Proof*** (i) By Definition 4.1, $\gamma_h(\mathfrak{Z})$ is the minimum membership sum of a *minimal* fuzzy hub dominating set. A set containing all vertices is never minimal, so its membership sum $M(\mathfrak{Z})$ is a strict upper bound; the same vertex membership function $\mu$ applies to $\bar{\mathfrak{Z}}$, so $\gamma_h(\bar{\mathfrak{Z}}) < M(\mathfrak{Z})$ as well.

(ii) Summing the two strict inequalities from (i) gives $\gamma_h(\mathfrak{Z}) + \gamma_h(\bar{\mathfrak{Z}}) < 2\,M(\mathfrak{Z})$.

(iii) Let $\mathfrak{D}$ be a minimum connected dominating set. For any $x, y \in \mathcal{V}\backslash\mathfrak{D}$, choose $d_x, d_y \in \mathfrak{D}$ adjacent to $x$ and $y$, respectively. Since $\langle\mathfrak{D}\rangle$ is connected, there is a $d_x$-$d_y$ path whose internal vertices lie in $\mathfrak{D}$; hence, the concatenated path $x - d_x - \cdots - d_y - y$ has all internal vertices in $\mathfrak{D}$. Thus, $\mathfrak{D}$ is also a connected-hub dominating set, and $\sum_{v\in\mathfrak{D}} \mu(v)$ is an admissible value for $\gamma_{hc}$, proving $\gamma_{hc}(\mathfrak{Z}) \le \gamma_c(\mathfrak{Z})$.

(iv) Let $S$ be a minimum fuzzy hub dominating set and let $\langle S\rangle$ have $k \ge 1$ components. Because $\mathfrak{Z}$ is connected, there exists a set $X \subseteq \mathcal{V}\backslash S$ with $|X| \le k-1$, such that $\langle S \cup X\rangle$ is connected (choose a minimal set of "Steiner" vertices that links the components of $\langle S\rangle$). Then, $S \cup X$ is a connected dominating set, so

$$\gamma_c(\mathfrak{Z}) \le \sum_{v\in S} \mu(v) + \sum_{x\in X} \mu(x).$$

Choosing $X$ to minimize the added membership yields the stated bound; in particular, one may take $X$ to be the $(k-1)$ vertices of $\mathcal{V}\backslash S$ with the smallest memberships. If $k = 1$, we may take $X = \emptyset$, giving $\gamma_c(\mathfrak{Z}) \le \gamma_h(\mathfrak{Z})$. □

We now explore *cardinality*-based upper bounds for the fuzzy hub domination number, mirroring the crisp case. Throughout this subsection, neighborhoods are taken in the support graph of $\mathfrak{Z}$ (i.e., $uv$ is an edge iff $\rho(u, v) > 0$). Thus, $N(v)$ denotes the set of neighbors of $v$, $d_N(v) := |N(v)|$, and $\Delta_N(\mathfrak{Z}) = \max_v d_N(v)$, $\delta_N(\mathfrak{Z}) = \min_v d_N(v)$. Similarly, $\Delta_E, \delta_E$ are the corresponding edge–degree parameters as defined in the preliminaries. Here, $p = |\mathcal{V}|$ is the (crisp) order.

**Theorem 4.8** *Let $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$ be a fuzzy graph of order $p$. Then*

*(i)* $\gamma_h(\mathfrak{Z}) \le p - \Delta_N(\mathfrak{Z})$;
*(ii)* $\gamma_h(\bar{\mathfrak{Z}}) \le p - \Delta_N(\bar{\mathfrak{Z}})$;
*(iii)* $\gamma_h(\mathfrak{Z}) \le p - \delta_N(\mathfrak{Z})$;
*(iv)* $\gamma_h(\mathfrak{Z}) \le p - \delta_E(\mathfrak{Z})$.

*Moreover, equality holds in* (i) *and* (ii) *when the support graph of $\mathfrak{Z}$ (resp. $\bar{\mathfrak{Z}}$) is a path $P_p$.*

***Proof*** (i) Let $v \in \mathcal{V}$ with $d_N(v) = \Delta_N(\mathfrak{Z})$. Set $S := \mathcal{V}\backslash N(v)$; note that $v \in S$. Every vertex in $\mathcal{V}\backslash S = N(v)$ is adjacent to $v \in S$, so $S$ dominates $\mathfrak{Z}$. For any two $x, y \in N(v)$, the 2–path $x - v - y$ has the intermediate vertex $v \in S$, so the hub property holds. Hence, $S$ is a hub dominating set and $\gamma_h(\mathfrak{Z}) \le |S| = p - \Delta_N(\mathfrak{Z})$.

(ii) Apply the argument of (i) to the complement fuzzy graph $\bar{\mathfrak{Z}}$ (with the same vertex set and membership function but edges given by the support complement). Taking a vertex of maximum degree in $\bar{\mathfrak{Z}}$ yields $\gamma_h(\bar{\mathfrak{Z}}) \le p - \Delta_N(\bar{\mathfrak{Z}})$.

(iii) Let $v \in \mathcal{V}$ with $d_N(v) = \delta_N(\mathfrak{Z})$. As in (i), $S := \mathcal{V}\backslash N(v)$ (which contains $v$) dominates $\mathfrak{Z}$ and ensures the hub property for any pair in $N(v)$ via $v$. Thus, $\gamma_h(\mathfrak{Z}) \le |S| = p - \delta_N(\mathfrak{Z})$.

(iv) Since $\delta_E(\mathfrak{Z}) \le \delta_N(\mathfrak{Z})$ by definition, (iii) immediately implies $\gamma_h(\mathfrak{Z}) \le p - \delta_E(\mathfrak{Z})$.

For the equality cases in (i)–(ii): if the support graph is a path $P_p$ ($p \ge 2$), then $\Delta_N = 2$ (for $p \ge 4$) and it is standard that $\gamma_h(P_p) = p - 2$; hence, (i) is tight. A symmetric argument holds when the support of $\bar{\mathfrak{Z}}$ is a path, yielding tightness for (ii). □

## Total Hub Domination Number of a Fuzzy Graph

Parallel to our investigation of total hub domination in crisp graphs, we now extend this concept to fuzzy graphs, measuring set sizes by membership sums.

**Definition 4.9** Let $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$ be a fuzzy graph (neighborhoods taken in the support graph $\rho > 0$). A subset $\mathfrak{D} \subseteq \mathcal{V}$ is a **minimal total hub dominating set** if:

(i) (*Domination*) Every vertex $\vartheta \notin \mathfrak{D}$ has a neighbor in $\mathfrak{D}$.
(ii) (*Total hub property*) For any two vertices $u, w \in \mathcal{V}\backslash\mathfrak{D}$ (adjacent or not), there exists a $u-w$ path whose *internal* vertices all lie in $\mathfrak{D}$.
(iii) (*Minimality*) No proper subset $\mathfrak{D}' \subsetneq \mathfrak{D}$ satisfies *both* (i) and (ii).

The **total hub domination number** $\gamma_{ht}(\mathfrak{Z})$ is the minimum membership sum $\sum_{\vartheta\in\mathfrak{D}} \mu(\vartheta)$ over all such sets $\mathfrak{D}$.

**Theorem 4.10** *For standard fuzzy graph families, $\gamma_{ht}$ is given by*

$$\gamma_{ht}(\mathfrak{Z}) = \begin{cases} \min\{\mu(\vartheta) : \vartheta \in \mathcal{V}\}, & \text{if } \mathfrak{Z} \cong K_p, \\ begineqnarray*6pt] \min\left\{ \sum_{i=1}^{p-2} \mu(v_i) : \{v_1, \ldots, v_{p-2}\} \subseteq \mathcal{V} \right\}, & \text{if } \mathfrak{Z} \cong C_p, \\ begineqnarray*6pt] \mu(\omega), & \text{if } \mathfrak{Z} \text{ is a star with center } \omega. \end{cases}$$

*Moreover, for every fuzzy graph $\mathfrak{Z}$ (not only cycles)*

$$\gamma(\mathfrak{Z}) \leq \gamma_{ht}(\mathfrak{Z}),$$

*where $\gamma(\mathfrak{Z})$ is the (fuzzy) domination number.*

***Proof*** *Complete fuzzy graphs.* Every vertex is adjacent to every other; any single vertex $\vartheta$ forms a total hub dominating set (it dominates all and lies on a 2-path between any two outside vertices). Minimizing the membership cost yields $\gamma_{ht}(K_p) = \min\{\mu(\vartheta)\}$.

*Cycle fuzzy graphs.* Let $\mathfrak{Z} \cong C_p$ with $p \geq 3$. ($\leq$) Pick any two distinct vertices $x, y$ to *omit* and let $\mathfrak{D} = \mathcal{V}\backslash\{x, y\}$ (so $|\mathfrak{D}| = p-2$). Then, $x$ and $y$ have a $x-y$ path whose internal vertices are precisely the $p-2$ remaining vertices (all in $\mathfrak{D}$); domination also holds, since each omitted vertex is adjacent to some vertex in $\mathfrak{D}$. Thus, $\mathfrak{D}$ is a total hub dominating set, and minimizing its membership sum over all choices of the two omitted vertices gives the stated expression. ($\geq$) If $|\mathcal{V}\backslash\mathfrak{D}| \geq 3$, choose two consecutive vertices $x, y$ among the non-hub vertices and let $z$ be another non-hub on the longer $x-y$ arc. Any $x-y$ path that avoids the short edge $xy$ necessarily uses the longer arc and hence includes $z$ as an *internal* vertex not in $\mathfrak{D}$, violating the total hub property. Hence, $|\mathfrak{D}| \geq p-2$. Together, this yields the formula above.

*Star fuzzy graphs.* Let $\omega$ be the center. Then, $\{\omega\}$ dominates all leaves and lies as the unique internal vertex on any path between two leaves. Minimality is evident; hence, $\gamma_{ht} = \mu(\omega)$.

*Domination inequality.* Every total hub dominating set is, in particular, a dominating set [condition (i)]; taking minima over membership sums gives $\gamma(\mathfrak{Z}) \leq \gamma_{ht}(\mathfrak{Z})$ for all fuzzy graphs. □

***Example 4.11*** Illustrative computations.

- *Complete fuzzy graph.* For $\mathcal{V} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$ with memberships $\{0.8, 0.6, 0.5, 0.9\}$ and $\rho(u, v) = \mu(u) \wedge \mu(v)$, we have $\gamma_{ht} = \min\{0.8, 0.6, 0.5, 0.9\} = 0.5$.

- *Cycle fuzzy graph.* For $C_5$ with memberships $\{0.1, 0.2, 0.1, 0.4, 0.1\}$, choosing $\{\vartheta_2, \vartheta_3, \vartheta_4\}$ (omit $\vartheta_1, \vartheta_5$) yields $\gamma_{ht} = 0.2 + 0.1 + 0.4 = 0.7$. (Any choice of three consecutive vertices achieves the minimum over all five ways of omitting two vertices; one may check the sum and pick the minimal configuration.)

**Proposition 4.12** *For every fuzzy graph* $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$

$$\gamma_h(\mathfrak{Z}) \leq \gamma_{ht}(\mathfrak{Z}).$$

***Proof*** Every total hub dominating set satisfies the domination condition and, by definition, the hub-path requirement for *all* pairs outside the set (adjacent or not). Hence, any total hub dominating set is a hub dominating set. Minimizing membership sums over a subset yields a value no smaller: $\gamma_h \leq \gamma_{ht}$. □

***Remark 4.13*** (Comparison on standard fuzzy families) Under the membership–sum convention used in this paper, we have the following:

- **Complete graphs** $K_p$: $\gamma_h(K_p) = 0$, while $\gamma_{ht}(K_p) = \min_{\vartheta \in \mathcal{V}} \mu(\vartheta)$. Thus, $\gamma_h < \gamma_{ht}$ unless some vertex has $\mu = 0$.
- **Cycles** $C_p$: $\gamma_h(C_p) = 0$, while $\gamma_{ht}(C_p) = \min\{\sum_{i=1}^{p-2} \mu(v_i)\}$. Hence, $\gamma_h < \gamma_{ht}$ (again strict if all memberships are positive).
- **Stars** $S_p$: $\gamma_h(S_p) = \mu(\omega) = \gamma_{ht}(S_p)$, where $\omega$ is the center.
- **Wheels** $W_p$: $\gamma_h(W_p) = \mu(\vartheta_h) = \gamma_{ht}(W_p)$, where $\vartheta_h$ is the hub.
- **Complete bipartite graphs** $K_{p,q}$:
  - If $p = 1$ or $q = 1$ (a star), $\gamma_h = \gamma_{ht} = \mu$(center).
  - If $p, q \geq 2$, write the bipartition as $A \cup B$. Let $u^\star \in A$ and $v^\star \in B$ minimize $\mu$ in their parts. Then, $\gamma_h = \mu(u^\star) + \mu(v^\star) = \gamma_{ht}$. *Proof sketch:* $S = \{u^\star, v^\star\}$ dominates both parts. For any two vertices in the same part, a length 2-path via the opposite chosen vertex has internal vertex in $S$. For a cross-part pair $(a, b) \in A \times B$, the path $a - v^\star - u^\star - b$ exists in $K_{p,q}$ and has all internal vertices in $S$. Minimality fails if either $u^\star$ or $v^\star$ is removed. Hence, $\gamma_{ht} \leq \mu(u^\star) + \mu(v^\star)$, and equality follows from Theorem 4.3.

In particular, $\gamma_h = \gamma_{ht}$ for stars, wheels, and $K_{p,q}$ with $p, q \geq 2$, whereas $\gamma_h < \gamma_{ht}$ for complete and cycle families (assuming all memberships are strictly positive).

## 5 Practical Applications of Hub Domination: Optimizing Telecommunication and Regional Transportation Networks

In network science and engineering, designing and optimizing complex networks efficiently are essential. Hub domination theory offers a powerful approach for addressing this, providing a strategic way to enhance connectivity and resource flow within networked systems. This section explores hub domination's practical applications, going beyond theory to show its real-world impact. We focus on its use in two key areas: telecommunication networks and regional transportation networks. These diverse but connected sectors highlight hub domination's broad applicability in optimizing network performance, enhancing resilience, and enabling strategic decisions. Through case studies and analyses, this section will demonstrate how the careful identification and strategic deployment of hub nodes, guided by hub domination principles, can significantly improve network efficiency, cost-effectiveness, and overall operation within these crucial infrastructure systems. In each case, we explicitly link computed quantities to the bounds and identities proved in Sections 3–4 (e.g., Theorems 4.7, 3.23, 4.3, 4.8, and Proposition 4.12), clarifying how theory informs operational decisions.

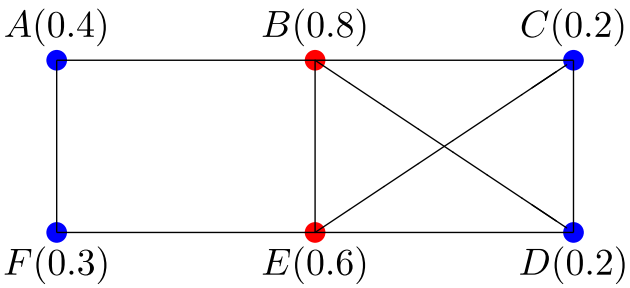


**Fig. 7** Telecommunication fraud network modeled as a fuzzy graph. Red vertices ($B$, $E$) denote hub actors forming a minimal hub dominating set (cost $\sum \mu$), while blue vertices denote non-hub actors. Vertex labels include membership values $\mu(v)$; edges are weighted by $\rho(u, v)$, the strength of collaboration

**Table 1** Computed fuzzy graph parameters for subnetworks $\Phi_i$ of the fraud network

| $i$ | Subnetwork $\Phi_i$ | $h(\Phi_i)$ | $\gamma_h(\Phi_i)$ | $\Delta(\Phi_i)$ | $\delta(\Phi_i)$ | $\beta(\Phi_i)$ |
|---|---|---|---|---|---|---|
| 1 | ABEF | 0.2 | 0 | 1.2 | 0.6 | 1.2 |
| 2 | ABCDEF | 0.8 | 1.4 | 1.4 | 0.7 | 1.1 |
| 3 | BCED | 0 | 0 | 1.0 | 0.6 | 0.2 |
| 4 | ABCEF | 1.4 | 1.4 | 1.2 | 0.6 | 0.6 |
| 5 | ABCDE | 0.8 | 0.8 | 1.4 | 0.4 | 1.0 |

$h(\Phi_i)$: hub number; $\gamma_h(\Phi_i)$: hub domination number; $\Delta(\Phi_i)$, $\delta(\Phi_i)$: max/min fuzzy degree; $\beta(\Phi_i)$: fuzzy independence number. All values are in membership units

### 5.1 Case 1: Identifying Central Actors in a Telecommunication Fraud Network Using Fuzzy Graphs

Figure 7 illustrates a telecommunication fraud network modeled as a fuzzy graph $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$. Here, vertices $\mathcal{V} = \{A, B, C, D, E, F\}$ represent suspected fraud teams. The fuzzy graph components are defined as follows:

- **Vertex Membership** ($\mu(v)$): For $v \in \mathcal{V}$, $\mu(v)$ quantifies the relative data access (or risk/cost weight) of team $v$, used in fuzzy cardinalities (e.g., $\sum_{v \in D} \mu(v)$).
- **Edge Membership** ($\rho(u, v)$): For $u, v \in \mathcal{V}$, $\rho(u, v)$ is the fraction of information flow between teams $u$ and $v$ (link strength/bandwidth/reliability in [0, 1]).

In this context, identifying a hub dominating set within this fuzzy graph can aid in pinpointing central fraud teams that act as key intermediaries in information exchange and potentially orchestrating fraudulent activities. Disrupting these hub teams would have a significant impact on the network's ability to operate effectively. Minimizing $\gamma_h(\mathfrak{Z})$ gives the lowest fuzzy cost of such an intervention, while $h(\mathfrak{Z})$ and degree-based bounds (Theorem 4.7) provide fast screening indicators of centralization.

Table 1 provides graph and hub parameters calculated for this fuzzy fraud network (and subnetworks derived from it, indexed by $i$). For example, row 2 representing network configuration *ABCDEF*:

- **Hub Number** ($h(\Phi_2)$): A value of **0.8**, consistent with degree-based bounds (cf. Theorem 4.7) and indicating moderate centralization.
- **Hub Domination Number** ($\gamma_h(\Phi_2)$): A value of **1.4**, indicating the minimum fuzzy cardinality of a hub dominating set. This is the estimated fuzzy cost (or risk mass) to neutralize central mediation.
- **Maximum Degree** ($\Delta(\Phi_2)$): A value of **1.4**, the largest fuzzy degree among the vertices.
- **Minimum Degree** ($\delta(\Phi_2)$): A value of **0.7**, the lowest fuzzy degree.
- **Independence Number** ($\beta(\Phi_2)$): A value of **1.1**, representing the fuzzy independence number, useful for assessing loosely coupled subgroups.

**Link to theory and operational guidance.** Rows with larger $\Delta(\Phi_i)$ and smaller $\beta(\Phi_i)$ tend to exhibit smaller feasible $\gamma_h(\Phi_i)$, aligning with degree-based bounds (Theorem 4.7). When $\gamma_h(\Phi_i)$ is close to the lower values predicted by the class results (Theorem 4.3), the network is tightly mediated by a few actors—priority targets for

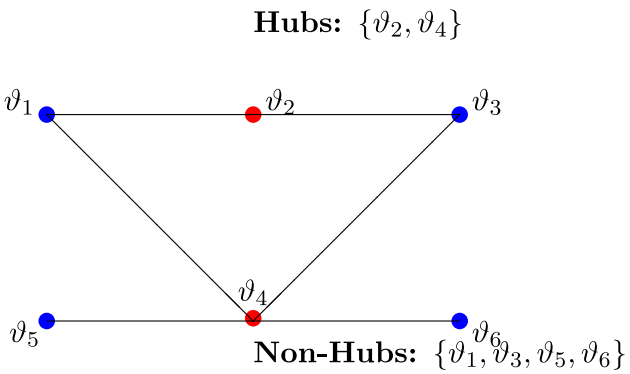


**Fig. 8** Telecommunication network as a fuzzy graph $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$. Vertices $\vartheta_i$ denote routers/servers; $\mu(\vartheta_i) \in [0, 1]$ encodes capacity/cost; $\rho(\vartheta_i, \vartheta_j) \in [0, 1]$ encodes link reliability/bandwidth. Red nodes $(\vartheta_2, \vartheta_4)$ constitute a minimal hub dominating set ensuring hub-mediated connectivity of all blue nodes

intervention. Conversely, $\gamma_h(\Phi_i) = 0$ (rows 1,3) indicates that no special hubs are required at threshold, suggesting lower immediate risk of centrally coordinated fraud.

**Managerial note and limitations.** The minimal fuzzy hub set provides a costed shortlist for investigation or disruption, while $h(\cdot)$ and degree statistics offer rapid triage on streaming data. Scalability to very large graphs can be addressed with the greedy heuristics of Algorithms 1–3, together with threshold tuning for $\rho$ to balance sensitivity and effort.

## 5.2 Case 2: Enhancing Telecommunication Network Efficiency and Resilience Through Hub Domination

The concept of hub domination offers significant advantages in the design and optimization of telecommunication networks. In a crisp-graph model $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$, vertices represent network nodes (servers, routers, or data centers) and edges represent communication links. In the fuzzy extension $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$, membership values enrich this model:

- $\mu(\vartheta_i)$ quantifies processing capacity, cost, or strategic importance of node $\vartheta_i$.
- $\rho(\vartheta_i, \vartheta_j)$ measures bandwidth, reliability, or latency of the link $(\vartheta_i, \vartheta_j)$.

In what follows, we explicitly link the network-design choices to the theoretical results of Sections 2–3. In particular, degree-based bounds and their fuzzy analogs (Theorems 3.15 and 4.7) inform quick screening for small hub sets, while class-wise identities (Theorem 4.3) and connected-hub relations (Theorem 4.8) guide resilience planning.

### 5.2.1 Strategic Hub Placement for Efficient Data Flow

A *hub dominating set* $\mathfrak{D} \subseteq \mathcal{V}$ represents high-capacity communication hubs ensuring that for any two non-hub vertices $u, v \in \mathcal{V} \setminus \mathfrak{D}$, there exists a path between them whose internal vertices lie entirely in $\mathfrak{D}$. In a fuzzy setting, we often minimize the total hub cost $\sum_{\vartheta \in \mathfrak{D}} \mu(\vartheta)$ subject to the hub-domination constraint, thereby operationalizing $\gamma_h(\mathfrak{Z})$. Bounds from Theorem 3.15/4.7 give implementable estimates for required hub budgets.

Figure 8 demonstrates this concept. In this example, $\mathfrak{D} = \{\vartheta_2, \vartheta_4\}$ is a hub dominating set. Even though $\vartheta_1$ and $\vartheta_3$ are not directly connected, they communicate efficiently via hub $\vartheta_2$.

### 5.2.2 Benefits of Hub Domination in Telecommunication Networks

- **Cost Optimization:** Minimizing $\gamma_h(\mathfrak{Z})$ reduces the number (or fuzzy cost) of hub upgrades. If $\mu(\vartheta_i)$ encodes upgrade cost, then $\sum_{\vartheta \in \mathfrak{D}} \mu(\vartheta)$ formalizes infrastructure expenditure. Degree-based bounds ( Theorem 3.15/4.7) provide quick estimates of achievable costs before detailed design.

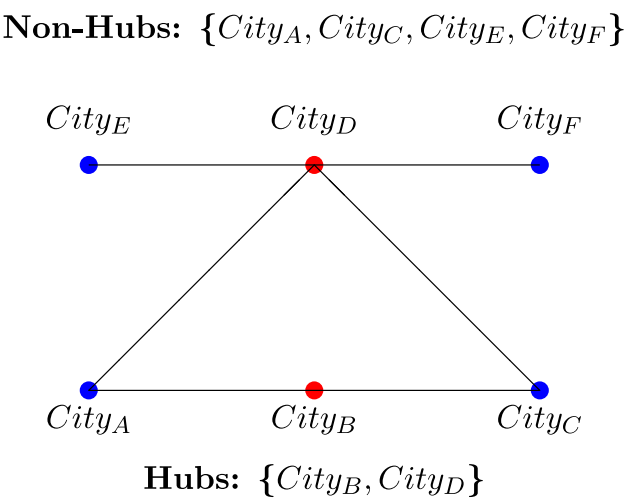


**Fig. 9** Regional transportation network $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$ with *red* hubs $(City_B, City_D)$ and *blue* non-hubs $(City_A, City_C, City_E, City_F)$. The set $\mathfrak{D} = \{City_B, City_D\}$ is a hub dominating set: any two non-hub cities (e.g., $City_A$ and $City_C$) are connected by a path whose internal vertices lie in $\mathfrak{D}$ (e.g., $City_A \to City_B \to City_C$)

- **Resource Efficiency:** Fewer hubs concentrate investment into high-performance routers/data centers, consistent with class-wise minima (Theorem 4.3) in highly structured topologies.
- **Critical Node Identification:** Hub dominating sets highlight nodes whose failure would critically affect connectivity. Connected-hub relations (Theorem 3.18) guide when augmenting a hub set yields a connected dominating core.
- **Enhanced Resilience:** Alternative hub sets can be identified, allowing redundancy planning and robustness against failures. Multiple minimal solutions create standby hub configurations to maintain service under outages.

**Novelty and limitations.** Compared with crisp models, the fuzzy framework captures heterogeneous node capacities and link qualities via $(\mu, \rho)$, yielding cost-aware hub selections and threshold-based designs (cf. Algorithm 3). Limitations include scalability to nation-scale backbones and sensitivity to thresholding; practical mitigation uses greedy/approximation schemes and cross-validation of $\alpha$ against SLAs. Future work will integrate dynamic demand, learning-based link forecasts, and multi-criteria objectives (cost–latency–reliability) to further operationalize hub domination in carrier networks.

### 5.3 Case 3: Strategic Hub Designation for Enhanced Connectivity

In this transportation network context, a *hub dominating set* $\mathfrak{D} \subseteq \mathcal{V}$ represents a set of locations strategically designated as transportation hubs. These hubs are designed to facilitate efficient movement of people and goods between locations within the region. The core property of a hub dominating set ensures that for any two locations $\omega, \vartheta \in \mathcal{V} \setminus \mathfrak{D}$ (locations *not* designated as hubs), there exists at least one path $P_{\omega\vartheta}$ in $\mathfrak{Z}$ connecting $\omega$ and $\vartheta$, such that all intermediate locations along $P_{\omega\vartheta}$ are within the hub set $\mathfrak{D}$. These design choices are consistent with the bounds in Theorem 3.15 (and their fuzzy counterparts in Theorem 4.7), which imply that sparse corridor-like networks typically require few hubs; if a connected backbone is additionally required, Theorem 3.18 (and Theorem 4.7 in the fuzzy setting) shows that at most one additional node (or its membership cost) suffices to ensure connected domination.

Consider a simplified example of a regional transportation network depicted in Fig. 9.

In this example, $\mathfrak{D} = \{City_B, City_D\}$ constitutes a hub dominating set. Cities $B$ and $D$ are designated as transportation hubs, perhaps by investing in enhanced infrastructure, such as major airports, railway terminals, or logistics centers. Even if Cities $A$ and $C$ are not directly connected, transportation between them can be efficiently facilitated via the hub cities (e.g., $City_A \to City_B \to City_C$). When costs $c(\vartheta)$ (or fuzzy memberships $\mu(\vartheta)$) are assigned to hub candidates, the design objective is

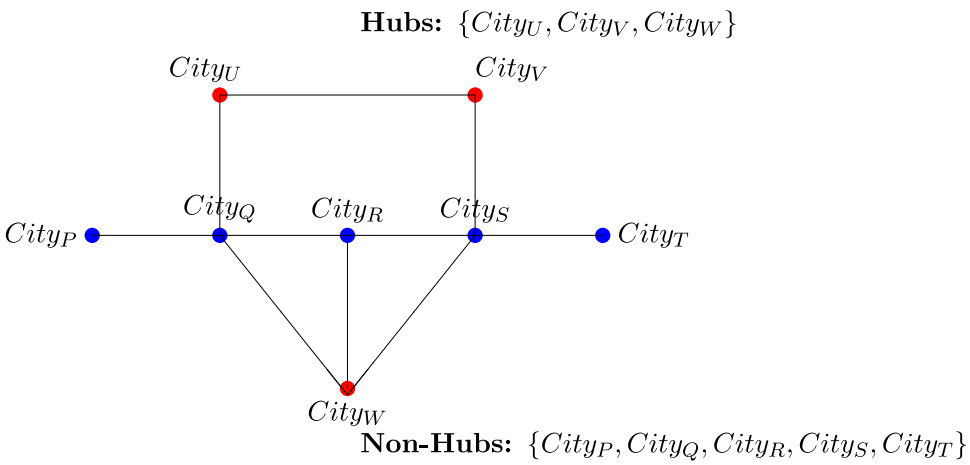


**Fig. 10** Linear-corridor network with bypass hubs ($City_U, City_V, City_W$). The hub set $\mathfrak{D}$ ensures any pair of non-hub cities remains connected via hub-internal paths, reflecting the small hub budgets predicted by degree-based bounds for sparse topologies (Theorem 3.15)

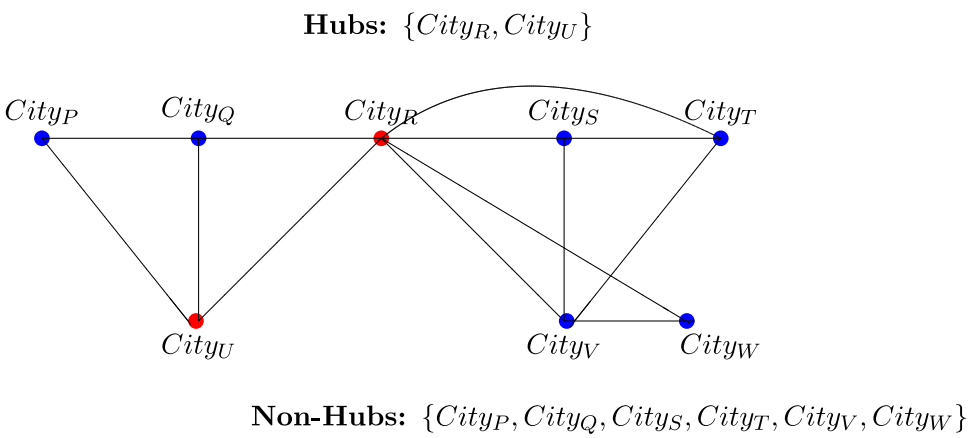


**Fig. 11** Alternative regional network with hub set $\mathfrak{D} = \{City_R, City_U\}$. For any two non-hub cities (e.g., $City_P$ and $City_S$), there exists a path whose internal vertices are hubs (e.g., $City_P \to City_U \to City_R \to City_S$). If a connected backbone is mandated, Theorem 3.18 ensures that augmenting $\mathfrak{D}$ by at most one node yields a connected dominating set

$$\text{Minimize} \quad \sum_{\vartheta \in \mathfrak{D}} c(\vartheta), \tag{1}$$

which aligns with minimizing $\gamma_h(\mathfrak{Z})$ (or its fuzzy analog) subject to service and resilience constraints.

To illustrate corridor-like geographies with bypass options, consider Fig. 10.

A second variant is shown in Fig. 11, emphasizing that multiple minimal hub configurations may exist depending on geography and budget.

In this configuration (Fig. 11), consider any two non-hub cities, for instance, $City_P$ and $City_S$. There exists a path $City_P \to City_U \to City_R \to City_S$. The intermediate cities on this path are $City_U$ and $City_R$, both of which belong to the hub set $\mathfrak{D}$. Similarly, for any other pair of non-hub cities, one can find a path whose internal vertices are hubs (e.g., $City_Q \to City_R \to City_W$). These examples demonstrate that hub dominating sets are not unique; alternative minimal configurations may exist, allowing planners to trade off cost, capacity, and redundancy within the theoretical bounds cited above.

## 5.4 Advantages of Hub Domination in Transportation Network Design

The design insights below follow directly from our degree-based bounds and connected-vs-hub relations (Theorems 3.15, 4.7, and 3.18), which quantify how sparse topologies admit small hub sets and how at most one additional node (or membership cost) can secure a connected backbone.

Employing hub domination principles in transportation network design offers several key advantages:

- **Optimized Infrastructure Investment:** By identifying a minimal hub dominating set, transportation planners can focus infrastructure investments in a limited number of strategically located hubs. This targeted approach is more resource-efficient than attempting to upgrade infrastructure across the entire network. Mathematically, minimizing $\gamma_h(\mathfrak{Z})$ or the fuzzy hub domination number directly addresses cost optimization in hub development. In sparse corridor-like networks, the bounds $\gamma_h(\mathfrak{Z}) \leq p - \delta_N$ and $\gamma_h(\mathfrak{Z}) \leq p - \Delta_N$ imply small hub budgets (Theorem 3.15).
- **Enhanced Regional Accessibility:** Hubs act as central connection points, improving accessibility between all locations within the region, even those not directly connected. For any two non-hub locations $\omega, \vartheta \in \mathcal{V} \setminus \mathfrak{D}$, the existence of a path through hubs in $\mathfrak{D}$ guarantees connectivity. If a connected backbone is required, Theorem 3.18 ensures $\gamma_c(\mathfrak{Z}) \leq \gamma_h(\mathfrak{Z}) + 1$ (crisp) and Theorem 4.8 gives the analogous fuzzy statement.
- **Increased Network Efficiency:** Concentrating traffic flow through strategically positioned hubs can lead to economies of scale and improved network efficiency. Hub locations can be designed to handle high volumes of traffic and facilitate seamless transfers between different modes of transportation. This concentrates capex/opex at high–leverage nodes while keeping $\sum_{\vartheta \in \mathfrak{D}} c(\vartheta)$ (or $\sum_{\vartheta \in \mathfrak{D}} \mu(\vartheta)$) minimal.
- **Improved Network Resilience and Redundancy:** While hubs are critical components, hub domination strategies can also enhance network resilience. By understanding minimal hub dominating sets, planners can identify alternative sets of hubs, providing redundancy and ensuring continued network functionality even if some hubs experience disruptions. Furthermore, the inherent structure of hub-and-spoke networks can offer a degree of robustness to failures in non-hub locations. Evaluating multiple minimal (or near-minimal) hub sets produces contingency playbooks with bounded overhead by the same theorems.

### 5.5 Algorithm for Finding a Minimal Hub Dominating Set

We now give a practical greedy heuristic that aligns with Definition 3.2: it grows a hub set by maximizing, at each step, the number of currently disconnected non-hub pairs that become connected via paths whose internal vertices lie entirely inside the hub set (Fig. 12).

**Explanation of Algorithm** 1:

1. **Initialization (Lines 1–2):** Start with $\mathfrak{D} = \emptyset$ and let $U$ be the set of unordered non-adjacent pairs. These pairs are not yet guaranteed to have a path whose *internal* vertices lie in $\mathfrak{D}$.
2. **Greedy Selection (Lines 3–16):** For each candidate $v$, tentatively form $\mathfrak{D}' = \mathfrak{D} \cup \{v\}$ and compute coverage: the number of pairs $\{u, w\} \in U$ for which there exist hubs $a \in N(u) \cap \mathfrak{D}'$ and $b \in N(w) \cap \mathfrak{D}'$ that are connected within the hub-induced subgraph $H' = \langle \mathfrak{D}' \rangle$. This certifies an $u$–$w$ path with hubs as internal vertices ($u - a \rightsquigarrow b - w$).
3. **Update and Pruning (Lines 17–31):** After adding the best vertex, recompute $U$ by removing pairs now connected via hubs in $H = \langle \mathfrak{D} \rangle$. When $U = \emptyset$, prune $\mathfrak{D}$ for minimality by testing whether any single vertex is redundant.

**Complexity and Variants.** The greedy loop is $O(p) \times O(|U|) \times$ (connectivity checks in $H$), where $p = |\mathcal{V}|$. Using a dynamic connectivity structure or precomputing components of $H$ each iteration yields an overall worst case of $O(p^3)$ for sparse graphs and $O(p^3 + |E|p)$ in general. A cost-aware variant for fuzzy graphs replaces the selection rule by maximizing a benefit–to–cost ratio, e.g., $\Delta\text{coverage}(v)/\mu(v)$ or $\Delta\text{coverage}(v)/c(v)$, to align with minimizing $\sum_{\vartheta \in \mathfrak{D}} \mu(\vartheta)$ (or $c(\vartheta)$).

**Link to Theory and Practice.** The greedy choices are guided by the same structural levers as our bounds: in corridor-like topologies, high-degree candidates rapidly connect many pairs, mirroring the small values predicted by Theorems 3.15 and 4.7. If a connected backbone is mandated, adding at most one more location (Theorems 3.18, 4.8) converts the hub set into a connected dominating set with controlled extra cost.

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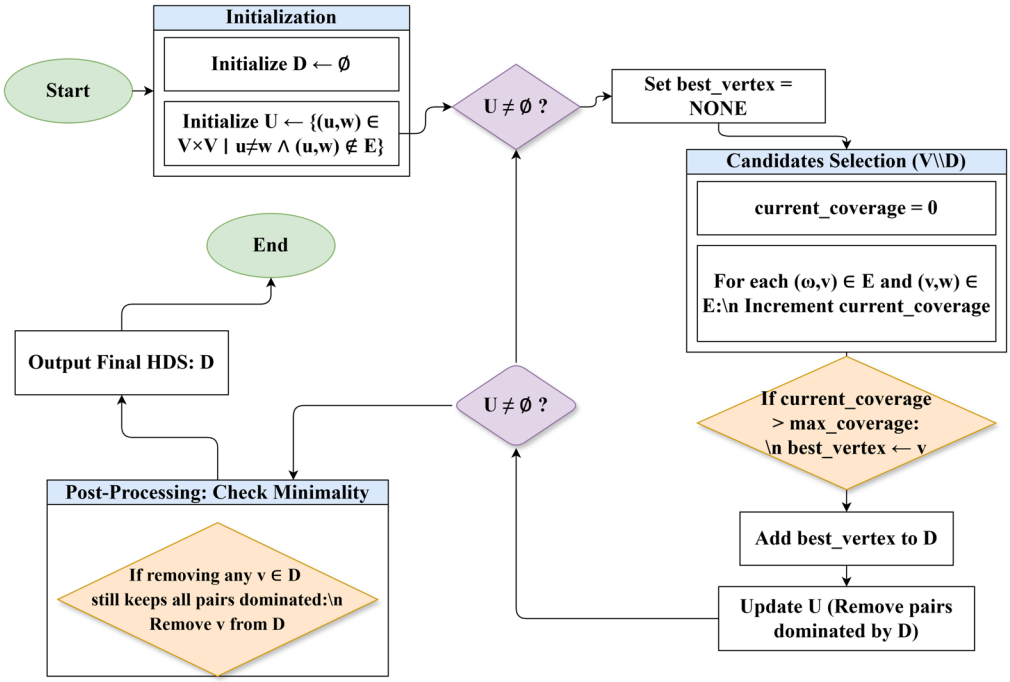


**Fig. 12** Flowchart of the greedy procedure for minimal hub dominating set (HDS) on a crisp graph $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$. At each iteration, the algorithm adds the vertex that connects the largest number of currently disconnected non-hub pairs via hub-internal paths. Symbols: $\mathfrak{D}$ (current hub set), $U$ (set of non-adjacent non-hub pairs not yet connected via hubs), $H = \langle \mathfrak{D} \rangle$ (hub-induced subgraph)

### 5.5.1 Fuzzy Greedy Heuristic and Cost-Aware Selection

We extend the greedy procedure to fuzzy graphs and incorporate a cost-aware selection rule. Throughout, adjacency is determined by positive edge membership: $(x, y) \in \mathcal{E}_\rho \iff \rho(x, y) > 0$. A pair $\{u, w\}$ is *unconnected via hubs* if $\rho(u, w) = 0$ and there is no $u$–$w$ path whose *internal* vertices lie in the current hub set.

**Scoring Rules.** *Coverage* selects the vertex that connects the largest number of currently unserved non-adjacent pairs via hub-internal paths; *ratio* selects by benefit per unit cost, using $c(v)$ (default $c(v) = \mu(v)$) (Table 2).

**Remark.** In budgeted settings, the ratio rule aligns with minimizing $\sum_{\vartheta \in \mathfrak{D}} c(\vartheta)$; when all costs are comparable, coverage often converges faster. Both variants retain the same minimality pruning and are compatible with the connected-backbone upgrade guaranteed by $\gamma_c \leq \gamma_h + 1$.

## 5.6 Algorithm for Finding a Minimal Hub Dominating Set in Fuzzy Graphs

Algorithm 3 presents a greedy heuristic adapted for finding a minimal hub dominating set in a fuzzy graph $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$. We use a thresholded adjacency model: for a fixed $\alpha \in (0, 1]$, an edge is considered present iff $\rho(x, y) \geq \alpha$. The procedure below corrects variable naming, allows *multi-hop* routing inside the hub set (via connectivity of the induced hub subgraph), and enforces minimality by pruning (Fig. 13).

**Input:** Graph $\mathfrak{Z} = (\mathcal{V}, \mathcal{E})$
**Output:** Minimal Hub Dominating Set $\mathfrak{D} \subseteq \mathcal{V}$

1: $\mathfrak{D} \leftarrow \emptyset$
2: $U \leftarrow \{\, \{u, w\} \subset \mathcal{V} : u \neq w,\ (u, w) \notin \mathcal{E} \,\}$ ▷ unordered non-adjacent pairs
3: **while** $U \neq \emptyset$ **do**
4: $best_v \leftarrow$ None, $best_cov \leftarrow 0$
5: **for** each $v \in \mathcal{V} \setminus \mathfrak{D}$ **do**
6: $\mathfrak{D}' \leftarrow \mathfrak{D} \cup \{v\}$, $H' \leftarrow \langle \mathfrak{D}' \rangle$
7: $cov \leftarrow 0$
8: **for** each $\{u, w\} \in U$ **do**
9: $A \leftarrow N(u) \cap \mathfrak{D}'$, $B \leftarrow N(w) \cap \mathfrak{D}'$
10: **if** $A \neq \emptyset$ and $B \neq \emptyset$ and some $a \in A$ is connected to some $b \in B$ in $H'$ **then**
11: $cov \leftarrow cov + 1$
12: **end if**
13: **end for**
14: **if** $cov > best_cov$ **then**
15: $best_cov \leftarrow cov$, $best_v \leftarrow v$
16: **end if**
17: **end for**
18: $\mathfrak{D} \leftarrow \mathfrak{D} \cup \{best_v\}$
19: $H \leftarrow \langle \mathfrak{D} \rangle$
20: $U \leftarrow \{\{u, w\} \in U : \neg\exists a \in N(u) \cap \mathfrak{D}, b \in N(w) \cap \mathfrak{D}$ with $a$ connected to $b$ in $H\}$
21: **end while**
22: **for** each $x \in \mathfrak{D}$ **do** ▷ prune for minimality
23: $\mathfrak{D}_{tmp} \leftarrow \mathfrak{D} \setminus \{x\}$, $H_{tmp} \leftarrow \langle \mathfrak{D}_{tmp} \rangle$
24: $ok \leftarrow$ True
25: **for** each unordered $\{u, w\} \subseteq \mathcal{V} \setminus \mathfrak{D}_{tmp}$ with $(u, w) \notin \mathcal{E}$ **do**
26: $A \leftarrow N(u) \cap \mathfrak{D}_{tmp}$, $B \leftarrow N(w) \cap \mathfrak{D}_{tmp}$
27: **if** $A = \emptyset$ or $B = \emptyset$ or no $a \in A$ connected to any $b \in B$ in $H_{tmp}$ **then**
28: $ok \leftarrow$ False; **break**
29: **end if**
30: **end for**
31: **if** $ok$ **then**
32: $\mathfrak{D} \leftarrow \mathfrak{D}_{tmp}$
33: **end if**
34: **end for**
35: **return** $\mathfrak{D}$

**Algorithm 1:** Greedy Heuristic for Minimal Hub Domination Set in Crisp Graphs (HDS)

Here, $N_\alpha(u) = \{y \in \mathcal{V} : \rho(u, y) \geq \alpha\}$. The coverage test (Lines 5–7) and the update of $U$ (Line 11) allow *multi-hop* hub routing by checking connectivity inside the induced hub subgraph, not just single-hub ($u$–$h$–$w$) paths. Setting $\alpha = 1$ recovers the crisp-graph heuristic. A cost-aware variant is given in Algorithm 2.
**Explanation.** The algorithm mirrors the crisp greedy strategy with three fuzzy-specific adjustments:

1. *Thresholded Non-adjacency:* Pairs with $\rho(u, w) < \alpha$ enter $U$ and must be connected via hub-internal paths.

 https://doi.org/10.1007/s44196-025-00991-5

**Input:** Fuzzy graph $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$; optional cost $c : \mathcal{V} \to \mathbb{R}_{>0}$ (default $c(v) = \mu(v)$); scoring rule $s \in \{\mathsf{coverage},\ \mathsf{ratio}\}$
**Output:** Minimal fuzzy hub dominating set $\mathfrak{D} \subseteq \mathcal{V}$

$\mathfrak{D} \leftarrow \emptyset$; $U \leftarrow \{\{u, w\} \subset \mathcal{V} :\ u \neq w,\ \rho(u, w) = 0\}$
**while** $U \neq \emptyset$ **do**
  $best_v \leftarrow$ None; $best_score \leftarrow -\infty$
  **for** each $v \in \mathcal{V} \setminus \mathfrak{D}$ **do**
    $\mathfrak{D}' \leftarrow \mathfrak{D} \cup \{v\}$; $H' \leftarrow \langle \mathfrak{D}' \rangle$ (edges with $\rho > 0$)
    $cov \leftarrow |\{\{u, w\} \in U :\ \exists a \in N_\rho(u) \cap \mathfrak{D}',\ \exists b \in N_\rho(w) \cap \mathfrak{D}'$ s.t. $a$ and $b$ are connected in $H'\}|$
    $score \leftarrow \begin{cases} cov, & s = \mathsf{coverage} \\ \dfrac{cov}{c(v)}, & s = \mathsf{ratio} \end{cases}$
    **if** $score > best_score$ **then**
      $best_score \leftarrow score$; $best_v \leftarrow v$
    **end if**
  **end for**
  $\mathfrak{D} \leftarrow \mathfrak{D} \cup \{best_v\}$; $H \leftarrow \langle \mathfrak{D} \rangle$
  $U \leftarrow \{\{u, w\} \in U :\ \neg\exists a \in N_\rho(u) \cap \mathfrak{D},\ \neg\exists b \in N_\rho(w) \cap \mathfrak{D}$ with $a$ connected to $b$ in $H\}$
**end while**
**for** each $x \in \mathfrak{D}$ **do** ▷ prune for minimality
  $\mathfrak{D}_{tmp} \leftarrow \mathfrak{D} \setminus \{x\}$; $H_{tmp} \leftarrow \langle \mathfrak{D}_{tmp} \rangle$; $ok \leftarrow$ True
  **for** each $\{u, w\} \subseteq \mathcal{V} \setminus \mathfrak{D}_{tmp}$ with $\rho(u, w) = 0$ **do**
    $A \leftarrow N_\rho(u) \cap \mathfrak{D}_{tmp}$; $B \leftarrow N_\rho(w) \cap \mathfrak{D}_{tmp}$
    **if** $A = \emptyset$ or $B = \emptyset$ or no $a \in A$ connected to any $b \in B$ in $H_{tmp}$ **then**
      $ok \leftarrow$ False; **break**
    **end if**
  **end for**
  **if** $ok$ **then**
    $\mathfrak{D} \leftarrow \mathfrak{D}_{tmp}$
  **end if**
**end for**
**return** $\mathfrak{D}$

**Algorithm 2:** Greedy Heuristic for Minimal Fuzzy Hub Dominating Set (Fuzzy–HDS)

2. *Hub-Internal Connectivity:* We require the endpoints' hub-neighbors to lie in the same connected component of the hub-induced subgraph (edges $\rho \geq \alpha$), thereby permitting multi-hop hub routing.
3. *Minimality Pruning:* Any redundant hub is removed while preserving all hub-mediated connections for pairs in $U$.

**Complexity and Practical Notes.** A naive implementation that recomputes connectivity per candidate has worst case $O(p^3)$ per outer iteration (with $p = |\mathcal{V}|$). Using a dynamic union—find over the hub-induced subgraph reduces checks to near-linear per update within each iteration. Higher $\alpha$ typically increases $|\mathfrak{D}|$ (stricter links), while lower $\alpha$ reduces it (more links count). In budgeted settings, select candidates by coverage-per-cost as in Algorithm 2.

**Table 2** Illustrative divergence between selection rules on a small fuzzy network

| Step | Candidate $v$ | $cov(v)$ | $\mu(v)$ | $cov(v)/\mu(v)$ |
|---|---|---|---|---|
| 1 | $a$ | 5 | 0.7 | 7.14 |
| | $b$ | 4 | 0.4 | 10.00 |
| | $c$ | 3 | 0.2 | 15.00 |
| | $d$ | 6 | 0.9 | 6.67 |
| | $e$ | 2 | 0.3 | 6.67 |
| Pick | coverage: $d$ | | ratio: $c$ | |
| 2 | $a$ | 3 | 0.7 | 4.29 |
| | $b$ | 3 | 0.4 | 7.50 |
| | $d/c$ | – | – | – |
| | $e$ | 2 | 0.3 | 6.67 |
| Pick | coverage: $a$ or $b$ (tie) | | ratio: $b$ | |

$cov(v)$ counts the number of currently unserved unordered non-adjacent pairs $\{u, w\}$ that become connected via hub-internal paths if $v$ is added. Costs default to $\mu(v)$. At Step 1, coverage picks $d$ (largest absolute coverage), while ratio picks $c$ (best benefit per unit cost). Values are illustrative but consistent with the scoring definitions

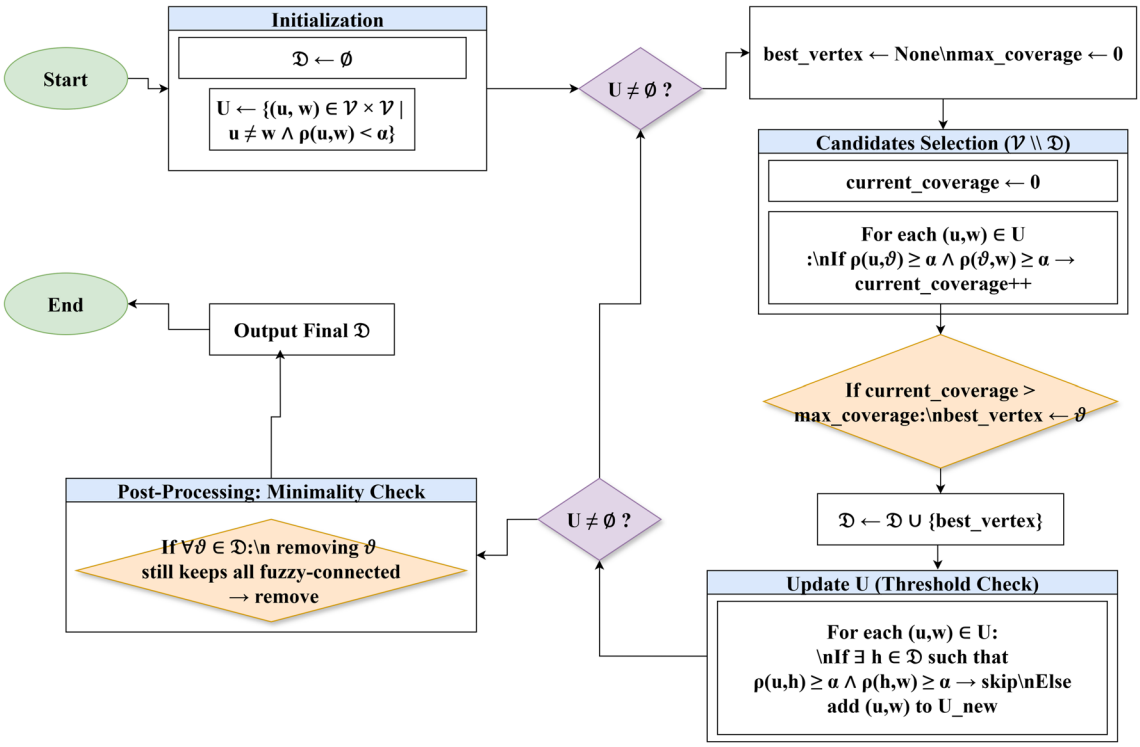


**Fig. 13** Flowchart of minimal hub domination set in fuzzy graphs

## Conclusion

This paper develops a unified framework for *hub domination* in both crisp and fuzzy graphs. We formalized hub and total hub domination, proved exact values for principal graph families (complete, cycle, star, wheel, and complete bipartite), and established consolidated bounds that relate $\gamma_h$ to classical invariants (e.g., independence and chromatic numbers) and degree parameters (via $\Delta_N$, $\Delta_E$, $\delta_N$, $\delta_E$). We further clarified structural links between hub, connected, and total versions (e.g., $\gamma_{hc} \leq \gamma_c$, $\gamma_h \leq \gamma_{ht}$), and gave complementary results on complements and extremal cases (paths).

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**Input:** Fuzzy graph $\mathfrak{Z} = (\mathcal{V}, \mu, \rho)$, threshold $\alpha \in (0, 1]$
**Output:** Minimal hub dominating set $\mathfrak{D} \subseteq \mathcal{V}$

$\mathfrak{D} \leftarrow \emptyset$; $U \leftarrow \{\{u, w\} \subset \mathcal{V} : u \neq w,\ \rho(u, w) < \alpha\}$ ▷ unordered non-adjacent pairs (at level $\alpha$)
**while** $U \neq \emptyset$ **do**
  $best_v \leftarrow$ None; $best_cov \leftarrow -\infty$
  **for** each $v \in \mathcal{V} \setminus \mathfrak{D}$ **do**
    $\mathfrak{D}' \leftarrow \mathfrak{D} \cup \{v\}$; $H' \leftarrow$ subgraph induced by $\mathfrak{D}'$ with edges $\{x, y\}$ iff $\rho(x, y) \geq \alpha$
    $cov(v) \leftarrow |\{\{u, w\} \in U :\ \exists a \in \mathfrak{D}' \cap N_\alpha(u),\ \exists b \in \mathfrak{D}' \cap N_\alpha(w)$ s.t. $a$ is connected to $b$ in $H'\}|$
    **if** $cov(v) > best_cov$ **then**
      $best_cov \leftarrow cov(v)$; $best_v \leftarrow v$
    **end if**
  **end for**
  **if** $best_v =$ None **then**
    **break**
  **end if**
  $\mathfrak{D} \leftarrow \mathfrak{D} \cup \{best_v\}$; $H \leftarrow$ subgraph induced by $\mathfrak{D}$ with edges $\rho \geq \alpha$
  $U \leftarrow \{\{u, w\} \in U :\ \neg(\exists a \in \mathfrak{D} \cap N_\alpha(u),\ \exists b \in \mathfrak{D} \cap N_\alpha(w)$ with $a$ connected to $b$ in $H)\}$
**end while**
**for** each $x \in \mathfrak{D}$ **do** ▷ prune to ensure minimality
  $\mathfrak{D}_{tmp} \leftarrow \mathfrak{D} \setminus \{x\}$; $H_{tmp} \leftarrow$ subgraph induced by $\mathfrak{D}_{tmp}$ with edges $\rho \geq \alpha$; $ok \leftarrow$ True
  **for** each $\{u, w\} \subseteq \mathcal{V} \setminus \mathfrak{D}_{tmp}$ with $\rho(u, w) < \alpha$ **do**
    $A \leftarrow \mathfrak{D}_{tmp} \cap N_\alpha(u)$; $B \leftarrow \mathfrak{D}_{tmp} \cap N_\alpha(w)$
    **if** $A = \emptyset$ or $B = \emptyset$ or no $a \in A$ is connected to any $b \in B$ in $H_{tmp}$ **then**
      $ok \leftarrow$ False; **break**
    **end if**
  **end for**
  **if** $ok$ **then**
    $\mathfrak{D} \leftarrow \mathfrak{D}_{tmp}$
  **end if**
**end for**
**return** $\mathfrak{D}$

**Algorithm 3:** Greedy Heuristic for Minimal Hub Dominating Set in Fuzzy Graphs (thresholded)

For applications, we connected these results to two practical settings: (i) telecommunication (including fraud-team networks) and (ii) regional transportation. In both, minimal (fuzzy) hub dominating sets support resource allocation, cost-aware infrastructure planning, and resilience analysis. The proposed greedy heuristics for crisp and fuzzy graphs (Algorithms 1 and 3) provide implementable procedures, while the fuzzy variant highlights how thresholding and membership weights enable optimization under uncertainty. Together, these contributions strengthen the theoretical basis of hub domination and demonstrate its utility for network design, monitoring, and intervention.

## Future Work

Several directions follow naturally from our results:

- **Dynamic and Directed Networks:** Extend $\gamma_h$ and $\gamma_{ht}$ to time-varying and directed graphs, including temporal/failure-robust formulations.
- **Large-scale computation:** Develop scalable approximation and decomposition methods for minimal (fuzzy) hub dominating sets on massive graphs; analyze worst-case and average-case performance of the greedy schemes.
- **Fuzzy Modeling Choices:** Study sensitivity to the edge-threshold $\alpha$ and to alternative path semantics (e.g., different $t$-norms/aggregators beyond min), including data-driven estimation of $\mu$ and $\rho$.
- **Comparative Analysis:** Benchmark hub domination against related variants (connected/independent domination) and fuzzy topological indices to delineate when each is most informative.
- **Broader Applications:** Apply the framework to social, supply-chain, and critical-infrastructure networks, incorporating multi-objective costs (e.g., capacity, risk, equity) and domain constraints.

Addressing these problems will further mature the theory and expand the practical impact of hub domination in complex, uncertain networked systems.

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**Conflict of interest** The authors declare no conflict of interest.

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