A DIFFERENTIAL CALCULUS ON THE SPACE IR_h^3

Assoc. Prof. Salih CELIK

Yildiz Technical University

ORCID NO: 0000-0002-6590-1032

Assis. Prof. Ozgur OKSUZ

Adiyaman University

ORCID NO: 0000-0001-5568-6116

ABSTRACT

This work is about an h-deformation of coordinate functions on the 3-dimensional space IR^3 and its geometry. Moreover, the geometry is going to be identified on this space.

In this work, we will denote by $F(IR^3)$ the $q-$ deformation of the coordinate functions on IR^3 satisfying the $q-$deformed commutation relations:

$$xy = qyx;\ \ yz = qzy;\ \ xz = qzx.$$

It is obvious that, the algebra of coordinate functions on IR^3 is going to be obtained by the limit when q goes to 1.

In this work, the quantum space IR_q^3 has been taken into account: using $q-$deformation of IR^3, it has been transformed to h – deformation, denoted by IR_h^3, by defining a singular g matrix that its elements consist of q and h. As a technical term, we can call it as a contraction. As a result, we have a quantum space $IR_{q,h}^3$ that its elements satisfy $q-$ and $h-$ deformed commutation relations. Then, we have obtained algebraic structures of two-parameter quantum space $IR_{q,h}^3$. We see that even the algebraic structure obtained in the first step is a singular in the limit q = 1. However, the singularity has been eliminated by using the algebraic structure of IR_q^3. By this method, we can obtain the $h-$deformed commutation relations:

$$xy = yx;\ \ yz = zy;\ \ xz = zx + hz^2.$$

It is clear that when we take the limit at h=0, we get the classical commutative algebraic relations on space IR^3.

Then, by the help these new h-deformed commutation relations, a differential calculus has been made towards to $h-$ deformation of three dimensional quantum space.

Keywords: q – deformation, the quantum space IR^3, h – deformation, the quantum space IR_h^3, differential algebra, non-commutative geometry.

1. INTRODUCTION

The theory of quantum groups has an important role in mathematical physics. Quantum groups is a genaralization of group concept. More specifically, a quantum group is a deformation of a group that it has a deformation parameter.

Quantum group terminology was mentioned by V. Drinfeld (Drinfeld, 1986) for the first time. Then, many researchers have looked it in a different perspective. Non-commutative differential geometry of quantum groups was introduced by Woronowicz (Woronowicz S. , 1987), and (Woronowicz S. L., 1989). Another improvement on quantum groups was made by Wess and Zumino (Wess & Zumino, 1991). Manin (Manin, 1988) introduced the primary structures of quantum space.

A quantum group is a deformation of a classical group. $h-$ deformation is also known as Jordanian deformations. Aghamohammadi and his friends (Aghamohammadi, Khorrami, & Shariati, 1995), using plane's q-deformation, they obtained h-deformation by the help of a singular g matrix. Later, many studies were conducted on this subject such as (Salih Celik, 1997), (Celik & Celik, 1998), (Celik & Celik, 1999), (Salih Celik, 2001), (Bulut & Celik, 2017), and (Salih Celik, 2020).

The aim of this study, using the q-deformation of the coordinate functions on the 3-dimensional space, with a singular transformation, is to develop the h-space and set a non-commutative differential calculus on it. Therefore, first a 3x3 singular matrix will be determined. Then, other steps are about differential calculus with the approach of Wess and Zumino.

Throughout this paper, there are prime and non-prime quantities. We denote q - deformed objects by the primed quantities. Unprimed quantities represent the transformed coordinates.

2. h –DEFORMATION OF SPACE IR^3

2.1 The Algebra $F(IR_q^3)$

Before delving into $h-$ deformation of space IR , we would like to introduce three dimensional quantum space.

A three dimensional quantum space was defined by Manin (Manin, 1988). Let $x', y',$ and z' be the coordinate functions over $\mathbb{C}$ (complex number set) and $q \in \mathbb{C} - \{0\}$, then we have

$$x'y' = qy'x';\ \ y'z' = qz'y';\ \ x'z' = qz'x'. \tag{2.1}$$

With these algebraic relations, a three dimensional space is a quantum three dimensional space. Moreover, it is represented as IR_q^3. $A_q = F(IR_q^3)$ is associative algebra that satisfies the relations at (2.1) and the generators of this algebra are $x', y',$ and z'. This associative algebra (over C) is known as algebra of polynomials over three dimensional quantum space. This algebra is commutative when the limit at $q = 1$. Moreover, it is known as algebra of polynomials $\mathbb{C}[x', y', z']$ over classical three dimensional space.

In A_q algebra, the deformation parameter is q. In this section, we want to find a new algebra using $q-$ commutation relations at (2.1). As a result, we have a new deformation parameter which we show it as h.

Now, we would like to explain the idea for the contraction as follows:

$X' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \in IR_q$, where x', y', z' satisfy the relations at (3.1). *Let* g be the 3x3 matrix which its elements consist of q and h.

Let $g = \begin{pmatrix} 1 & A & B \\ 0 & 1 & C \\ 0 & 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be, we can write

$$X' = g_{q,h}X. \tag{2.2}$$

Then we have,

$$x = x' - Ay' + (A\,C - B)z'$$
$$y = y' - Cz' \qquad (2.3)$$
$$z = z'$$

We have different options for elements A, B, and, C. These are:

(1) For $A = \frac{h}{q-1}. B = 1, C = 0$, $g = \begin{pmatrix} 1 & \frac{h}{q-1} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, we have

$$x = x' - \frac{h}{q-1}y' - z'$$
$$y = y'$$
$$z = z'$$

(2) For $B = \frac{h}{q-1}. A = C = 0$, $g = \begin{pmatrix} 1 & 0 & \frac{h}{q-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, we have

$$x = x' - \frac{h}{q-1}z'$$
$$y = y'$$
$$z = z'$$

(3) For $A = \frac{h}{q-1}. B = C = 0$, $g = \begin{pmatrix} 1 & \frac{h}{q-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, we have

$$x = x' - \frac{h}{q-1}y'$$
$$y = y'$$
$$z = z'$$

(4) For $C = \frac{h}{q-1}. A = B = 0$, $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{h}{q-1} \\ 0 & 0 & 1 \end{pmatrix}$. Then, we have

$$x = x'$$
$$y = y' - \frac{h}{q-1}z' \qquad (2.4)$$
$$z = z'$$

Thus, we have completed the first step to $h-$ deformation.

A note that every choice of matrix g is singular in the limit at $q = 1$. However, we will show that this singularity is going to be eliminated by the help of a tranformation. We will show this next subsection.

2.2 The Algebra $F(IR_h^3)$

We choose to use the last g matrix (the fourth one) for future calculations. The other ones will be examined later. We leave this as a future work.

With this g matrix, we are going to have a new algebra which it elements consist of q and h.

For example,

$$\begin{aligned} xy &= x'\left(y' - \frac{h}{q-1}z'\right) \\ &= x'y' - \tfrac{h}{q-1}x'z' \\ &= qy'x' - \tfrac{hq}{q-1}z'x' \\ &= q\left(y' - \tfrac{h}{q-1}z'\right)x' \\ &= qyx. \end{aligned}$$

The other relations can be found similarly. Then, we have the following relations:

$$\begin{aligned} xz &= qzx \\ xy &= qyx \\ yz &= qzy + hz^2. \end{aligned} \tag{2.5}$$

In (2.5), when we take the limit at q=1, we get

$$\begin{aligned} xz &= zx \\ xy &= yx \\ yz &= zy + hz^2, \end{aligned} \tag{2.6}$$

which these relations define h - space. As a note that in the limit at h=0, everything is going to be classic. We show the algebra in (2.6) as A_h.

In next section, we will build a new differential structure on space IR_h using the relations above.

3. DIFFERENTIAL ALGEBRA

In this section, our aim is to build a non-commutative diferential geometry on quantum space IR_h. Inside of this differential geometric structure, there are coordinate functions, and their differentials, and correlations between the coordinate functions with their diferentials. We first give the definition of operator d.

d is an operator that maps an algebra's elements to its differentials.

$d: u \to du$, where $u \in \{x, y, z\}$.

$d: A \to \Gamma$, $du \in \Gamma$, where Γ is a differential algebra that consists of the generators of algebra A's differentials.

One of the most important properites of differential calculus is Liebniz rule.

$$d(fg) = (df)g + (-1)^{\hat{f}} f(dg), \tag{3.1}$$

where f and g are two differentiable functions, and $\hat{f}$ is the Grassmann degree of f.

$$\hat{f} = \begin{cases} 1, & when\ f\ is\ odd\ variable \\ 0, & when\ f\ is\ even\ variable \end{cases}$$

In this paper, the degree of a coordinate function is 0 while the degree of a coordinate functions' differential is 1. Moreover,

$$d^2 = 0, \tag{3.2}$$

which d is a nilpotent operator.

3.1 q −Differential Algebra

From (Celik & Yasar, 2006), the differential relations of space IR_q are the followings:

$$\begin{aligned} &x'dx' = dx'x', x'dy' = qdy'x', x'dz' = qdz'x' \\ &y'dx' = q^{-1}dx'y', y'dy' = dy'y', y'dz' = qdz'y' \\ &z'dx' = q^{-1}dx'z', z'dy' = q^{-1}dy'z', z'dz' = dz'z'. \end{aligned} \tag{3.3}$$

3.1 h −Differential Algebra

We are going to compute differentials of (2.4), and we get

$$dx = dx', dy = dy' - \frac{h}{q-1}dz', dz = dz'. \tag{3.4}$$

Then, we apply (2.4) and (3.4) to (3.3), we obtain

$$\begin{aligned} &xdx = dxx, xdy = qdyx, xdz = qdzx \\ &ydx = q^{-1}dxy, ydy = dyy - hdzy + q^{-1}hdyz - q^{-1}h^2dzz, \\ &ydz = qdzy + hdzz,\ zdx = q^{-1}dxz, zdy = q^{-1}dyz - q^{-1}hdzz, zdz = dzz. \end{aligned} \tag{3.5}$$

We give an example how we obtain these relations:

$$\begin{aligned} ydz &= (y' - \frac{h}{q-1}z')dz' \\ &= y'dz' - \tfrac{h}{q-1}z'dz' \\ &= qdz'y' - \frac{h}{q-1}dz'z' \\ &= qdz(y + \frac{h}{q-1}z) - \frac{h}{q-1}dzz \\ &= qdzy + \frac{qh}{q-1}dzz - \frac{h}{q-1}dzz \\ &= qdzy + h(\frac{q}{q-1} - \frac{1}{q-1})dzz \\ &= qdzy + hdzz \end{aligned}$$

when we take the limit at $q = 1$, then we get

$$\begin{aligned} &xdx = dxx, xdy = dyx, xdz = dzx \\ &ydx = dxy, ydy = dyy - hdzy + hdyz - h^2dzz, ydz = dzy + hdzz \\ &zdx = dxz, zdy = dyz - hdzz, zdz = dzz. \end{aligned} \tag{3.6}$$

The realtions in (3.6) are $h-$ deformed relations. We represent the new algebra in (3.6) as Γ^1.

Now, we apply d operator to (3.6), then we obtain

$$dx \wedge dx = 0, dx \wedge dy = -qdy \wedge dx,$$
$$dy \wedge dy = hdz \wedge dy, dx \wedge dz = -qdz \wedge dx \quad (3.7)$$
$$dy \wedge dz = -qdz \wedge dy, dz \wedge dz = 0.$$

We show how we get these equalities with an example below.

$$ydz = qdzy + hdzz$$
$$d(ydz) = d(qdzy + hdzz)$$
$$dy \wedge dz = -qdz \wedge dy$$

when we take the limit at q=1, we get

$$dx \wedge dx = 0, dx \wedge dy = -dy \wedge dx,$$
$$dy \wedge dy = hdz \wedge dy, dx \wedge dz = -dz \wedge dx, \quad (3.8)$$
$$dy \wedge dz = -dz \wedge dy, dz \wedge dz = 0.$$

We show the algebra in (3.8) as Γ^2. As a result, we can write

$\Gamma = \Gamma^0 \cup \Gamma^1 \cup \Gamma^2$, where $\Gamma^0 \cong A_h$.

4. RELATIONS OF PARTIAL DERIVATIVES

In the previous sections, we have mentioned that x, y, z are coordinate functions. In this section, we introduce the commutation relations between the coordinates of IR_h quantum three dimensional space and their partial derivatives.

In the classical analysis, we know that differential operator d can be written as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz, \quad (4.1)$$

This equality in (4.1) is written in $q-$ deformation theory

$$d'f = (dx'\partial_{x'} + dy'\partial_{y'} + dz'\partial_{z'})f \text{ where } \partial_{u'} = \frac{\partial}{\partial u}. \quad (4.2)$$

This is because the algebra that we are working on is not commutative.

From the chain rule, we have

$$\partial_x = \partial_{x'} \ \partial_y = \partial_{y'} \ \partial_z = \partial_{z'} + \frac{h}{q-1}\partial_y. \quad (4.3)$$

As an important note that ∂_z is singular when we take the limit at $q = 1$.

Now we use the following relations (4.2), (4.3) and (3.4) all together, we obtain

$$d'f = (dx'\partial_{x'} + dy'\partial_{y'} + +dz'\partial_{z'})f$$
$$= (dx\partial_x f + (dy + \frac{h}{q-1}dz)\partial_y f + dz(\partial_z - \frac{h}{q-1}\partial_{y)})f$$
$$= (dx\partial_x f + dy\partial_y f + \frac{h}{q-1}dz\partial_y f + dz\partial_z f - \frac{h}{q-1}dz\partial_y f)$$
$$= (dx\partial_x + dy\partial_y + dz\partial_z) = df. \quad (4.4)$$

Thus, we can say that operator d satisfies the same equation for $h-$ deformation.

We now wish to obtain the relations of partial derivatives with coordinate functions. Replacing f with xf in (4.4), and applying (3.6), we have

$$\begin{aligned} d(xf) &= (dx)f + x(df) = (dx)f + x(dx\partial_x + dy\partial_y + dz\partial_z)f \\ &= (dx)f + (xdx)\partial_x f + (xdy)\partial_y f + (xdz)\partial_z f \\ &= (dx)f + (dxx\partial_x)f + (dyx\partial_y)f + (dzx\partial_z)f \\ &= dx(1 + x\partial_x)f + dy(x\partial_y)f + dz(x\partial_z)f. \end{aligned}$$

In addition, on the left hand side of the equality, we have

$$d(xf) = \big(dx\partial_x x + dy\partial_y x + dz\partial_z x\big)f.$$

Thus we obtain

$$\begin{gathered} \partial_x x = 1 + x\partial_x, \partial_x y = y\partial_x, \partial_x z = z\partial_x \\ \partial_y x = x\partial_y, \partial_y y = 1 + y\partial_y + hz\partial_y, \partial_y z = z\partial_y \\ \partial_z x = x\partial_z, \partial_z y = y\partial_z - h\big(y\partial_y + hz\partial_y - z\partial_z\big), \partial_z z = 1 + z\partial_z - hz\partial_y. \end{gathered} \tag{4.5}$$

In order to obtain the h-deformed relations between partial derivatives, we have to use the nilpotency of the operator d, we obtain following relations:

$$\partial_x\partial_y = \partial_y\partial_x, \partial_x\partial_z = \partial_z\partial_x, \partial_y\partial_z = \partial_z\partial_y - h\partial_y\partial_y. \tag{4.6}$$

This is because

$$\begin{aligned} 0 = d^2 f &= d\big(dx\partial_x + dy\partial_y + dz\partial_z\big)f = \big(-dxd\partial_x - dyd\partial_y - dzd\partial_z\big)f \\ &= [-dx\big(dx\partial_x + dy\partial_y + dz\partial_z\big)\partial_x - dy\big(dx\partial_x + dy\partial_y + dz\partial_z\big)\partial_y - dz\big(dx\partial_x + dy\partial_y + dz\partial_z\big)\partial_z]f \\ &= [-dxdy\partial_y\partial_x - dxdz\partial_z\partial_x - dydx\partial_x\partial_y - dydy\partial_y\partial_y - dydz\partial_z\partial_y - dzdx\partial_x\partial_z - dzdy\partial_y\partial_z]f \\ &= \big[-dxdy\partial_y\partial_x - dxdz\partial_z\partial_x + dxdy\partial_x\partial_y + hdydz\partial_y\partial_y - dydz\partial_z\partial_y + dxdz\partial_x\partial_z + dydz\partial_y\partial_z\big]f \\ &= [dxdy\big(-\partial_y\partial_x + \partial_x\partial_y\big) + dxdz(-\partial_z\partial_x + \partial_x\partial_z) + dydz\big(h\partial_y\partial_y - \partial_z\partial_y + \partial_y\partial_z\big]f. \end{aligned}$$

$d^2 = 0$ if and only if (4.6) exists.

Then we have a new algebra ¥ that its generators $\partial_x, \partial_y, \partial_z$ and it satisfies the relations in (4.5) and (4.6).

5. RELATIONS BETWEEN DIFFERENTIALS AND DERIVATIVES

In this section, we would like to connect algebra ¥ and Γ. In other words, we compute the commutation relations between differentials and derivatives.

From (Celik & Yasar, 2006), the commutation relations between differentials and derivatives of space IR_q^3 are the followings:

$$\begin{gathered} \partial_x, dx' = dx'\partial_{x'}, \partial_{x'}dy' = q^{-1}dy'\partial_{x'}, \partial_{x'}dz' = q^{-1}dz'\partial_{x'} \\ \partial_{y'}dx' = qdx'\partial_{y'}, \partial_{y'}dy' = dy'\partial_{y'}, \partial_{y'}dz' = q^{-1}dz'\partial_{y'} \\ \partial_{z'}dx' = qdx'\partial_{z'}, \partial_{z'}dy' = qdy'\partial_{z'}, \partial_{z'}dz' = dz'\partial_{z'}. \end{gathered} \tag{5.1}$$

Now, we use these relations with relations in (3.4) and (4.3), we get (after the limit at q=1)

$$\partial_x dx = dx\partial_x, \partial_x dy = dy\partial_x, \partial_x dz = dz\partial_x$$
$$\partial_y dx = dx\partial_y, \partial_y dy = dy\partial_y + hdz\partial_y, \partial_y dz = dz\partial_y \quad (5.2)$$
$$\partial_z dx = dx\partial_z, \partial_z dy = dy\partial_z - hdy\partial_y + hdz\partial_z - h^2 dz\partial_y, \partial_z dz = dz\partial_z - hdz\partial_y.$$

We show how we get the relations in (5.2) giving an example below:

$$\partial_y dy = \partial_{y'}(dy' - \frac{h}{q-1}dz')$$
$$= \partial_{y'}dy' - \frac{h}{q-1}\partial_{y'}dz'$$
$$= dy'\partial_{y'} - \frac{q^{-1}h}{q-1}dz'\partial_{y'} - \frac{h}{q-1}dz'\partial_{y'} + \frac{h}{q-1}dz'\partial_{y'}$$
$$= (dy' - \frac{h}{q-1}dz')\partial_{y'} + \frac{h}{q-1}(1-q^{-1})dz'\partial_{y'}$$
$$= (dy' - \frac{h}{q-1}dz')\partial_{y'} + \frac{q^{-1}h}{q-1}(q-1)dz'\partial_{y'}$$
$$= dy\partial_y + q^{-1}hdz\partial_y$$

When we take the limit at $q = 1$, we get

$\partial_y dy = dy\partial_y + hdz\partial_y$.

At last, we obtain the relations between partial derivatives and operator d using relations in (4.6) and (5.2) as follows:

$$\partial_x d = \partial_x(dx\partial_x + dy\partial_y + dz\partial_z)$$
$$= dx\partial_x^2 + dy\partial_x\partial_y + dz\partial_x\partial_z$$
$$= dx\partial_x^2 + dy\partial_y\partial_x + dz\partial_z\partial_x$$
$$= (dx\partial_x + dy\partial_y + dz\partial_z)\partial_x$$
$$= d\partial_x.$$

We can find other relations similarly. We get all relations as

$$\partial_x d = d\partial_x, \partial_y d = d\partial_y, \partial_z d = d\partial_z. \quad (5.3)$$

As an important note that these relations in (5.3) show that d is a nilpotent operatör:

$$d^2 f = d\big(dx\partial_x + dy\partial_y + dz\partial_z\big)f$$
$$= -dxd\partial_x f - dyd\partial_y f - dzd\partial_z f$$
$$= -dx\partial_x df - dy\partial_y df - dz\partial_z df$$
$$= -(dx\partial_x + dy\partial_y + dz\partial_z)df$$
$$= -d^2 f$$
$$d^2 f = 0.$$

6. CONCLUSION

In this paper, we use the q −deformation of the coordinate functions on the 3- dimensional space to obtain space IR . Moreover, a non-commutative differential calculus on the h -space is presented via a contraction of the q -space.

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