A new efficient method for solving nonlinear multi-point boundary value problems

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Abstract

In this work, a new homotopy type approach is described by means of the nonlinear multi-point boundary value problems. An efficient way of choosing the initial approximation and appropriate series representation for the nonlinear part is presented. The results demonstrate reliability and efficiency of the method.

Keywords: homotopy perturbation method, multi-point boundary value problems, series solution, ordinary differential equations

1. Introduction

Multi-point boundary value problems arise in various fields of applied science. We consider the multi-point boundary value problems of the type

$$u^{(k)} = f\left(x, u, u', \dots, u^{(k-1)}\right) + g(x), \ 0 < x < 1, \tag{1}$$

$$u^{(r)}(0) = \sum_{i=1}^{l} \alpha_{ri} u(\xi_i), \ u^{(m)}(1) = \sum_{i=1}^{l} \beta_{mi} u(\eta_i), \ r + m = k - 2, \tag{2}$$

where $f\left(x, u, u', \dots, u^{(k-1)}\right)$ is a continuous function of its arguments and $\xi_i, \eta_i \in (0,1)$. An application of these type of problems appear in the design of a large-size bridge with multipoint supports, where *u(x)* denotes the displacement of the bridge from the unloaded position [1]. Many problems in the theory of elastic stability can be handled by multi-point boundary value problems. For other applications of problem (1)-(2), see [2], [3], [4] and the references therein. Boundary conditions (2) includes various commonly discussed multi-point boundary conditions. In particular, the boundary conditions (2) is reduced to

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{l} \beta_i u(\eta_i),$$

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if k=2, r=m=0, and $\alpha_{0i} = 0$ for all i (see [5-14]), to the form

$$u(0) = \sum_{i=1}^{l} \alpha_i u(\xi_i), \quad u(1) = 0$$

if k=2, r=m=0, and $\beta_{0i} = 0$ for all i (see [15]), to the four-point boundary condition

$$u(0) = \alpha u(\xi), \qquad u(1) = \beta u(\eta)$$

if k=2, r=m=0, and l=1 (see [11], [15-17]), and to the two-point boundary condition

$$u(0) = u(1) = 0$$

if k=2, r=m=0, $\alpha_{0i} = 0$and $\beta_{0i} = 0$ for all i.

The study of multi-point boundary problems for linear second-order equations was initiated in [18,19] by Il'in and Moiseev. In [20] Gupta studied a three-point boundary problem for nonlinear second-order equations. Additional works that deal with the multi-point boundary problems can be found in [21-29].

There are some works that deal with the numerical methods for the solutions of multi-point boundary problems. In [30] Urabe studied the Chebyshev series approximating solutions of first-order multi-point boundary problems. In [31] the authors applied the finite difference technique for solving multi-point boundary problems for first-order systems of equations. A multiple shooting technique was applied for computing the solutions of nonlinear first-order multi-point boundary problems in [32]. Some other computational methods developed for the solution of first-order multi-point boundary problems in [33-35]. In [36-38] the authors gave several constructive methods for the solutions of multi-point discrete boundary value problems. In case of second-order multi-point boundary problems, there are only a few computational algorithms in the literature. The paper [39] deals with the reproducing kernel Hilbert space method for the solution of a second-order multi-point boundary problem. The papers [40-46] have focused on Numerov's method for nonlinear second-order multi-point boundary problem. The paper [47] deals with the finite difference method for nonlinear higher-order multi-point boundary value problems. An optimal homotopy asymptotic method applied for solving multi-point boundary problem in [48]. Geng and Cui [49,50] presented the solution of a second-order multi-point boundary value problems by combining homotopy perturbation and variational iteration methods when the nonlinear part is a finite degree polynomial of u. Chun and Sakthivel [51] used homotopy perturbation technique for solving two-point boundary value problems. Li and Wu [52] presented the solution of a fourth-order four-point boundary value problems by combining homotopy perturbation and reproducing kernel methods. In fact all numerical methods require huge computational works. In this paper we represent a simple and effective alternative to the homotopy perturbation method (HPM) for solving multi-point boundary problems.

Approximation methods like Adomian decomposition, HPM, etc. in standard form usually have some difficulties and the solution process requires huge computational works. These difficulties mainly related with the form of nonlinear part of the equation. If this part has a strong nonlinearity like u^{-k}, $(a+u^k)^{-m}$, etc. it becomes very difficult to use the methods with some standard algorithms. That is why there are only a few application of Adomian decomposition or HPM alone for solving multi-point boundary value problems.

The main idea of HPM is to introduce a homotopy parameter, say p, which takes values from 0 to 1 [55-57]. When p=0 the equation usually reduces to a sufficiently simplified form (linear or very easy nonlinear). As p increases to 1, the equation goes through a sequence of "deformations" (homotopics) and at p=1 takes the original form of the equation.

We rewrite the equation (1) as

$$Lu = Nu + g(x) \tag{3}$$

where $Lu \equiv u^{(k)}$ (with boundary conditions (2)), $Nu \equiv f\left(x, u, u', \ldots, u^{(k-1)}\right)$. We construct the homotopy

$$Lu - Lv_0 + p[Lv_0 - Nu - g(x)] = 0, \tag{4}$$

where v_0 is some initial approximation.

The main difficulty in defining v_0 and in the representation of $f(x, u(x), \ldots, u^{k-1}(x))$ in more suitable series form is that the solution of the corresponding differential equations (to find the components of series solution) becomes very difficult after one-two steps.

This work has two main purposes. The first purpose is to show that the HPM can be applied successfully for solving the multi-point boundary value problems by effective choice of the initial approximation and some appropriate representation of a nonlinear part: We assume that the solution of the problem (1-2) is in the form

$$u = u_0 + pu_1 + p^2u_2 + \cdots \tag{5}$$

and substituting (5) into the equation (4), where

$$Nu = \sum_{m=0}^{\infty} a_m(u - v_0)^m$$

and equating terms of the same powers of p we obtain a system of multi-point boundary value problems for $u_0, u_1, u_2, \ldots$ Solving these problems we obtain a solution in the form

$$u = u_0 + u_1 + u_2 + \ldots$$

The second purpose is to introduce an alternative homotopy method (AHM): We construct a new homotopy

$$\begin{aligned} L(u_0 + pu_1 + p^2u_2 + \ldots) - Lv_0 + p\{Lv_0 - Nu_0 - p[(N(u_0 + u_1) - N(u_0)] \\ -p^2[(N(u_0 + u_1 + u_2) - N(u_0 + u_1)] - \cdots \\ -p^n[(N(u_0 + u_1 + \ldots + u_n) - N(u_0 + u_1 + \ldots + u_{n-1})] - \ldots - g(x)\} = 0. \end{aligned} \tag{6}$$

The case $p = 0$ corresponds to the equation $Lu_0 - Lv_0 = 0$ and $p = 1$ corresponds to the main problem. The advantage of the formula (6) is that for some more complicated nonlinear forms of the operator *N* the computations of the expressions like $N(u_0 + u_1 + \cdots + u_n)$ will now become a fairly simple process rather than $N(u_0 + pu_1 + \cdots + pu_n)$ (and comparison of the like powers of *p* becomes more simple process).

To make our research more valuable and reliable we prove a new kind of existence and convergence theorems for the boundary value problems of second order equation.

2. Existence results for a boundary value problem of second order equation

In this section we shall prove the existence theorem for the problem

$$u'' = f(x, u) + g(x), \ 0 < x < 1, \tag{7}$$

$u(0) = 0, u(1) = \beta u(\eta), 0 \leq \beta, \eta < 1.$

Then we demonstrate the convergence of the alternative homotopy algorithm (6).

Let us denote by $v_0(x)$ the solution of the problem (7) when $f = 0$:

$$u'' = g(x), \ 0 < x < 1, \tag{8}$$

$$u(0) = 0, u(1) = \beta u(\eta), 0 \leq \beta, \eta < 1$$

and let $m(t)$ be an absolutely integrable function and

$$\int_0^1 m(x)dx \leq M.$$

Theorem 2.1. Suppose that $g(x)$ is continuous on (0,1], integrable on [0,1] and $f(x,u)$ is continuous for each $x \in (0,1]$ in u and

$$|f(x,u(x))| \leq m(x), g(x) \leq m(x) \tag{9}$$

for all $u(x)$ with $|u(x) - v_0(x)| \leq \frac{3M}{2(1-\beta\eta)}$. Then the boundary value problem (7) has a solution.

Proof. The problem (8) has a unique solution

$$v_0(x) = \int_0^x (x-s)g(s)ds + cx, \tag{10}$$

where

$$c = \frac{\beta \int_0^\eta (\eta - s)g(s)ds - \int_0^1 (1-s)g(s)ds}{1-\beta\eta}.$$

Let us consider the recurrence relations

$$u_0(x) = v_0(x), \tag{11}$$

$$u_1(x) = \int_0^x (x-s)f(s,u_0(s))ds + c_0 x + v_0(x),$$

$$u_2(x) = \int_0^x (x-s)f(s,u_1(s))ds + c_1 x + v_0(x), \cdots,$$

$$u_n(x) = \int_0^x (x-s)f(s,u_{n-1}(s))ds + c_{n-1} x + v_0(x), \ldots$$

where $c_n = c(u_n)$ is a constant that depends on u_n

$$c_n = c(u_n) = \frac{\beta \int_0^\eta (\eta - s)f(s,u_n(s))ds - \int_0^1 (1-s)f(s,u_n(s))ds}{1-\beta\eta}. \tag{12}$$

Let us show that the sequence $\{u_n(x)\}$ has a uniformly convergent subsequence. Indeed it follows from (9), (10) and (11) that

$$|u_1(x)-v_0(x)| = \left|\int_0^x (x-s)f(s,u_0(s))ds + c_0x\right|$$

$$\leq M + \frac{\beta\int_0^\eta(\eta-s)Mds + \int_0^1(1-s)Mds}{1-\beta\eta}$$

$$\leq M\left(1+\frac{1}{2}\frac{\beta\eta^2+1}{1-\beta\eta}\right) = M\frac{2(1-\beta\eta)+\beta\eta^2+1}{2(1-\beta\eta)} \leq \frac{3M}{2(1-\beta\eta)},$$

$$|u_2(x)-v_0(x)| = \left|\int_0^x (x-s)f(s,u_1(s))ds + c_1x\right| \leq \frac{3M}{2(1-\beta\eta)}, \ldots$$

and therefore $\{u_n(x)-v_0(x)\}$ is uniformly bounded sequence. On the other hand, for $|x-t| < \delta$ we have

$$|u_n(x)-v_0(x)-(u_n(t)-v_0(t))| = \left|\int_t^x (x-s)f(s,u_{n-1}(s))ds + v_0(x)-v_0(t)\right|$$

$$\leq M_1|x-t|$$

for some constant M_1. That is $\{u_n(x)-v_0(x)\}$ is uniformly bounded and equicontinuous sequence on [0,1] and it follows from Ascoli-Arzela lemma, that $\{u_n(x)\}$ has a uniformly convergent subsequence, say for simplicity $u_n(x)\to u(x)$. Then the sequence (c_n) is convergent and now it follows from the Lebesgue dominated convergence theorem that

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty}\int_0^x (x-s)f(s,u_{n-1}(s))ds + c'x + v_0(x)$$

$$= \int_0^x (x-s)f(s,u(s))ds + c'x + v_0(x),$$

where

$$c' = \lim_{n\to\infty} c_n = \frac{\beta\int_0^\eta(\eta-s)f(s,u(s))ds - \int_0^1(1-s)f(s,u(s))ds)}{1-\beta\eta}$$

We have therefore demonstrated the existence of a solution, $u(x)$, to the boundary value problem (7).

3. Convergence of the AHM

Theorem 3.1

Suppose that $g(x)$ is continuous on (0,1], integrable on [0,1] and $f(x,u)$ is continuous for each $x\in(0,1]$ in u and

$$|f(x,u(x))| \leq m(x),$$

for some integrable $m(x)$. Then the iterative algorithm (6) converges to the solution

$$u = u_0 + u_1 + u_2 + \ldots$$

Proof. Equating like powers of p in (6) we obtain

$$Lu_0 = Lv_0, \tag{13}$$

$$Lu_1 = -Lv_0 + Nu_0 + g(x),$$

$$Lu_2 = N(u_1 + u_0) - N(u_0), \ldots$$

$$Lu_{n+1} = N(u_n + \ldots + u_0) - N(u_{n-1} + \ldots + u_0), \ldots,$$

where $Lu \equiv u''$ and v_0 is the solution of the problem $Lu = g(x)$. Now calculating the successive functions u_k, determined by (13) we obtain successively

$$u_0(x) = \int_0^x (x-s)g(s)ds + C_0 x, \tag{14}$$

$$u_1(x) = \int_0^x (x-s)f(s,u_0(s))ds + C_1 x,$$

$$u_2(x) = \int_0^x (x-s)\left[f\left(s,u_0(s)+u_1(s)\right) - f\left(s,u_0(s)\right)\right]ds + C_2 x, \ \ldots$$

$$u_{n+1}(x) = \int_0^x (x-s)[f(s,u_0+u_1+\ldots+u_n) - f(s,u_0+u_1+\ldots+u_{n-1})]ds + C_{n+1}x, \cdots$$

where

$$C_0 = \frac{\beta\int_0^\eta (\eta-s)g(s)ds - \int_0^1 (1-s)g(s)ds}{1-\beta\eta},$$

$$C_1 = \frac{\beta\int_0^\eta (\eta-s)f(s,u_0)ds - \int_0^1 (1-s)f(s,u_0)ds}{1-\beta\eta}, \cdots$$

$C_n = C(u_n)$

$$= \frac{\beta\int_0^\eta (\eta-s)[f(s,u_{n-1}) - f(s,u_{n-2})]ds - \int_0^1 (1-s)[f(s,u_{n-1}) - f(s,u_{n-2})]ds}{1-\beta\eta},$$

$n > 1$. For the sequence of partial sums $y_k = u_0 + u_1 + \ldots + u_k$ we have

$$|y_k(x)| = \left|\int_0^x (x-s)f(s,u_0+u_1+\ldots+u_{k-1})ds + u_0(x) + (C_0+\ldots+C_k)x\right|$$

$$\leq \int_0^x m(s)ds + |u_0(x)| + (1-\beta\eta)^{-1}\beta \left| \int_0^\eta (\eta - s)g(s)ds + \int_0^\eta (\eta - s)f(s, u_{k-1})ds \right| x$$

$$+ (1-\beta\eta)^{-1} \left| \int_0^1 (1-s)g(s)ds + \int_0^1 (1-s)f(s, u_{k-1})ds \right| x \leq K$$

for some constant K. That is the sequence $\{y_k(x)\}$ is uniformly bounded sequence. In like manner, for $x_1, x_2 \in [0,1], x_1 < x_2$ we have

$$|y_k(x_2) - y_k(x_1)| \leq \int_{x_1}^{x_2} m(s)ds + |u_0(x_2) - u_0(x_1)|$$

$$+(x_2 - x_1)(1-\beta\eta)^{-1} \left[\beta \left| \int_0^\eta (\eta - s)g(s)ds + \int_0^\eta (\eta - s)f(s, u_{k-1})ds \right| \right.$$

$$\left. + \left| \int_0^1 (1-s)g(s)ds + \int_0^1 (1-s)f(s, u_{k-1})ds \right| \right] \leq (x_2 - x_1)K_1$$

for some constant K_1, which means that the sequence $\{y_k(x)\}$ is equicontinuous. The Ascoli-Arzela lemma yields that $\{y_n(x)\}$ has a uniformly convergent subsequence, say for simplicity $y_n(x) \to y(x)$. Now it follows from (14) that

$$u_0 + u_1 + \ldots + u_n = \int_0^x (x-s)f(s, u_0 + u_1 + \ldots + u_n)ds + \int_0^x (x-s)g(s)ds + x\sum_{k=1}^n C_k$$

and therefore the Lebesgue dominated convergence theorem yields

$$y = \sum_{k=1}^\infty u_k = \int_0^x (x-s)f(s,y)ds + \int_0^x (x-s)g(s)ds + Cx$$

where $C = \sum_{k=1}^\infty C_k$. Calculating y'' we obtain that y is the solution of the problem (7).

4. Applications

First we consider two special problems to explain the method and the advantageous of the proposed AHM algorithm (6).

Example 4.1.

Consider the problem

$$u'' = u^2 - x^2 - x^4 + 2x^3 - 2,$$

$$u(0) = 0, u(1) = 0.$$

The exact solution is $u = x - x^2$.
The algorithm (6) gives

$$v_0 = -\frac{x^4}{12} - \frac{x^6}{30} + \frac{x^5}{10} - x^2 + \frac{61}{60}x = u_0,$$

$$Lu_1 = Lv_0 - x^2 - x^4 + 2x^3 - 2 + {u_0}^2,$$

$$Lu_1 = \left(-\frac{x^4}{12} - \frac{x^6}{30} + \frac{x^5}{10} - x^2 + \frac{61}{60}x\right)^2$$

$$u_1 = \frac{1}{163800}x^{14} - \frac{1}{23400}x^{13} + \frac{7}{59400}x^{12} - \frac{1}{6600}x^{11} + \frac{53}{64800}x^{10} - \frac{241}{64800}x^9$$

$$+\frac{37}{5600}x^8 - \frac{61}{15120}x^7 + \frac{1}{30}x^6 - \frac{61}{600}x^5 + \frac{3721}{43200}x^4 - \frac{2257573}{129729600}x,$$

$$Lu_2 = N(u_1 + u_0) - N(u_0) = (u_1 + u_0)^2 - (u_0)^2 = (u_1)^2 + 2u_1u_0,$$

$$u_2 = -2.1565 \times 10^{-8}x^{21} + 6.8524 \times 10^{-8}x^{20} - 1.6485 \times 10^{-7}x^{19} + 2.0464 \times 10^{-7}x^{18}$$

$$+1.4994 \times 10^{-7}x^{17} - 1.1993 \times 10^{-6}x^{16} + 2.3604 \times 10^{-6}x^{15} - 1.6479 \times 10^{-5}x^{14}$$

$$+\frac{66\,062045641}{655705\,290240000}x^{13} - \frac{152\,156683403}{554827\,553280000}x^{12} + \frac{4592683177}{13318\,905600000}x^{11}$$

$$-\frac{3203\,780543819}{3530720\,793600000}x^{10} + \frac{61}{16200}x^9 - \frac{14\,747637587}{2179\,457280000}x^8 + \frac{44\,590552057}{10699\,153920000}x^7$$

$$+\frac{2257573}{1297296000}x^5 - \frac{590413\,916746631}{201957229\,393920000}x^4 + 7.6794 \times 10^{-4}x$$

Now the standard HPM gives the same for u_0 and u_1. But for u_2 we obtain from (4) and (5) (for $u = u_0 + pu_1 + p^2u_2 + \cdots$)

$$Lu_2 = 2u_0u_1$$

and

$$u_2 = -\frac{1}{1135134000}x^{22} + \frac{1}{103194000}x^{21} - \frac{113}{2464862400}x^{20} + \frac{7}{58687200}x^{19}$$

$$-\frac{113527}{297\,729432000}x^{18} + \frac{2501}{1272348000}x^{17} - \frac{639517}{93\,405312000}x^{16}$$

$$+\frac{16483}{1257379200}x^{15}-\frac{303077}{9081072000}x^{14}+\frac{315953}{2122848000}x^{13}-\frac{178291}{449064000}x^{12}$$
$$+\frac{1403}{2772000}x^{11}-\frac{161891}{163296000}x^{10}+\frac{529824613}{140\,107968000}x^{9}-\frac{248089159}{36\,324288000}x^{8}$$
$$+\frac{10826747}{2554051500}x^{7}+\frac{2257573}{1297296000}x^{5}-\frac{137711953}{46\,702656000}x^{4}+7.7663\times10^{-4}x.$$

The advantage of the algorithm (6) can be easily seen from the formulas for u_2. In case of (6) to find u_2 we use the term $(u_1)^2$ immediately, while in the standard approach the term $(u_1)^2$ arise in the next step to find u_3. The Table 4.1 demonstrates the absolute errors in $u_{exact}-(u_0+u_1+u_2)$ for HPM and AHM approaches.

Table 4.1. Error estimates for the Example 4.1.

x	HPM	AHM
0.0	0.0	0.0
0.1	1.4023×10^{-8}	1.1506×10^{-8}
0.2	2.1083×10^{-7}	1.7120×10^{-7}
0.4	2.8314×10^{-6}	2.2612×10^{-6}
0.5	6.1392×10^{-6}	4.8759×10^{-6}
0.7	1.7450×10^{-5}	1.378×10^{-5}
0.9	3.2935×10^{-5}	2.5982×10^{-5}
1.0	4.1144×10^{-5}	3.2462×10^{-5}

Example 4.2

Consider the problem

$$u''=u^3+2x(\ln x)u^2-4x^3(\ln x)^2+4x^3\ln x-x^3-\frac{2}{x}$$
$$u(0)=0,\qquad u(1)=\frac{\sqrt{e}}{2}u\left(\frac{1}{\sqrt{e}}\right),$$

with the exact solution, $u_{exact}=x-2x\ln x$. The algorithm (6) gives,

$$Lv_0=-4x^3(\ln x)^2+4x^3\ln x-x^3-\frac{2}{x},\ v_0(0)=0,\qquad v_0(1)=\frac{\sqrt{e}}{2}v_0\left(\frac{1}{\sqrt{e}}\right)$$

$$v_0=x(-0.60290+2)+(\ln x)\left(\frac{19}{50}x^5-2x\right)-\frac{1}{5}x^5(\ln x)^2-\frac{201}{1000}x^5=u_0$$

$$Lu_1={u_0}^3+2x(\ln x){u_0}^2,\qquad u_1(0)=0,\qquad u_1(1)=\frac{\sqrt{e}}{2}u\left(\frac{1}{\sqrt{e}}\right),$$

$$u_1 = -2.9412 \times 10^{-5}x^{17}(\ln x)^6 + 1.8906 \times 10^{-4}x^{17}(\ln x)^5 - 5.1865 \times 10^{-4}x^{17}(\ln x)^4$$

$$+7.765 \times 10^{-4}x^{17}(\ln x)^3 - 6.6898 \times 10^{-4}x^{17}(\ln x)^2 + 3.1453 \times 10^{-4}x^{17}(\ln x)$$

$$-6.3096 \times 10^{-5}x^{17} - 1.0256 \times 10^{-3}x^{13}(\ln x)^5 + 5.794 \times 10^{-3}x^{13}(\ln x)^4$$

$$-0.013431x^{13}(\ln x)^3 + 1.5968 \times 10^{-2}x^{13}(\ln x)^2 - 0.0097416x^{13}(\ln x)$$

$$+2.4419 \times 10^{-3}x^{13} - 1.1111 \times 10^{-2}x^9(\ln x)^4 + 6.2652 \times 10^{-2}x^9(\ln x)^3$$

$$-0.12895x^9(\ln x)^2 + 0.11778x^9(\ln x) - 4.0574 \times 10^{-2}x^9 + 0.27942x^5(\ln x)^2$$
$$-0.64186x^5(\ln x) + 0.39724x^5 - 0.61398x$$

$$Lu_2 = (u_1 + u_0)^3 + 2x(\ln x)(u_1 + u_0)^2 - u_0^3 - 2x(\ln x)u_0^2$$

$$u_2 = -1.2266 \times 10^{-2}x^{13}(\ln x)^4 + 2.3055 \times 10^{-2}x^{13}(\ln x)^3$$

$$-2.3251 \times 10^{-2}x^{13}(\ln x)^2 + 1.2189 \times 10^{-2}x^{13}(\ln x)$$

$$-2.639 \times 10^{-3}x^{13} + 1.552 \times 10^{-2}x^9(\ln x)^4 - 0.08827x^9(\ln x)^3$$

$$+0.18208x^9(\ln x)^2 - 0.1645x^9(\ln x) + 5.5144 \times 10^{-2}x^9$$

$$-0.1228x^5(\ln x)^2 + 0.37824x^5(\ln x) - 0.27026x^5 + 0.37271x.$$

We used the Latex tools to solve the corresponding differential equations.
If we use the standard HPM scheme (4)-(5), we have the same for u_1 and u_0, but for u_2 we obtain (by equating the like powers of p in (4), where $u = u_0 + pu_1 + p^2u_2 + \cdots$)

$$Lu_2 = 3u_0^2u_1 + 4x(\ln x)u_0u_1$$

and therefore

$$u_2 = -1.2944 \times 10^{-2}x^{17}(\ln x)^3 + 1.0854 \times 10^{-2}x^{17}(\ln x)^2 - 5.058 \times 10^{-3}x^{17}(\ln x)$$
$$+1.0148 \times 10^{-3}x^{17} - 2.849 \times 10^{-4}x^{13}(\ln x)^6 + 5.5423 \times 10^{-3}x^{13}(\ln x)^5$$
$$-0.028102x^{13}(\ln x)^4 + 6.5775 \times 10^{-2}x^{13}(\ln x)^3 - 8.1185 \times 10^{-2}x^{13}(\ln x)^2$$
$$+5.1761 \times 10^{-2}x^{13}(\ln x) - 1.3545 \times 10^{-2}x^{13} + 1.5523 \times 10^{-2}x^9(\ln x)^4$$
$$-0.10734x^9(\ln x)^3 + 0.2581x^9(\ln x)^2 - 0.26768x^9(\ln x) + 0.10271x^9$$
$$-0.1228x^5(\ln x)^2 + 0.45363x^5(\ln x) - 0.37162x^5 + 0.48334x$$

Table 4.2 represents the absolute errors in the Example 4.2.

Table 4.2. Error estimates for the Example 4.2.

x	HPM	AHM
	$\lvert u_{exact} - (u_0 + u_1 + u_2)\rvert$	$\lvert u_{exact} - (u_0 + u_1 + u_2)\rvert$
0.0	0.0	0.0
0.1	2.6638×10^{-2}	1.5577×10^{-2}
0.2	5.3102×10^{-2}	3.1047×10^{-2}
0.3	7.8808×10^{-2}	4.6080×10^{-2}
0.5	0.12341	7.2589×10^{-2}
0.7	0.14969	9.0294×10^{-2}
0.9	0.15092	9.6649×10^{-2}
1.0	0.14306	9.6120×10^{-2}

Example 4.3 [46]
Consider the problem

$$u'' = u(u-1) - q(x), 0 < x < 1,$$

$$u(0) = \frac{1}{9}u\left(\frac{1}{2}\right), \qquad u(1) = \frac{1}{8}u\left(\frac{1}{4}\right),$$

where

$$q(x) = 8 + \frac{\pi^2 \sin(2\pi x)}{2} + \left(4x(1-x) + \frac{1+\sin(2\pi x)}{8}\right)\left(1 - 4x(1-x) - \frac{1+\sin(2\pi x)}{8}\right).$$

The exact solution is $u = 4x(1-x) + \frac{1+\sin(2\pi x)}{8}$. For simplicity first we rewrite the equation as

$$u'' = Nu + g(x),$$

where

$$g(x) = -8 - \frac{\pi^2 \sin(2\pi x)}{2}$$

and

$$Nu \equiv u(u-1) - \left(4x(1-x) + \frac{1+\sin(2\pi x)}{8}\right)\left(4x(1-x) + \frac{1+\sin(2\pi x)}{8} - 1\right)$$

$$= \frac{d}{du}u(u-1)\Big|_{u=4x(1-x)+\frac{1+\sin(2\pi x)}{8}}\left(u - \left(4x(1-x) + \frac{1+\sin(2\pi x)}{8}\right)\right)$$

$$+\frac{1}{2}\frac{d^2}{du^2}u(u-1)\Big|_{u=4x(1-x)+\frac{1+\sin(2\pi x)}{8}}\left(u - \left(4x(1-x) + \frac{1+\sin(2\pi x)}{8}\right)\right)^2 + \cdots$$

$$\equiv a_1(x)(u - v_0) + a_2(x)(u - v_0)^2 + \ldots$$

for some continuous functions $a_1(x)$, $a_2(x)$,... It follows from (4), and (5) that

$$L(u_0 + pu_1 + p^2u_2 + \ldots) - Lv_0 + p(Lv_0 - Nu - g(x)) = 0,$$

and

$$L(u_0 + pu_1 + p^2u_2 + \cdots) - Lv_0 + p\left(Lv_0 - a_1(x)(pu_1 + p^2u_2 + \cdots) - a_2(x)(pu_1 + p^2u_2 + \cdots)^2 + \cdots + 8 + \frac{\pi^2 \sin(2\pi x)}{2}\right) = 0$$

and equating the like powers of p we obtain the system of equations

$$Lu_0 - Lv_0 = 0, \qquad u(0) = \frac{1}{9}u\left(\frac{1}{2}\right), \qquad u(1) = \frac{1}{8}u\left(\frac{1}{4}\right),$$

$$Lu_1 + Lv_0 + 8 + \frac{\pi^2 \sin(2\pi x)}{2} = 0,$$

$$Lu_2 - a_1(x)u_1 = 0,$$

$$Lu_3 - a_2(x)u_2 = 0, \ldots$$

Solving correspondingly we obtain

$$u_0 = v_0 = 4x(1-x) + \frac{1 + \sin(2\pi x)}{8},$$

$$u_1 = 0, u_2 = 0, u_3 = 0, \ldots$$

Thus we attain an exact solution $u = u_0$. Applying the alternative algorithm (6) we obtain the same results, since $u_1 = u_2 = \ldots = 0$.

This example demonstrates the importance of the choice of the function $f(x, u, u', \ldots, u^{k-1}) = Nu$ and its appropriate series representation.

Example 4.4

We consider the problem

$$u'' = \frac{\cos^2\frac{\pi}{2}x}{1-u^2} - \frac{\pi^2}{4}\sin\frac{\pi}{2}x - 1$$

$$u(0) = 0, u(1) = \sqrt{2}u\left(\frac{1}{2}\right).$$

Let us take $v_0 = 0$ and apply the standard HPM: It follows from (4) and (5) that

$$Lu - Lv_0 + p(Lv_0 - \frac{\cos^2\frac{\pi}{2}x}{1-u^2} + \frac{\pi^2}{4}\sin\frac{\pi}{2}x + 1) = 0,$$

$$L(u_0 + pu_1 + p^2u_2 + \ldots) - Lv_0 + p(Lv_0 + \frac{\pi^2}{4}\sin\frac{\pi}{2}x + 1$$

$$-(\cos^2\frac{\pi}{2}x)((1 + (u_0 + pu_1 + p^2u_2 + \ldots)^2 + (u_0 + pu_1 + p^2u_2 + \ldots)^4 + \ldots)) = 0.$$

Equating the like powers of p we obtain

$$Lu_0 - Lv_0 = 0, u_0 = 0,$$

$$Lu_1 = -\frac{\pi^2}{4}\sin\frac{\pi}{2}x - 1 + \cos^2\frac{\pi}{2}x, \qquad u_1(0) = 0, u_1(1) = \sqrt{2}u_1\left(\frac{1}{2}\right)$$

$$u_1 = \frac{1}{2\pi^2} - \frac{1}{4\pi^2}\left(2\cos(\pi x) - 4\pi^2 \sin\left(\frac{1}{2}\pi x\right) + \pi^2 x^2 + 4(-0.45046)\pi^2 x\right)$$
$$= 2.0213x - 0.64596\,x^3 - 0.20562x^4 + 7.9693\times 10^{-2}\,x^5 + 6.7645\times 10^{-2}x^6$$
$$-4.6818\times 10^{-3}\,x^7 - 1.1922\times 10^{-2}x^8 + 1.6044\times 10^{-4}\,x^9 + 1.3074\times 10^{-3}x^{10}.$$

$$Lu_2 = 0, u_2 = 0$$

$$Lu_3 = cos^2\left(\frac{\pi}{2}x\right){u_1}^2$$
$$= (cos^2\frac{\pi}{2}x)\left(-\frac{1}{2\pi^2} + \frac{1}{4\pi^2}\left(2cos\pi x - 4\pi^2\sin\frac{1}{2}\pi x + \pi^2x^2 + 4(-0.45046)\pi^2 x\right)\right)^2,$$

$$u_3 = -1.7725\times 10^{-4}x^{19} + 3.6029\times 10^{-5}\,x^{18} + 9.7934\times 10^{-4}x^{17}$$
$$+5.1967\times 10^{-4}x^{16} - 4.2213\times 10^{-3}\,x^{15} - 5.331\times 10^{-3}x^{14} + 1.3447\times 10^{-2}x^{13}$$
$$+2.9593\times 10^{-2}\,x^{12} - 2.8958\times 10^{-2}x^{11} - 0.11034x^{10} + 3.5972\times 10^{-2}\,x^9$$
$$+0.2763x^8 - 1.9791\times 10^{-2}x^7 - 0.42307\,x^6 + 0.34046x^4 - 0.28485x,$$

$$Lu_4 = 0,$$

$$Lu_5 = \left(cos^2\frac{\pi}{2}x\right)(u_2^2 + 2u_1u_3 + u_1^4)$$
$$u_5 = -3.1958\times 10^{-4}x^{26} - 2.1765\times 10^{-4}x^{25} + 1.2264\times 10^{-3}x^{24}$$
$$+1.7856\times 10^{-3}x^{23} - 3.5413\times 10^{-3}x^{22} - 9.1612\times 10^{-3}x^{21}$$
$$+6.2224\times 10^{-3}x^{20} + 3.4766\times 10^{-2}x^{19} + 3.0021\times 10^{-3}x^{18}$$
$$-0.10026x^{17} - 6.8175\times 10^{-2}x^{16} + 0.21575x^{15} + 0.27483x^{14}$$
$$-0.32913x^{13} - 0.67836x^{12} + 0.32418x^{11} + 1.1199x^{10} - 0.17592x^9$$
$$-1.1752x^8 + 3.5560\times 10^{-2}x^7 + 0.66333x^6 - 9.5958\times 10^{-2}x^4 - 0.14773x.$$

Table 4.3 shows that the maximum error between the exact and approximate solution is less than $2{,}9\times 10^{-2}$.

Table 4.3. Error estimates for Example 4.4.

x	Exact solution $=\sin(\pi/2)x$	Series solution $u_o+u_1+\ldots+u_5$	Error
0.0	0	0	0
0.1	0.15643	0.15823	1.7946×10^{-3}
0.2	0.30902	0.31268	3.6606×10^{-3}
0.3	0.45985	0.45399	5.8577×10^{-3}
0.4	0.5967	0.58779	8.9168×10^{-3}
0.5	0.72055	0.70711	1.3441×10^{-2}
0.7	0.91702	0.89101	2.6012×10^{-2}
0.9	1.0167	0.98769	2.8999×10^{-2}
1.0	1.0191	1	1.9059×10^{-2}

If we rewrite the equation in the Example 4.4 as

$$u'' = \frac{cos^2(\pi x/2) - 1 + u^2}{1 - u^2} - \frac{\pi^2}{4}\sin\frac{\pi}{2}x$$

and apply the HPM in this form by taking $v_0 = sin(\pi/2)x$ as a solution of the problem

$$u'' = -\frac{\pi^2}{4}\sin\frac{\pi}{2}x,$$

$$u(0) = 0, u(1) = \sqrt{2}u\left(\frac{1}{2}\right)$$

we obtain the exact solution in the first step!

Example 4.5. Consider the problem

$$u^{(4)} = \frac{cos^2(\pi/2)x}{1-u^2} - \frac{4}{\pi^2}(u')^2 + \frac{\pi^4}{16}sin\frac{\pi}{2}x - sin^2\frac{\pi}{2}x \tag{15}$$

$$u(0) = 0, u'(0) = \frac{\pi}{2}, u(1) = \sqrt{2}u\left(\frac{1}{2}\right), u'(1) = 0u'\left(\frac{1}{2}\right).$$

We have two choices for v_0: 1) $v_0 = 0$ and 2) v_0 as the solution of the problem

$$u^{(4)} = \frac{\pi^4}{16}\sin\frac{\pi}{2}x - sin^2\frac{\pi}{2}x,$$

$$u(0) = 0, u'(0) = \frac{\pi}{2}, u(1) = \sqrt{2}u\left(\frac{1}{2}\right), u'(1) = 0u'\left(\frac{1}{2}\right). \tag{16}$$

In the first case the standard HPM algorithm

$$L(u_0 + pu_1 + p^2u_2 + \cdots) - Lv_0 + p(Lv_0 - \frac{\pi^4}{16}\sin\frac{\pi}{2}x + sin^2\frac{\pi}{2}x \tag{17}$$

$$-cos^2(\pi/2)x(1 + (u_0 + pu_1 + p^2u_2 + \cdots)^2 + (u_0 + pu_1 + p^2u_2 + \cdots)^4 + \cdots)$$

$$+\frac{4}{\pi^2}(u_0' + pu_1' + p^2u_2' + \ldots)^2) = 0$$

gives

$$Lu_0 - Lv_0 = 0, u_0 = 0$$

$$Lu_1 = \frac{\pi^4}{16}\sin\frac{\pi}{2}x - sin^2\frac{\pi}{2}x + cos^2\frac{\pi}{2}x,$$

$$u(0) = 0, u'(0) = \frac{\pi}{2}, u(1) = \sqrt{2}u\left(\frac{1}{2}\right), u'(1) = 0.$$

$$u_1 = -\frac{1}{\pi^4} + \frac{1}{6\pi^4}\left(6cos\pi x + 6\pi^4\sin\frac{1}{2}\pi x - \frac{24}{\pi^4}\pi^4x^3 + 3\frac{12}{\pi^4}\pi^4x^2\right),$$

$$Lu_2 = 0, u_2 = 0$$

$$Lu_3 = \left(cos^2\frac{\pi}{2}x\right)u_1^2 - \frac{4}{\pi^2}(u_1')^2, u(0) = 0, u'(0) = 0, u(1) = \sqrt{2}u\left(\frac{1}{2}\right), u'(1) = 0.$$

$$u_3 = 2.8944 \times 10^{-6}x^{17} - 2.3988 \times 10^{-5}x^{16} - 1.1440 \times 10^{-5}x^{15}$$
$$+1.4660 \times 10^{-4}x^{14} + 3.2344 \times 10^{-5}x^{13} - 6.7717 \times 10^{-4}x^{12}$$
$$-6.6922 \times 10^{-5}x^{11} + 2.3321 \times 10^{-3}x^{10} + 1.3466 \times 10^{-4}x^9$$
$$-6.2369 \times 10^{-3}x^8 - 1.6824 \times 10^{-4}x^7 + 1.4143 \times 10^{-2}x^6$$
$$-2.3206 \times 10^{-4}x^5 - 4.1667 \times 10^{-2}x^4 + 5.9663 \times 10^{-2}x^3 - 0.03131x^2$$

and therefore we obtain an approximate solution $u = u_0 + u_1 + u_2 + u_3$.

Note that the AHM can also be applied in the case $v_0 = 0$: If we use the algorithm (6) we obtain by taking

$$Nu = \frac{cos^2\frac{\pi}{2}x}{1-u^2} - \frac{4}{\pi^2}(u')^2 + \frac{\pi^4}{16}\sin\frac{\pi}{2}x - sin^2\frac{\pi}{2}x, \qquad g(x) = 0$$

and equating the like powers of p in (6):

$$u_0(x) = 0,$$

$$Lu_1 = -Lu_0 + Nu_0 = cos\pi x + \frac{\pi^4}{16}\sin\frac{\pi}{2}x,$$

$$u(0) = 0, u'(0) = \frac{\pi}{2}, u(1) = \sqrt{2}u\left(\frac{1}{2}\right), u'(1) = 0$$

$$u_1(x) = \pi^{-4}cos\pi x + \sin\frac{\pi}{2}x - \left(\frac{4}{\pi^4}\right)x^3 + \left(\frac{6}{\pi^4}\right)x^2 - \pi^{-4}.$$

In the second case we find v_0 from (16):

$$v_0 = -\frac{1}{2\pi^4} + \frac{1}{48\pi^4}\left(24cos\pi x + 48\pi^4\sin\frac{1}{2}\pi x - \pi^4x^4 + 1.618\pi^4x^3 - 0.42703\pi^4x^2\right)$$
$$= u_0.$$

And now it follows from (17) that

$$Lu_1 = cos^2\frac{\pi}{2}x - \frac{4}{\pi^2}(u_0')^2, u_1(0) = 0, u_1'(0) = 0, u_1(1) = \sqrt{2}u_1\left(\frac{1}{2}\right), u_1'(1) = 0.$$

Solving we obtain

$$u_1 = -\frac{1}{2\pi^4} + \frac{1}{48\pi^4}(24cos\pi x + \pi^4x^4 + 8(-0.44864)\pi^4x^3 + 24(0.14098)\pi^4x^2).$$

For the alternative algorithm (6) we have the same for u_0 and u_1.

Table 4.4 shows the maximum error between the exact and approximate solutions in the case $v_0 = 0$ and in the case when v_0 is a solution of corresponding problem with $f = 0$. It is worth a mention that in the second case we considered only two-term approximation.

Table 4.4. Error estimates for the Example 4.5.

x	$\lvert u_{exact} - (u_0 + \ldots + u_3)\rvert$ $v_0 = 0$, HPM	$\lvert u_{exact} - (u_0 + u_1)\rvert$ $v_0 = 0$, AHM	$\lvert u_{exact} - (u_0 + u_1)\rvert$ v_0 is the solution for $f = 0$
0.0	0	0	0
0.1	1.8515×10^{-4}	7.2442×10^{-4}	7.2417×10^{-5}
0.2	6.6624×10^{-4}	1.7562×10^{-5}	1.7459×10^{-4}
0.3	1.332×10^{-3}	2.0312×10^{-4}	2.0287×10^{-4}
0.4	2.0727×10^{-3}	1.3363×10^{-4}	1.3318×10^{-4}
0.5	2.7833×10^{-3}	0	7.3225×10^{-7}
0.7	3.7714×10^{-3}	2.0312×10^{-4}	2.0466×10^{-4}
0.8	3.9589×10^{-3}	1.7470×10^{-4}	1.7678×10^{-4}
1.0	3.9365×10^{-3}	0	3.4645×10^{-6}

Example 4.6 (see [46] for the solution of this problem by Numerov's method)

$$u'' = \frac{-\theta u}{1+u} - \left(\frac{4\pi^2}{9} - \frac{\theta}{1+\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)}\right)\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right), 0<x<1,$$

$$u(0) = \frac{\sqrt{3}}{6}u\left(\frac{1}{4}\right) + \frac{1}{4}u\left(\frac{1}{2}\right), \qquad u(1) = \frac{1}{4}u\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{6}u\left(\frac{3}{4}\right),$$

where θ is constant. Since the function $f(x,u) = -\frac{u}{1+u}$ can not be easily represented as a limit of rapidly convergent finite sum of polynomials of u^k, the methods like Adomian decomposition, variational iteration or HPM can not be applied in standard scheme. In [46] the authors applied Numerov's method for solving this problem and received an approximate solution. Note that in [46] the authors assumed $\theta>0$ for the applicability of Numerov's scheme. We shall show that even the exact solution of this problem can be received easily by using HPM for all θ. Indeed let us rewrite the differential equation as

$$u'' = \frac{\theta\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right) - \theta u}{(1+u)\left(1+\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)\right)} - \frac{4\pi^2}{9}\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)$$

and apply the HPM taking

$$f(x,u) \equiv \frac{\theta\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right) - \theta u}{(1+u)\left(1+\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)\right)},$$

$$g(x) = -\frac{4\pi^2}{9}\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right).$$

First we find v_0 from the equation

$$u'' = -\frac{4\pi^2}{9}\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)$$

with the same boundary conditions: $v_0 = \sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)$. It follows from (4) and (5) that

$$L(u_0 + pu_1 + p^2u_2 + \dots) - Lv_0 + p(Lv_0 - Nu - g(x)) = 0,$$

and using the representation

$$Nu = \left.\frac{df}{du}\right|_{u=\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)}\left(u-\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)\right) + \frac{1}{2}\left.\frac{d^2f}{du^2}\right|_{u=\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)}\left(u-\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)\right)^2 + \cdots$$

we obtain

$$L(u_0 + pu_1 + p^2u_2 + \dots) - Lv_0 + p\left(Lv_0 - b_1(x)(pu_1 + p^2u_2 + \dots) - b_2(x)(pu_1 + p^2u_2 + \dots)^2 + \cdots + \frac{4\pi^2}{9}\sin\left(\frac{2\pi}{3}x+\frac{\pi}{6}\right)\right) = 0$$

and therefore solving the system of equations

$$Lu_0 - Lv_0 = 0, \qquad u(0) = \frac{\sqrt{3}}{6}u\left(\frac{1}{4}\right) + \frac{1}{4}u\left(\frac{1}{2}\right),$$

$$u(1) = \frac{1}{4}u\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{6}u\left(\frac{3}{4}\right),$$

$$Lu_1 + Lv_0 + \frac{4\pi^2}{9}\sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right) = 0,$$

$$Lu_2 - b_1(x)u_1 = 0,$$

$$Lu_3 - b_2(x)u_1 = 0, \ldots$$

we obtain $u_0 = \sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right)$, $u_1 = u_2 = \ldots = 0$ and

$$u_{exact} = u_0 + u_1 + u_2 + \ldots = \sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right).$$

Example 4.7 (see [46] for the solution of this problem by Numerov's method)

$$u'' = \theta(u^4(x) - q^4(x), 0 < x < 1,$$

$$u(0) = \frac{\sqrt{2}}{8}u\left(\frac{1}{8}\right) + \frac{\sqrt{3}}{12}u\left(\frac{1}{4}\right) + \frac{1}{4}u\left(\frac{1}{2}\right),$$

$$u(1) = \frac{\sqrt{3}}{12}u\left(\frac{1}{4}\right) + \frac{1}{4}u\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{12}u\left(\frac{3}{4}\right),$$

where θ is positive constant and $q(x)$ is a given function. For simplicity let us take $\theta = 1$. To accommodate the exact solution of $\sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right)$ we let

$$q(x) = \left(\frac{4\pi^2}{9}\sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right) + sin^4\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right)\right)^{1/4}$$

As in the previous example we rewrite the equation as

$$u'' = \left(u^4(x) - sin^4\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right) - \frac{4\pi^2}{9}\sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right)\right)$$

and then we find the first iteration

$$u_0 = L^{-1}\left(-\frac{4\pi^2}{9}\sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right)\right) = \sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right).$$

For the second term we have

$$u_1 = L^{-1}NL^{-1}\sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right) = L^{-1}(0) = 0 = u_2 = \cdots$$

and so $u_{exact} = u_0$.

In like manner we can find an exact (or approximate-if f is not taken in more appropriate form) solution of the next problem

Example 4.8 (see [47] for the solution of this problem by compact finite difference method)

$$u^{(4)} = \sigma(x)\left(\frac{u''}{1+u}\right) + q(x), \qquad 0 < x < 1,$$

$$u(0) = \frac{1}{2}u\left(\frac{1}{2}\right), u(1) = \frac{1}{4}u\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{6}u\left(\frac{3}{4}\right),$$

$$u''(0) = \frac{\sqrt{3}}{3}u''\left(\frac{1}{4}\right), u''(1) = \frac{\sqrt{3}}{12}u''\left(\frac{1}{4}\right) + \frac{\sqrt{3}}{4}u''\left(\frac{3}{4}\right),$$

where $\sigma(x)$ and $q(x)$ are continuous functions

$$q(x) = \frac{4\pi^2}{9}\left(\frac{4\pi^2}{9} + \frac{\sigma(x)}{1+\sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right)}\right)\sin\left(\frac{2\pi}{3}x + \frac{\pi}{6}\right).$$

5. Concluding remarks

In this paper, the HPM and the AHM has been successfully used for finding the solution of multi-point boundary problems with strong nonlinearity. The solutions are very rapidly convergent by utilizing the HPM. Also, the illustrations show that the rate of rapid convergence depend on the choice of initial approximation and the form of the series representation for nonlinear part of the equation. The proposed alternative approach can be applied for solving a wide range of initial/boundary problems.

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