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Mathematical Methods in the Applied Sciences

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On Asymptotics of the Sum of the Fourth Power of the Negative Eigenvalues of the Singular Sturm–Liouville Operator

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Received: 9 August 2024 | **Accepted:** 5 August 2025

Keywords: asymptotic formulation | eigenvalues | self-adjoint operator | singular Sturm–Liouville operator

ABSTRACT

In this study, we obtain the asymptotic formula for the sum of fourth power of the negative eigenvalues of the operator L, which we created with the differential expression $\ell y := -(p(x)y\prime(x))' - q(x)y(x)$ and boundary condition $y(0) = 0$ in the space $L_2[0,\infty)$. It is known that the operator L is self-adjoint, semibounded below, and the negative part of its spectrum is discrete. is self-adjoint, semibounded below, and the negative part of its spectrum is discrete. Additionally, there are some asymptotic formulas for the number of negative eigenvalues of the operator. Our result applies to a class of mathematical physics equations.

MSC2020 Classification: 34L20, 34B24

1 | Introduction

We consider self-adjoint operator L in the space $L_2[0,\infty)$ generated by the differential expression

$$\ell(y) = -(p(x)y'(x))' - q(x)y(x) \tag{1}$$

and the boundary condition

$$y(0) = 0.$$

It is supposed that the functions $p(x)$ and $q(x)$ satisfy the following conditions:

1. $p(x)$ is a continuous, nondecreasing function with bounded derivative on $[0,\infty)$. There are positive constants c_1, c_2 such that $c_1 \le p(x) \le c_2$.
2. $q(x)$ is a continuous, decreasing, and positive-valued function on $[0,\infty)$.
3. $\lim_{x\to\infty} q(x) = 0$.
4. $\lim_{x\to\infty} q(x)x^{k-t} = \lim_{x\to\infty}[q(x)x^{k+t}]^{-1} = 0$ for every $t > 0$, where k is a constant that belongs to the interval $(0, \frac{2}{9})$.

It is known that the operator L is semibounded below and negative part of its spectrum is discrete [1]. $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$ is negative eigenvalue of the operator L.

For $\varepsilon > 0$, let $N(\varepsilon)$ denote the sum of the fourth power of negative eigenvalues of the operator L, which is smaller than $-\varepsilon$.

In this study, we find an asymptotic formula for the operator L in the form:

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$N(\varepsilon) = \frac{1}{315\pi}[1 + O(\varepsilon^{t_0})] \int\limits_{q(x)\geq\varepsilon} \sqrt{\frac{q(x)-\varepsilon}{p(x)}}[128q^4(x) + 64\varepsilon q^3(x) + 48\varepsilon^2 q^2(x) + 40\varepsilon^3 q(x) + 35\varepsilon^4]dx \qquad \varepsilon \to +0$, where $t_0 > 0$ is a constant.

Skacek [2] obtains

$$N(\varepsilon) = \frac{1 + o(\varepsilon^{\tau})}{\pi} \int\limits_{q(x)\geq\varepsilon} \sqrt[2n]{q(x) - \varepsilon}\, dx, \quad (\tau > 0),$$

for the number of the negative eigenvalues of the operator defined by

$$\ell(y) = (-1)^n y^{(2n)}(x) - q(x)y(x) \tag{2}$$

with boundary conditions

$$y(0) = y'(0) = \cdots = y^{(n-1)}(0) = 0,$$

in the space $L_2[0, \infty)$. In expression (2), $q(x)$ satisfies some conditions.

The asymptotic formulas for the number of the negative eigenvalues of scalar and abstract differential operators were obtained in the papers [3–14].

In the paper [15], the asymptotic formulas for the sum of squares of negative eigenvalues of the singular Sturm–Liouville operator has been found.

The present paper is devoted to obtaining asymptotics for the sum of fourth powers of the eigenvalues. For it, we use Courant's variational principle, which is a very useful tool for numerical investigation of eigenvalues of self-adjoint operators and operator functions.

Note that the study of the asymptotics of the number of negative eigenvalues of operators is a topical issue in modern mathematics and physics. This topic has wide application in various field such as quantum mechanics and theory of differential equations.

2 | Some Inequalities for the Negative Eigenvalues

Let g be inverse function of the function q. We consider the following operators where $\varepsilon \in (0, q(0))$:

a. Let L_0 and L_1 be operators in the space $L_2[0, g(\varepsilon)]$ generated by expression (1) with boundary conditions

$$y(0) = y(g(\varepsilon)) = 0,$$

$$y'(0) = y'(g(\varepsilon)) = 0,$$

respectively.
Let the partition points of the interval $[0, g(\varepsilon)]$ be $0 = x_0 < x_1 < \cdots < x_m = g(\varepsilon)$.

b. Let L_{0i} and L_{1i} be operators in the space $L_2[x_{i-1}, x_i]$ generated by expression (1) with boundary conditions

$$y(x_{i-1}) = y(x_i) = 0 \tag{3}$$

$$y'(x_{i-1}) = y'(x_i) = 0 \tag{4}$$

respectively.

c. Let $\overline{L}_{0i}$ be an operator in the space $L_2[x_{i-1}, x_i]$ generated by the expression

$$-p(x_i)y''(x) - q(x_i)y(x),$$

with boundary condition (3), and $\overline{L}_{1i}$ be an operator in the space $L_2[x_{i-1}, x_i]$ generated by the expression

$$-p(x_{i-1})y''(x) - q(x_{i-1})y(x),$$

with boundary condition (4).

Divide the interval $[0, g(\varepsilon)]$ into the intervals with length

$$\delta = \frac{g(\varepsilon)}{[|g^{\alpha}(\varepsilon)|] + 1} \tag{5}$$

Here, $\alpha \in (0, 1)$ and ε is a constant number that satisfies the condition $\varepsilon \in (0, g(\varepsilon))$, $g^{\alpha}(\varepsilon) \geq 2$.

Let $n_{0i}(\lambda)$ and $\overline{n}_{0i}(\lambda)$ be numbers of eigenvalues smaller than $-\lambda(\lambda \in (0, \infty))$ of the operators L_{0i} and $\overline{L}_{0i}$, respectively.

Instead of $n_{0i}(\varepsilon)$ and $\overline{n}_{0i}(\varepsilon)$, we will simply write n_{0i} and $\overline{n}_{0i}$, respectively. Let $\mu_{i1} \leq \mu_{i2} \leq \mu_{i3} \leq \cdots$ be the eigenvalues of the operator $\overline{L}_{0i}$.

Theorem 2.1. *For the eigenvalues smaller than $-\varepsilon$ of the operator $\overline{L}_{0i}$, the following inequality holds:*

$$\sum_{m=1}^{\overline{n}_{0i}} \mu_{im}^4 > \frac{\delta}{315\pi} \sqrt{\frac{q(x_i) - \varepsilon}{p(x_i)}}[128q^4(x_i) + 64q^3(x_i)\varepsilon + 48q^2(x_i)\varepsilon^2 + 40q(x_i)\varepsilon^3 + 35\varepsilon^4] - 2q^4(x_i).$$

Proof. It is easy to see that the eigenvalues of the operator $\overline{L}_{0i}$ have the form

$$\mu_{im} = p(x_i)\left(\frac{m\pi}{x_i - x_{i-1}}\right)^2 - q(x_i) \qquad (m = 1, 2, \cdots).$$

Then, we obtain

$$\begin{aligned} \sum_{m=1}^{\overline{n}_{0i}} \mu_{im}^4 &= \sum_{m=1}^{\overline{n}_{0i}} [p(x_i)\left(\frac{m\pi}{x_i - x_{i-1}}\right)^2 - q(x_i)]^4 \\ &= \sum_{m=1}^{\overline{n}_{0i}} \left[q(x_i) - p(x_i)\left(\frac{m\pi}{\delta}\right)^2\right]^4. \end{aligned} \tag{6}$$

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From the inequality $\left(\frac{m\pi}{\delta}\right)^2 \le \left(\frac{t\pi}{\delta}\right)^2 \quad (m \le t \le m+1)$, we get

$$\int_m^{m+1}\left[q(x_i)-p(x_i)\left(\frac{m\pi}{\delta}\right)^2\right]^4 dt > \int_m^{m+1}\left[q(x_i)-p(x_i)\left(\frac{t\pi}{\delta}\right)^2\right]^4 dt,$$

or

$$\left[q(x_i)-p(x_i)\left(\frac{m\pi}{\delta}\right)^2\right]^4 > \int_m^{m+1}\left[q(x_i)-p(x_i)\left(\frac{t\pi}{\delta}\right)^2\right]^4 dt \quad (1 \le m \le \overline{n}_{0i}-1).$$

From these inequalities and (6), we find

$$\begin{aligned}\sum_{m=1}^{\overline{n}_{0i}}\mu_{im}^4 &= \sum_{m=1}^{\overline{n}_{0i}}\left[p(x_i)\left(\frac{m\pi}{x_i-x_{i-1}}\right)^2 - q(x_i)\right]^4 \\ &\ge \sum_{m=1}^{\overline{n}_{0i}-1}\left[q(x_i)-p(x_i)\left(\frac{m\pi}{\delta}\right)^2\right]^4 \\ &> \sum_{m=1}^{\overline{n}_{0i}-1}\int_m^{m+1}\left[q(x_i)-p(x_i)\left(\frac{t\pi}{\delta}\right)^2\right]^4 dt \\ &= \int_1^{\overline{n}_{0i}}\left[q(x_i)-p(x_i)\left(\frac{t\pi}{\delta}\right)^2\right]^4 dt.\end{aligned}$$

Moreover, from the relation $p(x_i)\left(\frac{t\pi}{\delta}\right)^2 - q(x_i) < -\varepsilon$, we get

$$\frac{\delta}{\pi}\sqrt{\frac{q(x_i)-\varepsilon}{p(x_i)}} - 1 \le \overline{n}_{0i} < \frac{\delta}{\pi}\sqrt{\frac{q(x_i)-\varepsilon}{p(x_i)}}.$$

From here and (6), we obtain

$$\begin{aligned}\sum_{m=1}^{\overline{n}_{0i}}\mu_{im}^4 &> \int_0^{a-1}\left[q(x_i)-p(x_i)\left(\frac{t\pi}{\delta}\right)^2\right]^4 dt - q^4(x_i) \\ &> \int_0^{a}\left[q(x_i)-p(x_i)\left(\frac{t\pi}{\delta}\right)^2\right]^4 dt - 2q^4(x_i) \\ &= [128q^4(x_i)+64q^3(x_i)\varepsilon+48q^2(x_i)\varepsilon^2 \\ &\quad +40q(x_i)\varepsilon^3+35\varepsilon^4] - q^4(x_i),\end{aligned}$$

where $a = \frac{\delta}{\pi}\sqrt{\frac{q(x_i)-\varepsilon}{p(x_i)}}$.

The theorem has been proven. □

Let $\gamma_{i1} \le \gamma_{i2} \le \cdots$ and $\overline{\gamma}_{i1} \le \overline{\gamma}_{i2} \le \cdots$ be eigenvalues of the operators L_{1i} and $\overline{L}_{1i}$, respectively, and denote

$$n_{1i}(\lambda) = \sum_{\gamma_{im}<-\lambda} 1, \quad \overline{n}_{1i}(\lambda) = \sum_{\gamma_{im}<-\lambda} 1, n_{1i}(\lambda) = n_{1i}, \quad \overline{n}_{1i}(\lambda) = \overline{n}_{1i},$$

$$f(x,\varepsilon) = \sqrt{\frac{q(x)-\varepsilon}{p(x)}}[128q^4(x)+64q^3(x)\varepsilon + 48q^2(x)\varepsilon^2+40q(x)\varepsilon^3+35\varepsilon^4].$$

Theorem 2.2. *For the eigenvalues smaller than $-\varepsilon$ of the operator $\overline{L}_{1i}$, the following inequality holds:*

$$\sum_{m=1}^{\overline{n}_{1i}}(\overline{\gamma}_{im})^4 < \frac{1}{325\pi}f(x_{i-1},\varepsilon) + q^4(x_{i-1}) \tag{7}$$

Proof. The eigenvalues of the operator $\overline{L}_{1i}$ are of the form

$$\overline{\gamma}_{im} = p(x_{i-1})\left(\frac{(m-1)\pi}{x_i-x_{i-1}}\right)^2 - q(x_{i-1}) \qquad (m=1,2,\cdots) \tag{8}$$

Using the inequalities $\left(\frac{(m-1)\pi}{x_i-x_{i-1}}\right)^2 \ge \left(\frac{t\pi}{\delta}\right)^2, \quad (m-2 \le t \le m-1; m=2,3,\cdots)$, and (8), we obtain

$$\begin{aligned}&\sum_{m=1}^{\overline{n}_{1i}}(\overline{\gamma}_{im})^4 \\ &= \sum_{m=1}^{\overline{n}_{1i}}\left[q(x_{i-1})-p(x_{i-1})\left(\frac{(m-1)\pi}{\delta}\right)^2\right]^4 \\ &= q^4(0) + \sum_{m=2}^{\overline{n}_{1i}}[q(x_{i-1})-p(x_{i-1})\left(\frac{(m-1)\pi}{\delta}\right)^2]^4 \\ &< q^4(x_{i-1}) + \sum_{m=2}^{\overline{n}_{1i}}\int_{m-2}^{m-1}\left[q(x_{i-1})-p(x_{i-1})\left(\frac{t\pi}{\delta}\right)^2\right]^4 dt \\ &< q^4(x_{i-1}) + \int_0^{\overline{n}_{1i}-1}\left[q(x_{i-1})-p(x_{i-1})\left(\frac{t\pi}{\delta}\right)^2\right]^4 dt \quad (2 \le m \le \overline{n}_{0i}).\end{aligned} \tag{9}$$

From the inequality $p(x_{i-1})\left(\frac{(m-1)\pi}{\delta}\right)^2 - q(x_{i-1}) < -\varepsilon$, we get $\overline{n}_{1i} < \frac{\delta}{\pi}\sqrt{\frac{q(x_{i-1})-\varepsilon}{p(x_{i-1})}} + 1$. By the last inequality and (9), we obtain

$$\sum_{m=1}^{\overline{n}_{1i}}(\overline{\gamma}_{im})^4 < \int_0^{b}\left[q(x_{i-1})-p(x_{i-1})\left(\frac{t\pi}{\delta}\right)^2\right]^4 dt + q^4(x_{i-1}) \tag{10}$$

where $b = \sqrt{\frac{q(x_{i-1})-\varepsilon}{p(x_{i-1})}}$.

From (10), we obtain (7), so Theorem 2.2 has been proven. □

Let $N(\lambda)$, $N_0(\lambda)$, and $N_1(\lambda)$ be numbers of eigenvalues smaller than $-\lambda(\lambda \in (0,\infty))$ of the operators L, L_0, and L_1, respectively.

The following inequalities

$$N_0(\varepsilon) \le N(\varepsilon) \le N_1(\varepsilon)$$

and

$$L_{0i} \le \overline{L}_{0i}, \quad L_{1i} \ge \overline{L}_{1i}$$

are proved in [2]. In exactly the same way the inequalities

$$N_0(\lambda) \le N(\lambda) \le N_1(\lambda), \qquad (\lambda \ge \varepsilon) \tag{11}$$

can be proved.

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Because $q(x_i) \le q(x) \le q(x_{i-1})$ in the interval $[x_{i-1}, x_i]$, then $L_{0i} \le \overline{L}_{0i}$ and $L_{1i} \ge \overline{L}_{1i}$. From [16], it is known that

$$n_{0i}(\lambda) \ge \overline{n}_{0i}(\lambda), \qquad n_{1i}(\lambda) \le \overline{n}_{1i}(\lambda) \tag{12}$$

On the other hand, from the variation of Courant and Hilbert [17], we obtain

$$N_0(\lambda) \ge \sum_{i=1}^{M} n_{0i}(\lambda), \qquad N_1(\lambda) \le \sum_{i=1}^{M} n_{1i}(\lambda) \tag{13}$$

From (11–13), we have

$$\sum_{i=1}^{M} \overline{n}_{0i}(\lambda) \le N(\lambda) \le \sum_{i=2}^{M} \overline{n}_{0i} + n_{11}(\lambda).$$

By using the last inequality, the

$$\sum_{i=1}^{M}\sum_{m=1}^{\overline{n}_{0i}} \mu_{im}^4 \le \sum_{j=1}^{N(\varepsilon)} \lambda_j^4 \le \sum_{i=2}^{M}\sum_{m=1}^{\overline{n}_{1i}} (\overline{\gamma}_{im})^4 + \sum_{m=1}^{n_{1i}} \gamma_{1m}^4 \tag{14}$$

can be proved.

Theorem 2.3. *If the functions $p(x)$ and $q(x)$ satisfy Conditions 1–3, then for small positive of ε, we have*

$$\sum_{j=1}^{N(\varepsilon)} \lambda_j^4 > \frac{1}{315\pi} \int_{\delta}^{g(\varepsilon)} f(x,\varepsilon)dx - cg^{\alpha}(\varepsilon) \tag{15}$$

$$\sum_{j=1}^{N(\varepsilon)} \lambda_j^4 < \sum_{m=1}^{n_{11}} \gamma_{1m}^4 + \frac{1}{315\pi} \int_{\delta}^{g(\varepsilon)} f(x,\varepsilon)dx + cg^{\alpha}(\varepsilon). \tag{16}$$

Here,

$$f(x,\varepsilon) = \sqrt{\frac{q(x)-\varepsilon}{p(x)}}[128q^4(x) + 64q^3(x)\varepsilon + 48q^2(x)\varepsilon^2 + 40q(x)\varepsilon^3 + 35\varepsilon^4],$$

and c is a positive constant.

Proof. Condition 2 follows that function $f(x,\varepsilon)$ is monotonous decreasing with respect to x for every ε satisfying the conditions $\varepsilon \in (0, g(\varepsilon))$, $g^{\alpha}(\varepsilon) \ge 2$.

Therefore, we have

$$\delta f(x_i,\varepsilon) = \int_{x_i}^{x_{i+1}} f(x_i,\varepsilon)dx > \int_{x_i}^{x_{i+1}} f(x,\varepsilon)dx, \qquad (1 \le i \le M-1) \tag{17}$$

From Theorem 2.1 and (16), we obtain

$$\sum_{m=1}^{\overline{n}_{0i}} \mu_{im}^4 > \frac{1}{315\pi} \int_{x_i}^{x_{i+1}} f(x,\varepsilon)dx - 2q^4(0) \tag{18}$$

From (13) and (17), we get

$$\begin{aligned} \sum_{j=1}^{N(\varepsilon)} \lambda_j^4 &> \sum_{i=1}^{M-1}\left[\frac{1}{315\pi}\int_{x_i}^{x_{i+1}} f(x,\varepsilon)dx - 2q^4(0)\right] \\ &> \frac{1}{315\pi}\int_{x_1}^{x_M} f(x,\varepsilon)dx - 2q^4(0)M. \end{aligned} \tag{19}$$

From (5) for positive values of ε, we obtain

$$M = \frac{g(\varepsilon)}{\delta} = [|g^{\alpha}(\varepsilon)|] + 1 < 2g^{\alpha}(\varepsilon) \tag{20}$$

From (18) and (19) for $x_1 = \delta$ and $x_M = g(\varepsilon)$, we obtain inequality (14). Now, we prove inequality (15). Function $f(x,\varepsilon)$ is monotonous decreasing with respect to x for every $\varepsilon \in (0, g(\varepsilon))$, $g^{\alpha}(\varepsilon) \ge 2$; then, we get

$$\delta f(x_{i-1},\varepsilon) = \int_{x_{i-2}}^{x_{i-1}} f(x_{i-1},\varepsilon)dx > \int_{x_{i-2}}^{x_{i-1}} f(x,\varepsilon)dx, \qquad (2 \le i \le M) \tag{21}$$

From Theorem 2.2 and (20), we get

$$\sum_{m=1}^{\overline{n}_{1i}} (\overline{\gamma}_{im})^4 < \frac{1}{315\pi}\int_{x_{i-2}}^{x_{i-1}} f(x,\varepsilon)dx + q^4(0) \tag{22}$$

From (13) and (21), we have

$$\begin{aligned} \sum_{j=1}^{N(\varepsilon)} \lambda_j^4 &< \sum_{m=1}^{n_{11}} \overline{\gamma}_{1m}^4 + \sum_{i=2}^{M}\left[\frac{1}{315\pi}\int_{x_{i-2}}^{x_{i-1}} f(x,\varepsilon)dx + q^4(0)\right] \\ &= \sum_{m=1}^{n_{11}} \gamma_{1m}^4 + \frac{1}{315\pi}\int_{0}^{x_{M-1}} f(x,\varepsilon)dx + (M-1)q^4(0) \\ &< \sum_{m=1}^{n_{11}} \gamma_{1m}^4 + \frac{1}{315\pi}\int_{0}^{x_{M-1}} f(x,\varepsilon)dx + Mq^4(0). \end{aligned} \tag{23}$$

By (19) and (22), we get inequality (15). Theorem 2.3 has been proven. □

For the sum $\sum_{m=1}^{n_{11}} \gamma_{1m}^4$, the inequality

$$\sum_{m=1}^{n_{11}} \gamma_{1m}^4 < c_1 \int_{0}^{\delta} f(x,\varepsilon)dx - c_1 g^{\alpha}(\varepsilon) \tag{24}$$

can be proved. From (15) and (22), we find

$$\sum_{j=1}^{N(\varepsilon)} \lambda_j^4 < \frac{1}{315\pi}\int_{0}^{g(\varepsilon)} f(x,\varepsilon)dx + c_2 \int_{0}^{\delta} f(x,\varepsilon)dx + c_2 g^{\alpha}(\varepsilon). \tag{25}$$

Here, $c_2 > 0$ is a constant.

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3 | The Asymptotic Formulas for the Sum of the Fourth Power of the Negative Eigenvalues

In this section, we will find a formula for the sum $\sum_{\lambda_j<-\varepsilon}\lambda_j^4$ as $\varepsilon \to +0$.

Theorem 3.1. *If the functions $p(x)$ and $q(x)$ satisfy Conditions 1–4, then the asymptotic formula*

$$\sum_{\lambda_j<-\varepsilon}\lambda_j^4 = \frac{1}{315\pi}[1+O(\varepsilon^{s_0})]\int_{q(x)\geq\varepsilon}\sqrt{\frac{q(x)-\varepsilon}{p(x)}}[128q^4(x)+64q^3(x)\varepsilon + 48q^2(x)\varepsilon^2+40q(x)\varepsilon^3+35\varepsilon^4]dx$$

is satisfied as $\varepsilon \to +0$. Here, $s_0>0$ is a positive constant.

Proof. From Theorem 2.3, for small positive of ε, we obtain

$$\sum_{j=1}^{N(\varepsilon)}\lambda_j^4 > \frac{1}{315\pi}\int_0^{g(\varepsilon)} f(x,\varepsilon)dx - \frac{1}{315\pi}\int_0^{\delta} f(x,\varepsilon)dx - cg^{\alpha}(\varepsilon) \quad (26)$$

For the proof of theorem, we will limit each term on the right side of inequality (25). Because $p(\varepsilon)$ is monotonous decreasing and $p(x)\leq c_2$, we find

$$\begin{aligned}&\int_0^{g(\varepsilon)} f(x,\varepsilon)dx\\ &> \int_0^{g(2\varepsilon)}\sqrt{\frac{q(x)-\varepsilon}{p(x)}}[128q^4(x)+64q^3(x)\varepsilon+48q^2(x)\varepsilon^2\\ &\quad +40q(x)\varepsilon^3+35\varepsilon^4]dx \qquad (27)\\ &> \int_0^{g(2\varepsilon)}\sqrt{\frac{q(x)-\varepsilon}{p(x)}}[128\varepsilon^4+64\varepsilon^4+48\varepsilon^4+40\varepsilon^4+35\varepsilon^4]dx\\ &= 315\varepsilon^{\frac{9}{2}}\int_0^{g(2\varepsilon)}\frac{1}{\sqrt{p(x)}}dx > \varepsilon^{\frac{9}{2}}\sqrt{c_2^{-1}}g(2\varepsilon).\end{aligned}$$

From Condition 4 and $\lim_{\varepsilon\to\infty} g(\varepsilon)=\infty$, we get

$$\lim_{\varepsilon\to 0} q(g(2\varepsilon))(g(2\varepsilon))^{k+t}=\infty.$$

Hence, we obtain

$$g(2\varepsilon) > \varepsilon^{-\frac{1}{k+t}} \quad (28)$$

for the small values with respect to t of ε.

From (27) and (28), we get

$$\int_0^{g(\varepsilon)} f(x,\varepsilon)dx > \sqrt{c_2^{-1}}\varepsilon^{\frac{9}{2}-\frac{1}{(k+t)}} \quad (29)$$

Now, we estimate the integral $\int_0^{\delta} f(x,\varepsilon)dx$. From Condition 4, it follows that there is a function $f_1(t)$ such that

$$q(x)\leq f_1(t)x^{t-k} \qquad (0<t<k) \quad (30)$$

where $f_1(t)$ has positive values. Using Condition 1, we obtain

$$\begin{aligned}\int_0^{\delta} f(x,\varepsilon)dx &> \int_0^{\delta}\sqrt{\frac{q(x)}{p(x)}}[128q^4(x)\\ &\quad +64q^3(x)\varepsilon+48q^2(x)\varepsilon^2+40q(x)\varepsilon^3\\ &\quad +35\varepsilon^4]dx\\ &> \int_0^{\delta}\frac{1}{\sqrt{p(x)}}[128q^{\frac{9}{2}}(x)+64q^{\frac{7}{2}}\varepsilon+48q^{\frac{5}{2}}\varepsilon^2\\ &\quad +40q^{\frac{3}{2}}\varepsilon^3+35q^{\frac{1}{2}}\varepsilon^4]dx \qquad (31)\\ &< \int_0^{\delta} q^{\frac{9}{2}}(x)dx\\ &< c_3+\frac{315}{\sqrt{c_1}}f_1^{\frac{9}{2}}(t)\int_1^{\delta}x^{\frac{9}{2}(t-k)}dx\\ &< f_3(t)\delta^{(t-k)+1},\end{aligned}$$

where $f_3(t)$ is a positive-valued function of t $(0<t<k)$. From (4) for small values of ε, we get

$$\delta < g^{1-\alpha}(\varepsilon) \qquad (\alpha\in(0,1)) \quad (32)$$

From (30), we have

$$g(\varepsilon)\leq [f_1(t)]^{\frac{1}{k-t}}\varepsilon^{-\frac{1}{k-t}} \quad (33)$$

From (31–33), we obtain

$$\int_0^{\delta} f(x,\varepsilon)dx < f_4(t)\varepsilon^{-\frac{(1-\alpha)[9(t-k)+2]}{2(k-t)}} \quad (34)$$

where $f_4(t)$ is a positive-valued function of t.

From (29), (33), and (34), we obtain

$$\frac{\int_0^{\delta} f(x,\varepsilon)dx}{\int_0^{g(\varepsilon)} f(x,\varepsilon)dx} < f_5(t)\varepsilon^{-\frac{(1-\alpha)[9(t-k)+2]}{2(k-t)}-\frac{9}{2}+\frac{1}{k+t}} \quad (35)$$

$$\frac{g^{\alpha}(\varepsilon)}{\int_0^{g(\varepsilon)} f(x,\varepsilon)dx} < f_5(t)\varepsilon^{-\frac{\alpha}{k-t}-\frac{9}{2}+\frac{1}{k+t}} \quad (36)$$

Because $k>0$, the function $-\frac{(1-\alpha)[9(t-k)+2]}{2(k-t)}-\frac{9}{2}+\frac{1}{k+t}$ and $-\frac{\alpha}{k-t}-\frac{9}{2}+\frac{1}{k+t}$ are continuous with respect to t at the point $t=0$.

Therefore, for every $s>0$ as $0<t<\omega$, there is a number $\omega>0$ such that

$$-\frac{(1-\alpha)[9(t-k)+2]}{2(k-t)}-\frac{9}{2}+\frac{1}{k+t} > \frac{\alpha(2-9k)}{2k}-s \quad (37)$$

$$-\frac{\alpha}{k-t}-\frac{9}{2}+\frac{1}{k+t} > \frac{2-9k-2\alpha}{2k}-s \quad (38)$$

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Here, if we take $\alpha = \frac{2-9k}{4}$, $s = s_0 = \min\left\{\frac{(2-9k)^2}{16k}, \frac{2-9k}{8k}\right\}$, then from (37) and (38), we have

$$-\frac{(1-\alpha)[9(t-k)+2]}{2(k-t)} - \frac{9}{2} + \frac{1}{k+t} > \frac{(2-9k)^2}{8k} - s_0 > \frac{(2-9k)^2}{16k} \geq s_0, \tag{39}$$

$$-\frac{\alpha}{k-t} - \frac{9}{2} + \frac{1}{k+t} > \frac{2-9k}{4k} - s_0 \geq \frac{2-9k}{8k} \geq s_0 \tag{40}$$

From (35), (36), (39), and (40), we obtain

$$\frac{\int_0^{\delta} f(x,\varepsilon)dx}{\int_0^{g(\varepsilon)} f(x,\varepsilon)dx} < c_5\varepsilon^{s_0} \tag{41}$$

$$\frac{g^{\alpha}(\varepsilon)}{\int_0^{g(\varepsilon)} f(x,\varepsilon)dx} < c_5\varepsilon^{s_0} \tag{42}$$

where $c_5 = f_5(t_0) \in (0,\infty)$ is a constant. From (26), (41), and (42), we have

$$\frac{\sum_{j=1}^{N(\varepsilon)} \lambda_j^4}{\frac{1}{315\pi}\int_0^{g(\varepsilon)} f(x,\varepsilon)dx} < 1 - c_6\varepsilon^{s_0} \tag{43}$$

From (25), (41), and (42), we find

$$\frac{\sum_{j=1}^{N(\varepsilon)} \lambda_j^4}{\frac{1}{315\pi}\int_0^{g(\varepsilon)} f(x,\varepsilon)dx} < 1 + c_7\varepsilon^{s_0} \tag{44}$$

From (43) and (44), we obtain the asymptotic formula

$$\frac{\sum_{\lambda_j<-\varepsilon} \lambda_j^4}{\frac{1}{315\pi}\int_0^{g(\varepsilon)} f(x,\varepsilon)dx} - 1 = O(\varepsilon^{s_0})$$

or

$$\sum_{\lambda_j<-\varepsilon} \lambda_j^4 = \frac{1}{315\pi}[1+O(\varepsilon^{s_0})] \int_{q(x)\geq\varepsilon} \sqrt{\frac{q(x)-\varepsilon}{p(x)}}[128q^4(x)+64q^3(x)\varepsilon + 48q^2(x)\varepsilon^2 + 40q(x)\varepsilon^3 + 35\varepsilon^4]dx,$$

as $\varepsilon \to +0$.

The theorem is proved. □

Author Contributions

Azad M. Bayramov: methodology, investigation, validation, writing – original draft, writing – review and editing. **Mehmet Bayramoglu:** methodology, investigation, validation. **Seda Kizilbudak:**

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

1. M. A. Naimark, *Linear Differential Operators, Part I, II* (Harrap, 1968).

2. B. Y. Skacek, *In Asymptod of Negative Part of Spectrum of One Dimensioned Differential Operators* (Priblizhennye Metody Resheniya Differentsialnykh Uravneniy, 1963).

3. B. Y. Skachek, "On Asymptotics of the Negative Part of the Spectrum of Many Dimensional Differential Operators," *Doklady Akademii Nauk Ukrainskoi SSR* I (1964): 1417.

4. N. Rosenfeld, "The Eigenvalues of a Class of Singular Differential Operators," *Communications on Pure and Applied Mathematics* 13, no. 3 (1960): 395405.

5. M. G. Gasymov, V. V. Zhikov, and B. M. Levitan, "Discreteness and Finiteness Conditions of the Negative Spectrum of Schrödinger's Operator Equation," *Matematicheskie Zametki* 2, no. 5 (1967): 531538.

6. D. R. Yafaev, "On negative Spectrum of Schrödinger's Operator Equation," *Matematicheskie Zametki* 7, no. 6 (1970): 753763.

7. G. I. Rosenbloom, "Asymptotics of Negative Discrete Spectrum of Schrödinger's Operator," *Matematicheskie Zametki* 21, no. 3 (1977): 399407.

8. F. G. Maksudov, M. Bayramoglu, and E. E. Adiguzelov, "On Asymptotics of Spectrum and Trace of High Order Differential Operator With Operator Coefficients," *Doğa Turkish Journal of Mathematics* 17, no. 2 (1993): 113–128.

9. A. Laptev, "The Negative Spectrum of a Class of Two-Dimensional Schrödinger Operators With Potentials Depending Only on Radius," *Functional Analysis and its Applications* 34, no. 4 (2000): 305–307.

10. E. E. Adiguzelov, O. Baksi, and A. Bayramov, "The Asymptotic Behaviour of the Negative Part of the Spectrum of Sturm-Liouville operator With the Operator Coefficient Which Has Singularity," *International Journal of Differential Equations and Applications* 6, no. 3 (2002): 315–329.

11. J. Behrndt and C. Trunk, "On The Negative Squares of Indefinite Sturm–Liouville operators," *Journal of Differential Equations* 238, no. 2 (2007): 491–519.

12. S. Malchonov and B. Vainberg, "On Negative Spectrum of Schrödinger Type Operator," *Analysis, Partial Differential Equations and Applications, Operator Theory, Advances and Applications* 193 (2009): 197–214.

13. N. Goloschapova and L. Oridoroga, "On the Negative Spectrum of One-Dimensional Schrödinger Operators With Point Interactions," *Integral Equations and Operator Theory* 67, no. 1 (2010): 1–14.

14. H. I. Aslanov and N. A. Gadirli, "On Asymptotic Distribution of Negative Eigen Values of Second Order Equation With Operator Coefficients on a Semi-Axis," *Transactions of NAS of Azerbaijan, Issue Mathematics* 37, no. 1 (2017): 4452.

15. E. Adiguzelov, S. Sengul, and M. Akyol, "The Asymptotic Formulas for the Sum of Squares of Negative Eigenvalues of the Singular Sturm-Liouville Operator," *International Journal of Contemporary Mathematical Sciences* 1, no. 7 (2006): 341–358.

16. V. I. Smirnov, *A Course of Higher Mathematics*, vol. 5 (Pergamon Press, 1964).

17. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 1 (Wiley-VCH Verlag GmbH & Co. KGaA, 1970).

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