YTU FACULTY OF ELECTRICAL & ELECTRONICS ENGINEERING
DEPARTMENT OF CONTROL & AUTOMATION ENGINEERING
KOM3751-2 CONTROL SYSTEMS, MIDTERM EXAM

Name, Surname:
Student number:
Signature:

Solutions

Date: November 08, 2019
Duration: 75 mins.

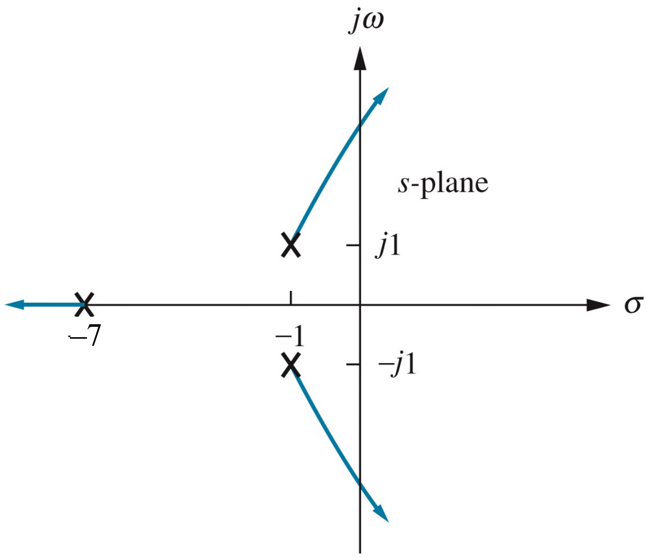
Marking:

Expected:

Problem 1: 70
Problem 2: 30

Problem 1. Considering the root-locus given, which is plotted for a unity feedback system for $K > 0$,

(a) Obtain the open-loop transfer function. **(10 pts)**
(b) Obtain the closed-loop transfer function. **(5 pts)**
(c) Find the value of gain and closed-loop poles at the imaginary axis crossings. **(10 pts)**
(d) Write the range of K for which the closed loop system is stable. **(5 pts)**
(e) Write the value of gain that makes the system marginally stable. **(5 pts)**
(f) What would be the period of oscillation in seconds when the system is marginally stable? **(5 pts)**
(g) What would be the settling time, peak time and percent overshoot at the gain of $K = 15$? **(15 pts)**
Method: For $K = 15$, the closed-loop poles appear at $-7.36, -0.82 \pm j1.81$. Show if the 2nd order approximation is valid. Then use the formula given at the footer.



(h) Calculate the steady-state error when the input is $r(t) = 0.62u(t)$ at the same gain ($K = 15$). **(15 pts)**

Solution 1. Considering it as a unity feedback system,

(a) The open-loop transfer function will be,

$$G(s) = \frac{K}{(s+7)(s^2+2s+2)} = \frac{K}{s^3+9s^2+16s+14}$$

(b) The closed-loop transfer function for the unity feedback system will be,

$$T(s) = \frac{K}{s^3+9s^2+16s+14+K}$$

(a) The Routh Table,

s^3	1	16
s^2	9	$14+K$
s^1	$130-K$	0
s^0	$14+K$	

The imaginary axis crossings occur for $\boldsymbol{K = 130}$ (see the highlighted raw, which is a Row of Zeros (RoZ) for $K = 130$)
Then the even polynomial is taken from the raw above the RoZ as,

$$9s^2 + 14 + K = 0, \text{for } K = 130$$

The poles at imaginary axis crossings: $s^2 = -\frac{144}{9}$ ➔ $\boldsymbol{s_{1,2} = \pm j4}$

(b) The range of K for which the closed loop system is stable: $-14 < K < 130$.
Or for positive values of gain: $0 < K < 130$

(c) When the system is marginally stable, $K = 130$

(d) When the system is marginally stable the frequency of oscillation, $\omega = \frac{2\pi}{T} = 4$ rad/s; $T = 1.57$ sec

(e) For $K = 15$, the closed-loop poles are at $-7.36, -0.82 \pm j1.81$. Since $|-7.36| > 5 \cdot |-0.82|$; the 2nd order approximation is valid. Therefore, the dominant poles of $-0.82 \pm j1.81$ can be used to estimate the time response performance characteristics: $T_s \cong \frac{4}{\zeta\omega_n} \cong \frac{4}{|\text{Re(poles)}|} = \frac{4}{0.82} = 4.88$ sec.

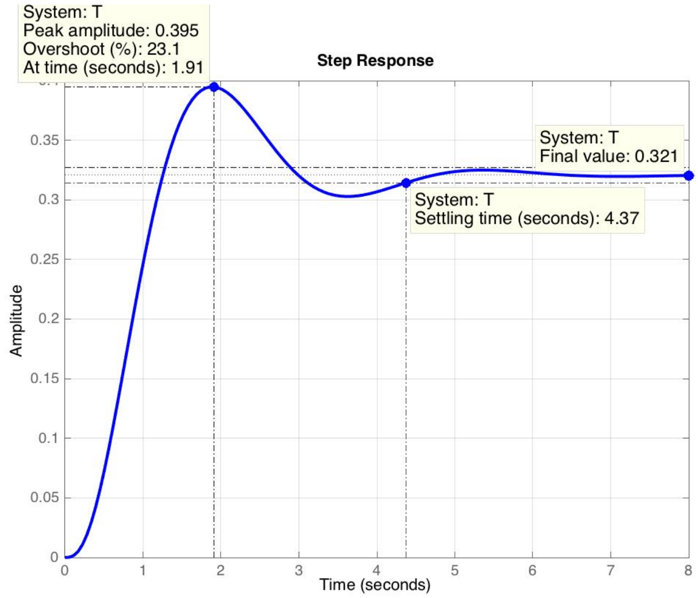
$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{|\text{Im(poles)}|} = \frac{\pi}{1.81} = 1.736$ sec and,

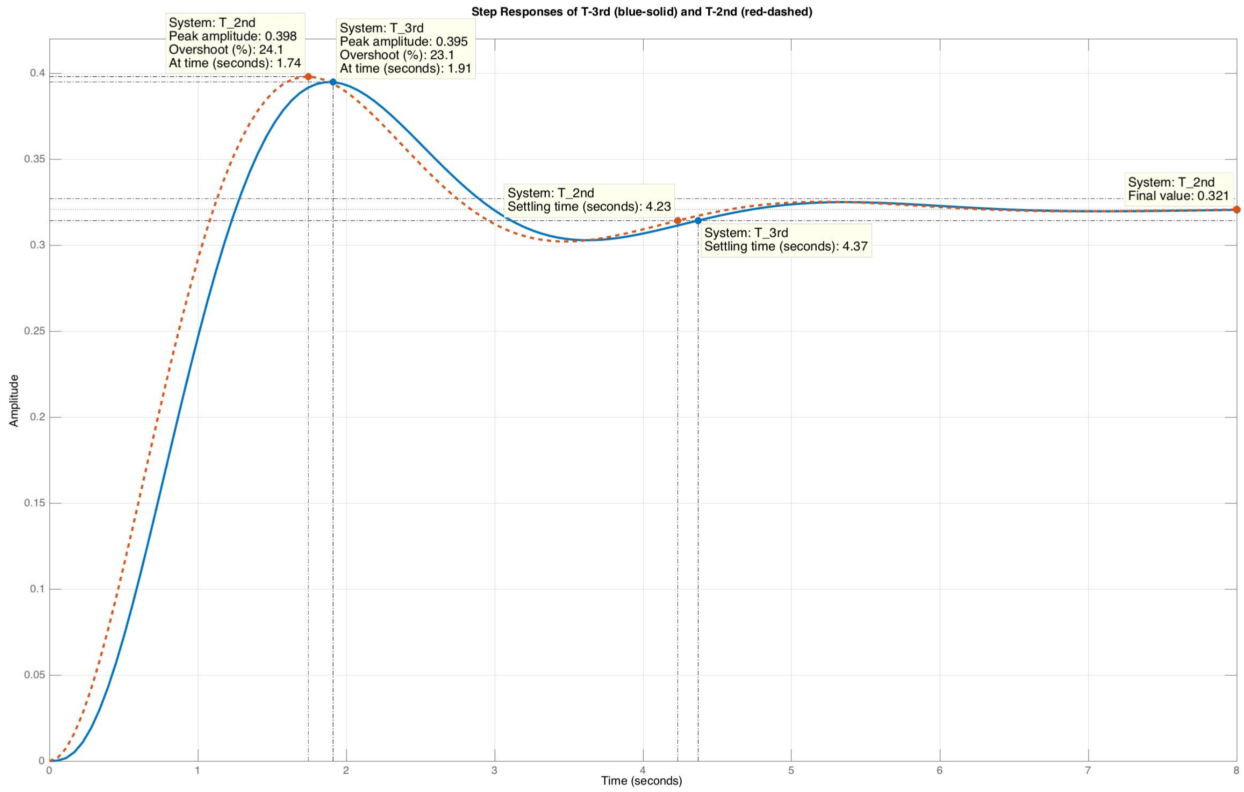
$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$, $T_s \cong \frac{4}{\zeta\omega_n}$, $\%OS = 100.e^{-\zeta\pi/\sqrt{1-\zeta^2}}$, $\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}}$, *Good luck! Şeref Naci Engin*

$\zeta = \cos\theta = \frac{0.82}{\sqrt{0.82^2+1.81^2}} = 0.413$ ➔ $\%OS = 100.e^{-\zeta\pi/\sqrt{1-\zeta^2}} = 24$

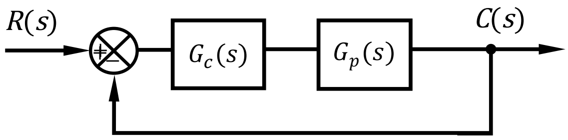
(f) $K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{15}{s^3+9s^2+16s+14} = \frac{15}{14} = 1.0714;\ e_{ss} = \frac{0.62}{1+K_p} = 0.3;\ c_{ss} = 0.62 - 0.3 = 0.32$

- If we simulate the 3rd order system as it is and get its step response of amplitude of 0.62, we would get the following plot. The second figure shows the two plots, output of the 3rd order (blue solid line) and its 2nd order approximation (red dashed line) together on the same plane for comparison purpose.
- Please note that the computed transient response vales of the system with 2nd order approximation are quite close to that of the simulated 3rd order system.
- Hence, it proves that the 2nd order approximation is valid for this system.





Problem 2. A cascaded control system as seen on the right has a plant transfer function,



$$G_p(s) = \frac{1}{(s-1)(s-3)}$$

(a) When the controller is $G_c(s) = K$, which is a simple P, i.e. proportional controller, sketch the root locus to show that the closed loop system is always unstable. **(10 pts)**

(b) When the controller has a zero and a pole as given below sketch the new root locus **(10 pts)**

$$G_c(s) = \frac{K(s+2)}{s+20}$$

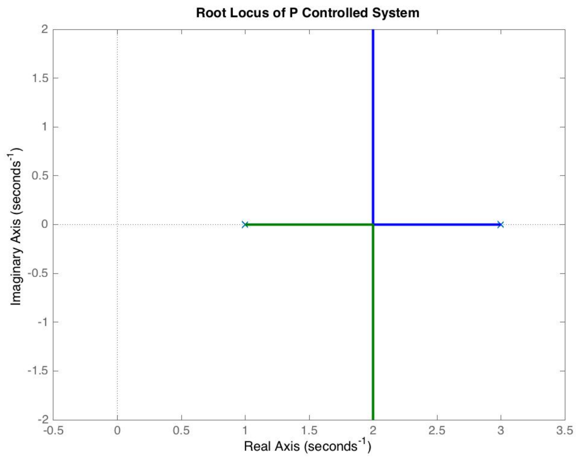
and determine the range of K for which the closed loop system is stable. **(5 pts)**

(c) Determine the value of K and the imaginary poles at $j\omega$ crossings. **(5 pts)**

Hint: When sketching the root locus, if necessary, make use of the asymptotes finding σ_a and θ_a that are the intersecting point and angles with the real axis, respectively, with the following formula,

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} \text{ and } \theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}, \text{where } k = 0, \pm 1, \pm 2, \ldots$$

Solution 2. The original system is an open-loop unstable system, since the open-loop poles are located in the right half of the s-plane.



(a) The root locus of the system with the following open-loop transfer function is plotted,

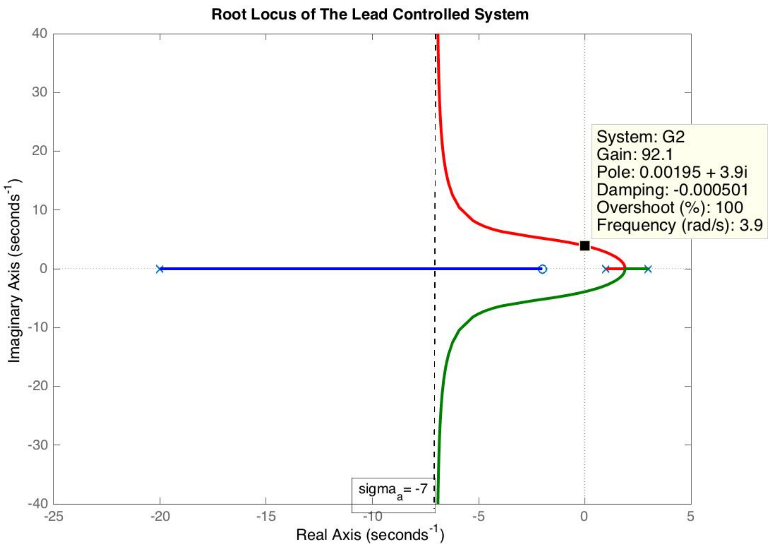
$$G_c(s)G_p(s) = \frac{K}{(s-1)(s-3)}$$

As seen from the plot, the root locus is in the right half of the s-plane for all gain values. Hence, the closed loop system is always unstable.

(b) Now, the system has a new controller, namely a lead controller, to make this open-loop unstable system stable for some values of the gain. The root locus of the system with the following open-loop transfer function is plotted below after finding σ_a and θ_a that are the intersecting point and angles with the real axis, respectively.

$$G_c(s)G_p(s) = \frac{K(s+2)}{(s+20)(s-1)(s-3)}$$

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}} = \frac{-20+1+3-(-2)}{3-1} = \frac{-14}{2} = -7; \quad \theta_a = \frac{(2k+1)\pi}{2} = \pm\frac{\pi}{2}$$



Now, let's find the breakaway and break-in points so that we can plot the root locus more intuitively,

$$G_c(s)G_p(s) = \frac{K(s+2)}{(s+20)(s-1)(s-3)} = \frac{K(s+2)}{s^3+16s^2-77s+60}$$

The characteristic equation is then, $1 + KG(s)H(s) = 0$ and $K = -\frac{1}{G(s)H(s)}$

$\therefore K = -\left.\frac{(s+20)(s-1)(s-3)}{s+2}\right|_{s=\sigma}$; $\frac{dK}{d\sigma} = -\frac{\sigma^3+16\sigma^2-77\sigma+60}{\sigma+2} = 0$; $2\sigma^3 + 22\sigma^2 + 64\sigma - 214 = 0$

The breakaway and break-in points will be the roots of the equation found above, and they are as follows:

$$\sigma_{1,2,3} = 1.9;\ -6.45 \pm j3.86$$

Since there is only one real root, there is only a breakaway point, no break-in points are found.

(c) The value of K and the imaginary poles at $j\omega$ crossings can be found from the Routh-Hurwitz criteria as follows. First, let's get the closed-loop transfer function for this new controller.

$$T(s) = \frac{K(s+2)}{s^3+16s^2+(K-77)s+2K+60}$$

s^3	1	$K-77$
s^2	~~16~~ 8	~~$2K+60$~~ $K+30$
s^1	$(7K-646)/$~~8~~	0
s^0	$K+30$	

The imaginary axis crossings occur for $\boldsymbol{K} = \frac{\mathbf{646}}{\mathbf{7}} = \mathbf{92.286}$

Then the even polynomial: $8s^2 + K + 30 = 0$, for $K = 92.286$

The poles at imaginary axis crossings: $s^2 = -\frac{122.286}{8}$ ➔ $\boldsymbol{s = \pm j3.91}$

The computed values are very close to those that can be read on the root-locus plot.