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Research Article

A Numerical Approach Based on Taylor Polynomials for Solving a Class of Nonlinear Differential Equations

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In this study, a matrix method based on Taylor polynomials and collocation points is presented for the approximate solution of a class of nonlinear differential equations, which have many applications in mathematics, physics and engineering. By means of matrix forms of the Taylor polynomials and their derivatives, the technique we have used reduces the solution of the nonlinear equation with mixed conditions to the solution of a matrix equation which corresponds to a system of nonlinear algebraic equations with the unknown Taylor coefficients. On the other hand, to illustrate the validity and applicability of the method, some numerical examples together with residual error analysis are performed and the obtained results are compared with the existing results in literature.

1. Introduction

In this study, we consider the high order differential equation with nonlinear terms in the form

$$\sum_{k=0}^{m} P_k(x)\, y^{(k)}(x) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x)\, y^{(p)}(x)\, y^{(q)}(x) = g(x) \tag{1}$$

subject to the mixed conditions;

$$\sum_{k=0}^{m-1} \left(a_{kj} y^{(k)}(a) + b_{kj} y^{(k)}(b) + c_{kj} y^{(k)}(c)\right) = \lambda_j \qquad a \le c \le b,\ \ j = 0, 1, \ldots, m-1 \tag{2}$$

Here $P_k(x), Q_{pq}(x)$ and $g(x)$ are the functions defined on the interval $a \le x \le b$; $a_{kj}, b_{kj}, c_{kj}, c$ and λ_j are suitable real constants; $y(x)$ is an unknown function to be determined.

In recent years, it is well known that there exists an increasing interest in the application of the models to nonlinear problems of (1)-(2) type in biology, physics and engineering. Also, the numerical solution methods of them have been developed very rapidly and intensively by many authors [1–10]. In [11], they present a new analytical technique for obtaining the analytical approximate solutions for system of Fredholm-Integral equations based on the use of residual power series method (RPSM). A new general form fractional power series is introduced in the sense of the Caputo fractional derivative in [12]. They implement a relatively new analytic iterative technique to get approximate solutions of differential algebraic equations system based on generalized Taylor series formula and the analysis proposes an analytical-numerical approach for providing solutions of a class of nonlinear fractional Klein-Gordan equation subjected to appropriate initial conditions in Caputo sense by using the Fractional Reduced Differential Transform Method (FRDTM) in [13, 14], respectively. In this study, by means of the matrix methods based on collocation points given by M. Sezer and coworkers [15–17], we developed a new numerical method to find the approximate solutions of (1) in the truncated Taylor series form

$$y(x) \cong y_N(x) = \sum_{n=0}^{N} a_n x^n, \quad a \le x \le b \tag{3}$$

where $a_n, n = 0, 1, \ldots, N \ge m$, are unknown coefficients to be determined; $y_N(x)$ is a Taylor polynomial of degree N on the interval $a \le x \le b$.

2. Operational Matrix Relations

We now consider (1) and find the matrix forms of each term in the equation. We first convert the Taylor polynomial equation defined by (3) to matrix form

$$y(x) \cong \boldsymbol{X}(x)\boldsymbol{A} \tag{4}$$

where

$$\boldsymbol{X}(x) = \begin{bmatrix} 1 & x & \cdots & x^N \end{bmatrix},$$
$$\boldsymbol{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T. \tag{5}$$

Besides, it is clearly seen that the recurrence relation between the matrix $X(x)$ and its derivatives $X^{(k)}(x)$ is

$$X^{(k)}(x) = \boldsymbol{X}(x)\boldsymbol{B}^k, \quad k = 0, 1, \ldots, m \tag{6}$$

where [18]

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)x(N+1)},$$
$$\boldsymbol{B}^0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(N+1)x(N+1)}. \tag{7}$$

From (4) and (6), we have the matrix relation

$$y(x)^{(k)} = \boldsymbol{X}(x)^{(k)}\boldsymbol{A} = \boldsymbol{X}(x)\boldsymbol{B}^k\boldsymbol{A}, \quad k = 0, 1, \ldots, m. \tag{8}$$

In addition, we can obtain the following matrix forms of the expressions $(y^{(0)}(x))^2$, $y^{(1)}(x)y^{(0)}(x)$, $(y^{(1)}(x))^2$, $y^{(2)}(x)y^{(1)}(x)$, $y^{(2)}(x)y^{(0)}(x)$ and $(y^{(2)}(x))^2$, by similar computations as (4)–(8) [15, 17]:

$$\left(y^{(0)}(x)\right)^2 = \boldsymbol{X}(x)\overline{\boldsymbol{X}}(x)\overline{\boldsymbol{A}} \tag{9a}$$

$$y^{(1)}(x)\,y^{(0)}(x) = \boldsymbol{X}(x)\boldsymbol{B}\overline{\boldsymbol{X}}(x)\overline{\boldsymbol{A}} \tag{9b}$$

$$\left(y^{(1)}(x)\right)^2 = \boldsymbol{X}(x)\boldsymbol{B}\overline{\boldsymbol{X}}(x)\overline{\boldsymbol{B}}\,\overline{\boldsymbol{A}} \tag{9c}$$

$$y^{(2)}(x)\,y^{(1)}(x) = \boldsymbol{X}(x)\boldsymbol{B}^2\overline{X}(x)\overline{\boldsymbol{B}}\,\overline{\boldsymbol{A}} \tag{9d}$$

$$y^{(2)}(x)\,y^{(0)}(x) = \boldsymbol{X}(x)\boldsymbol{B}^2\overline{\boldsymbol{X}}(x)\overline{\boldsymbol{A}} \tag{9e}$$

$$\left(y^{(2)}(x)\right)^2 = \boldsymbol{X}(x)\boldsymbol{B}^2\overline{\boldsymbol{X}}(x)\overline{\boldsymbol{B}^2} \tag{9f}$$

where

$$y^{(0)}(x) = y(x),$$
$$y^{(1)}(x) = y'(x),$$
$$y^{(2)}(x) = y''(x)$$
$$\overline{\boldsymbol{X}}(x) = \operatorname{diag}\left[\boldsymbol{X}(x)\ \ \boldsymbol{X}(x)\ \ \cdots\ \ \boldsymbol{X}(x)\right];\ \overline{\boldsymbol{X}}(x)_{(N+1)x(N+1)^2}$$
$$\overline{\boldsymbol{B}} = \operatorname{diag}\left[\boldsymbol{B}\ \ \boldsymbol{B}\ \ \cdots\ \ \boldsymbol{B}\right];\ \overline{\boldsymbol{B}}_{(N+1)^2x(N+1)^2}$$
$$\overline{\boldsymbol{B}^2} = \operatorname{diag}\left[\boldsymbol{B}^2\ \ \boldsymbol{B}^2\ \ \cdots\ \ \boldsymbol{B}^2\right];\ \overline{\boldsymbol{B}^2}_{(N+1)^2x(N+1)^2} \tag{10}$$
$$\boldsymbol{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix}_{(N+1)x1},$$
$$\overline{\boldsymbol{A}} = \begin{bmatrix} a_0\boldsymbol{A} \\ a_1\boldsymbol{A} \\ \vdots \\ a_N\boldsymbol{A} \end{bmatrix}_{(N+1)^2x1}.$$

Now we can define the collocation points as

$$x_i = a + \frac{b-a}{N}i, \quad i = 0, 1, \ldots, N;\ a \le x_0 < x_1 < \cdots < x_N = b. \tag{11}$$

By substituting the collocation points (11) into (1), we get the system of matrix equations, for $i = 0, 1, \ldots, N$

$$\sum_{k=0}^{m} P_k(x_i)\,y^{(k)}(x_i) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x_i)\,y^{(p)}(x_i)\,y^{(q)}(x_i) = g(x_i) \tag{12}$$

or briefly

$$\sum_{k=0}^{m} \boldsymbol{P}_k\boldsymbol{Y}^{(k)} + \sum_{p=0}^{2}\sum_{q=0}^{p} \boldsymbol{Q}_{pq}\boldsymbol{Y}^{(p,q)} = \boldsymbol{G} \tag{13}$$

where

$$\boldsymbol{P}_k = \operatorname{diag}\left[P_k(x_0)\ \ P_k(x_1)\ \ \cdots\ \ P_k(x_N)\right]$$
$$\boldsymbol{Q}_{pq} = \operatorname{diag}\left[Q_{pq}(x_0)\ \ Q_{pq}(x_1)\ \ \cdots\ \ Q_{pq}(x_N)\right]$$

$$\boldsymbol{Y}^{(k)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix},$$

$$\boldsymbol{Y}^{(p,q)} = \begin{bmatrix} y^{(p)}(x_0)\, y^{(q)}(x_0) \\ y^{(p)}(x_1)\, y^{(q)}(x_1) \\ \vdots \\ y^{(p)}(x_N)\, y^{(q)}(x_N) \end{bmatrix},$$

$$\boldsymbol{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}. \tag{14}$$

We now put the collocation points (11) into the relation (8) and obtain the matrix equation, for $i = 0, 1, \ldots, N$,

$$y^{(k)}(x_i) = \boldsymbol{X}(x_i)\boldsymbol{B}^k\boldsymbol{A} \Longrightarrow \boldsymbol{Y}^{(k)} = \boldsymbol{X}\boldsymbol{B}^k\boldsymbol{A} \tag{15}$$

where

$$\boldsymbol{Y}^{(k)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix},$$

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}(x_0) \\ \boldsymbol{X}(x_1) \\ \vdots \\ \boldsymbol{X}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & \ldots & {x_0}^N \\ 1 & x_1 & \ldots & {x_1}^N \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \ldots & {x_N}^N \end{bmatrix}. \tag{16}$$

On the other hand, we can write the nonlinear part of (13) as

$$\begin{aligned} \sum_{p=0}^{2}\sum_{q=0}^{p}\boldsymbol{Q}_{pq}\boldsymbol{Y}^{(p,q)} &= \boldsymbol{Q}_{00}\boldsymbol{Y}^{(0,0)} + \boldsymbol{Q}_{10}\boldsymbol{Y}^{(1,0)} + \boldsymbol{Q}_{11}\boldsymbol{Y}^{(1,1)} \\ &\quad + \boldsymbol{Q}_{20}\boldsymbol{Y}^{(2,0)} + \boldsymbol{Q}_{21}\boldsymbol{Y}^{(2,1)} \\ &\quad + \boldsymbol{Q}_{22}\boldsymbol{Y}^{(2,2)}. \end{aligned} \tag{17}$$

By substituting the collocation points (11) into the relations (9a), (9b), (9c), (9d), (9e), and (9f), respectively, the matrices $\boldsymbol{Y}^{(0,0)}$, $\boldsymbol{Y}^{(1,0)}$, $\boldsymbol{Y}^{(1,1)}$, $\boldsymbol{Y}^{(2,0)}$, $\boldsymbol{Y}^{(2,1)}$, and $\boldsymbol{Y}^{(2,2)}$ are obtained from the following matrix forms:

$$\begin{aligned} \boldsymbol{Y}^{(0,0)} &= \boldsymbol{X}^*_{0,0}\overline{\boldsymbol{A}} \\ \boldsymbol{Y}^{(1,0)} &= \boldsymbol{X}^*_{1,0}\overline{\boldsymbol{A}} \\ \boldsymbol{Y}^{(1,1)} &= \boldsymbol{X}^*_{1,1}\overline{\boldsymbol{A}} \\ \boldsymbol{Y}^{(2,0)} &= \boldsymbol{X}^*_{2,0}\overline{\boldsymbol{A}} \\ \boldsymbol{Y}^{(2,1)} &= \boldsymbol{X}^*_{2,1}\overline{\boldsymbol{A}} \\ \boldsymbol{Y}^{(2,2)} &= \boldsymbol{X}^*_{2,2}\overline{\boldsymbol{A}} \end{aligned} \tag{18}$$

where

$$\boldsymbol{X}^*_{0,0} = \begin{bmatrix} \boldsymbol{X}(x_0)\,\overline{\boldsymbol{X}}(x_0) \\ \boldsymbol{X}(x_1)\,\overline{\boldsymbol{X}}(x_1) \\ \vdots \\ \boldsymbol{X}(x_N)\,\overline{\boldsymbol{X}}(x_N) \end{bmatrix}$$

$$\boldsymbol{X}^*_{1,0} = \begin{bmatrix} \boldsymbol{X}(x_0)\,\boldsymbol{B}\overline{\boldsymbol{X}}(x_0) \\ \boldsymbol{X}(x_1)\,\boldsymbol{B}\overline{\boldsymbol{X}}(x_1) \\ \vdots \\ \boldsymbol{X}(x_N)\,\boldsymbol{B}\overline{\boldsymbol{X}}(x_N) \end{bmatrix}$$

$$\boldsymbol{X}^*_{1,1} = \begin{bmatrix} \boldsymbol{X}(x_0)\,\boldsymbol{B}\overline{\boldsymbol{X}}(x_0)\,\overline{\boldsymbol{B}} \\ X(x_1)\,\boldsymbol{B}\overline{\boldsymbol{X}}(x_1)\,\overline{\boldsymbol{B}} \\ \vdots \\ \boldsymbol{X}(x_N)\,\boldsymbol{B}\overline{\boldsymbol{X}}(x_N)\,\overline{\boldsymbol{B}} \end{bmatrix}$$

$$\boldsymbol{X}^*_{2,0} = \begin{bmatrix} \boldsymbol{X}(x_0)\,\boldsymbol{B}^2\overline{\boldsymbol{X}}(x_0) \\ X(x_1)\,\boldsymbol{B}^2\overline{\boldsymbol{X}}(x_1) \\ \vdots \\ X(x_N)\,\boldsymbol{B}^2\overline{\boldsymbol{X}}(x_N) \end{bmatrix}$$

$$\boldsymbol{X}^*_{2,1} = \begin{bmatrix} \boldsymbol{X}(x_0)\,\boldsymbol{B}^2\overline{\boldsymbol{X}}(x_0)\,\overline{\boldsymbol{B}} \\ \boldsymbol{X}(x_1)\,\boldsymbol{B}^2\overline{\boldsymbol{X}}(x_1)\,\overline{\boldsymbol{B}} \\ \vdots \\ \boldsymbol{X}(x_N)\,\boldsymbol{B}^2\overline{\boldsymbol{X}}(x_N)\,\overline{\boldsymbol{B}} \end{bmatrix}$$

$$\boldsymbol{X}^*_{2,2} = \begin{bmatrix} \boldsymbol{X}(x_0)\,\boldsymbol{B}^2\overline{\boldsymbol{X}}(x_0)\,\overline{\boldsymbol{B}^2} \\ \boldsymbol{X}(x_1)\,\boldsymbol{B}^2\overline{\boldsymbol{X}}(x_1)\,\overline{\boldsymbol{B}^2} \\ \vdots \\ \boldsymbol{X}(x_N)\,\boldsymbol{B}^2\overline{\boldsymbol{X}}(x_N)\,\overline{\boldsymbol{B}^2} \end{bmatrix} \tag{19}$$

$$\overline{\boldsymbol{A}} = \left[a_0\boldsymbol{A} \;\; a_1\boldsymbol{A} \;\; \cdots \;\; a_N\boldsymbol{A}\right]^T.$$

By substituting the matrix relations (15) and (18) into (13), we obtain the fundamental matrix equation

$$\sum_{k=0}^{m} \boldsymbol{P}_k \boldsymbol{X} \boldsymbol{B}^k \boldsymbol{A} + \sum_{p=0}^{2} \sum_{q=0}^{p} \boldsymbol{Q}_{pq} \boldsymbol{X}_{p,q}^{*} \overline{\boldsymbol{A}} = \boldsymbol{G} \tag{20}$$

or briefly

$$\boldsymbol{W}\boldsymbol{A} + \boldsymbol{V}\overline{\boldsymbol{A}} = \boldsymbol{G} \tag{21}$$

where

$$\begin{aligned} \boldsymbol{W} &= \sum_{k=0}^{m} \boldsymbol{P}_k \boldsymbol{X} \boldsymbol{B}^k = \left[w_{ij}\right]; \quad i, j = 0, 1, \ldots, N \\ \boldsymbol{V} &= \sum_{p=0}^{2} \sum_{q=0}^{p} \boldsymbol{Q}_{pq} \boldsymbol{X}_{p,q}^{*} = \left[v_{mn}\right]; \\ & \qquad m = 0, 1, \ldots, N \; n = 0, 1, \ldots, (N+1)^2 \end{aligned} \tag{22}$$

Also we can write the fundamental matrix equation (21) in the augmented matrix form

$$[\boldsymbol{W}; \boldsymbol{V} : \boldsymbol{G}] \tag{23}$$

or clearly

$$[\boldsymbol{W}; \boldsymbol{V} : \boldsymbol{G}] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N}; & v_{00} & v_{01} & \cdots & v_{0(N+1)^2} : & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N}; & v_{10} & v_{11} & \cdots & v_{1(N+1)^2} : & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N0} & w_{N1} & \cdots & w_{NN}; & v_{N0} & v_{N1} & \cdots & v_{N(N+1)^2} : & g(x_N) \end{bmatrix}. \tag{24}$$

Besides, by using the matrix relation (8), the fundamental matrix equation of the mixed conditions (2) becomes

$$\sum_{k=0}^{m-1} \left(a_{kj}\boldsymbol{X}(a) + b_{kj}\boldsymbol{X}(b) + c_{kj}\boldsymbol{X}(c)\right) \boldsymbol{B}^k \boldsymbol{A} = \left[\lambda_j\right] \tag{25}$$

or briefly

$$\boldsymbol{U}\boldsymbol{A} + \boldsymbol{O}^{*}\overline{\boldsymbol{A}} = \boldsymbol{\lambda} \Longrightarrow [\boldsymbol{U}; \boldsymbol{O}^{*} : \boldsymbol{\lambda}] \tag{26}$$

or clearly

$$[\boldsymbol{U}; \boldsymbol{O}^{*} : \boldsymbol{\lambda}] = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N}; & 0 & 0 & \cdots & 0 : & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N}; & 0 & 0 & \cdots & 0 : & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N}; & 0 & 0 & \cdots & 0 : & \lambda_{m-1} \end{bmatrix} \tag{27}$$

so that

$$\begin{aligned} \boldsymbol{U} &= \left[u_{j0} \;\; u_{j1} \;\; \cdots \;\; u_{jN}\right]; \quad j = 0, 1, \ldots, m-1 \\ &= \sum_{k=0}^{m-1} \left(a_{kj}\boldsymbol{X}(a) + b_{kj}\boldsymbol{X}(b) + c_{kj}\boldsymbol{X}(c)\right) \boldsymbol{B}^k \\ \boldsymbol{\lambda} &= \left[\lambda_0 \;\; \lambda_1 \;\; \cdots \;\; \lambda_{m-1}\right]^T \\ \boldsymbol{O}^{*} &= \left[0 \;\; 0 \;\; \cdots \;\; 0\right] \; (\textit{zero matrix}); \; \boldsymbol{O}^{*}{}_{mx(N+1)^2}. \end{aligned} \tag{28}$$

Consequently, to find the Taylor coefficients, a_n ($n = 0, 1, \ldots, N$), related to the approximate solution (3) of problem (1)-(2), by replacing the m row matrices (26) by the last m rows (or any m rows) of the augmented matrix (23), we obtain new augmented matrix.

$$\left[\widetilde{\boldsymbol{W}} : \widetilde{\boldsymbol{V}} : \widetilde{\boldsymbol{G}}\right] = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N}; & v_{00} & v_{01} & \cdots & v_{0(N+1)^2}; & g(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N}; & v_{10} & v_{11} & \cdots & v_{1(N+1)^2}; & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N}; & v_{N-m,0} & v_{N-m,1} & \cdots & v_{N-m,(N+1)^2}; & g(x_N) \\ u_{00} & u_{01} & \cdots & u_{0N} & 0 & 0 & \cdots & 0 & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & 0 & 0 & \cdots & 0 & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,0} & \cdots & u_{m-1,N} & 0 & 0 & \cdots & 0 & \lambda_{m-1} \end{bmatrix}. \tag{29}$$

From this nonlinear system, that is, the matrix equation $\widetilde{\boldsymbol{W}}\boldsymbol{A} + \widetilde{\boldsymbol{V}}\overline{\boldsymbol{A}} = \boldsymbol{G}$, the unknown coefficients, a_n ($n = 0, 1, \ldots, N$), are determined; therefore the Taylor polynomial solution is obtained as

$$y_N(x) = \sum_{n=0}^{N} a_n x^n. \tag{30}$$

3. Accuracy of Solutions and Residual Error Estimation

We can check the accuracy of the obtained solutions as follows [18]. Since the truncated Taylor series (3) is an approximate solution of Eq. (1), then the solution $y_N(x)$ and

its derivatives are substituted in Eq. (1), the resulting equation must be satisfied approximately; that is, for

$$x = x_l \in [a,b], \quad l = 0,1,2,\ldots$$
$$R_N(x_l) = \left| \sum_{k=0}^{m} P_k(x_l)\, y^{(k)}(x_l) + \sum_{p=0}^{2}\sum_{q=0}^{p} Q_{pq}(x_l)\, y^{(p)}(x_l)\, y^{(q)}(x_l) - g(x_l) \right| \cong 0 \tag{31}$$

or

$$R_N(x_l) \le 10^{-k_l}; \quad k_l \text{ is any positive numbers.} \tag{32}$$

If $\max 10^{-k_l} = 10^{-k}$ is prescribed, then the truncation limit N is increased until the difference $R_N(x_l)$ at each of the points becomes smaller than the prescribed 10^{-k}. Therefore, if $R_N(x_l) \longrightarrow 0$ when N is sufficiently large enough, then the error decreases.

On the other hand, by means of the residual function, $R_N(x)$, and the mean value of the function, $|R_N(x)|$, on the interval $[a,b]$, the accuracy of the solution can be controlled and the error can be estimated [19].

Thus, the upper bound of the mean error, $\overline{R}_n$, is as follows:

$$\left| \int_a^b R_N(x)\,dx \right| \le \int_a^b \left| R_N(x) \right| dx \tag{33}$$

and

$$\left| \int_a^b R_N(x)\,dx \right| = (b-a)\left| R_N(c) \right|, \quad a \le c \le b$$
$$\left| \int_a^b R_N(x)\,dx \right| = (b-a)\left| R_N(c) \right| \le \int_a^b \left| R_N(x) \right| dx \tag{34}$$
$$\left| R_N(c) \right| \le \frac{\int_a^b \left| R_N(x) \right| dx}{b-a} = \overline{R}_n$$

Moreover we use different error norms for measuring errors. These are defined as follows:

(1) $L_2 = (\sum_{i=0}^{n} (e_i)^2)^{1/2}$

(2) $L_\infty = \max(e_i),\ 0 \le i \le n$

(3) $RMS = \sqrt{\sum_{i=1}^{n} (e_i)^2/(n+1)}$

where $e_i = |y(x_i) - y_N(x_i)|$; also y and y_N are the exact and approximate solutions of the problem, respectively [20].

4. Numerical Examples

The method of this study is useful in finding the solutions of a class of nonlinear equations in terms of Taylor polynomials and the accuracy. We illustrate it by the several numerical examples and perform all of them on the computer using a program written separately in MATLAB R2017b.

Example 1. Consider the second-order nonlinear differential equation

$$y''(x) - 2y'(x) + y(x) + y^2(x) - y''(x)\,y(x) = g(x) \tag{35}$$

with the initial and boundary conditions

$$y(0) = 3, \quad y'(0) = 2 \tag{36}$$
$$0 \le x \le 1 \tag{37}$$

where $g(x) = 2 + 2e^x$.

While the exact solution is $y(x) = 1 + 2e^x$, the proposed method is applied and the approximate solutions of (35) under conditions (36) are obtained as $y_2(x) = 3 + 2x + x^2$, $y_3(x) = 3+2x+x^2+0.3942x^3$, $y_4(x) = 3+2x+x^2+0.3251x^3+0.1072x^4$, $y_5(x) = 3 + 2x + x^2 + 0.334149x^3 + 0.079586x^4 + 0.022571x^5$ for N = 2,3,4,5, respectively.

In Tables 1 and 2, we see that absolute errors and L_2, L_∞, and RMS errors are calculated for $N = 2, 3, 4, 5$, respectively.

Example 2. Consider the second-order nonlinear differential equation of the form

$$\frac{d^2y}{d^2x} + \left(\frac{dy}{dx}\right)^2 + y^2(x) = 1 - \sin(x) \tag{38}$$

with the initial conditions

$$y(0) = 0, \quad y'(0) = 1. \tag{39}$$

See [21].

Similarly, while the exact solution is $y(x) = \sin x$, the approximate solutions of (38) under conditions (39) are obtained as $y_2(x) = x$, $y_3(x) = x - 0.16398x^3$, $y_4(x) = x-0.16971x^3+0.00987x^4$, $y_5(x) = x-0.16673x^3+0.00030x^4+0.00793x^5$ for N = 2,3,4,5, respectively.

Considering $N = 2, 3, 4, 5$, the obtained approximate solutions are compared with the exact solution in Figure 1 and the absolute errors are demonstrated in Table 3.

The upper bound of the mean error, $\overline{R}_n$, of Example 2 can be calculated as $\overline{R}_2 = 7.930306e-01$, $\overline{R}_3 = 4.02186e-02$, $\overline{R}_4 = 1.706537e-02$, $\overline{R}_5 = 4.145337e-04$ as in the method given and $\overline{R}_n$ of Example 2 is illustrated in Figure 2 for $N = 2, 3, 4, 5$.

Example 3. Consider Volterra's Population Model, a model for population growth of a closed system introduced by Volterra.

The model is defined by the nonlinear integro-differential equation [22]

$$\kappa \frac{du}{dx} = u(x) - u^2(x) - u(x)\int_0^t u(x)\,dx, \quad u(0) = 0.15 \tag{40}$$

Table 1: Absolute error of Example 1 for N = 2, 3, 4, 5 and h = 0.1.

x_l	Absolute errors			
	e_2	e_3	e_4	e_5
0	0	0	0	0
0.1	3.41836e-04	5.23638e-05	6.01615e-06	4.98080e-07
0.2	2.80552e-03	3.48083e-04	3.31963e-05	2.24337e-06
0.3	9.71762e-03	9.25784e-04	7.15951e-05	3.92686e-06
0.4	2.36494e-02	1.57940e-03	9.86752e-05	4.72833e-06
0.5	4.74425e-02	1.83245e-03	1.05041e-04	5.66754e-06
0.6	8.42376e-02	9.09599e-04	1.22880e-04	8.24884e-06
0.7	1.37505e-01	2.29481e-03	2.57394e-04	1.01147e-05
0.8	2.11082e-01	9.25145e-03	7.21536e-04	2.60625e-06
0.9	3.09206e-01	2.18344e-02	1.87440e-03	6.66051e-05
1	4.36564e-01	4.23636e-02	4.26365e-03	2.56735e-04

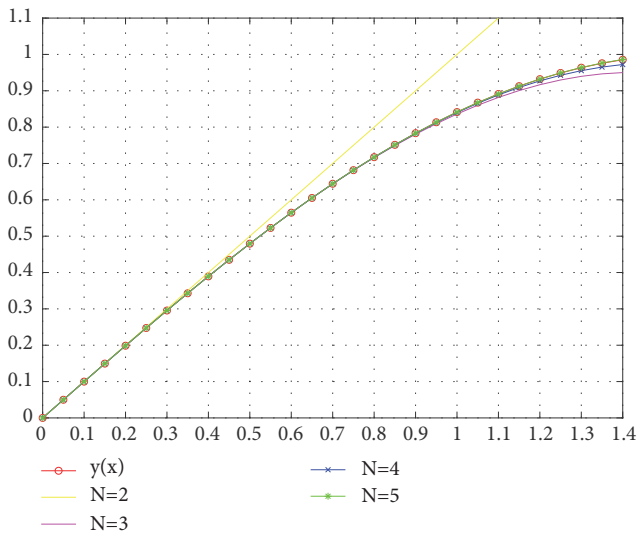


Figure 1: Comparison of the Taylor polynomial solutions and exact solution of Example 2 for N=2-5.

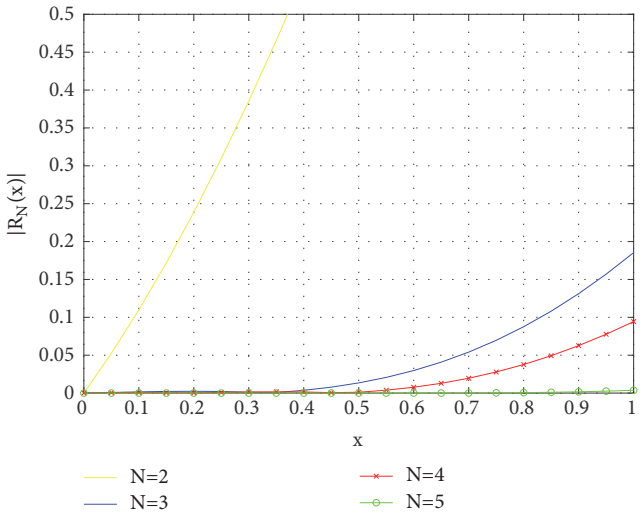


Figure 2: The residual error functions of Example 2 for N=2-5

Table 2: L_2, L_∞, and RMS errors of Example 1 for N = 3, 4, 5.

N	L_2-error	L_∞-error	RMS-error
3	4.86820e-02	4.23636e-02	2.43410e-02
4	4.72452e-03	4.26365e-03	2.11287e-03
5	2.65709e-04	2.56735e-04	1.08475e-04

where κ is a prescribed parameter. In order to examine the mathematical behavior of the individual's scale population u (t), we first set

$$y(t) = \int_0^t u(x)\,dx,$$
$$y'(t) = u(t), \tag{41}$$
$$y''(t) = u'(t)$$

Substituting (41) into (40) and using $\kappa = 0.25$ for numerical purpose, we find

$$y''(t) - 4y'(t) + 4y'(t)^2 + 4y(t)\,y'(t) = 0 \tag{42}$$

with the initial conditions

$$y(0) = 0,$$
$$y'(0) = 0.15 \tag{43}$$

See [23].

Because the problem has not exact solution, we follow the same procedure and obtain the approximate solutions of (42) under the conditions (43). For N=5, the obtained solution using our method is $y_5(x) = 0.15x + 0.2550x^2 + 0.27107596x^3 - 0.12335678x^4 - 0.108288366x^5$. In Table 4, the results of the approximate solutions of the problem that is obtained with Runge-Kutta method [24], Adomian decomposition method [23], and the proposed method on $x \in [0, 1]$ are presented in Table 4.

Table 3: Absolute errors of Example 2 for N = 2, 3, 4, 5 and h = 0.1.

x_n	Absolute errors			
	e_2	e_3	e_4	e_5
0	0	0	0	0
0.1	1.66583e-04	2.60335e-06	2.13964e-06	3.73468e-08
0.2	0.00133	1.88292e-05	1.12187e-05	1.53195e-07
0.3	0.00448	5.23333e-05	2.24296e-05	2.16761e-07
0.4	0.01058	8.69376e-05	2.71103e-05	1.79108e-07
0.5	0.02057	7.69613e-05	2.24136e-05	2.26104e-07
0.6	0.03536	6.21533e-05	2.06813e-05	6.36595e-07
0.7	0.05578	4.62827e-04	5.84302e-05	1.25213e-06
0.8	0.08264	1.31385e-03	2.04858e-04	4.68499e-07
0.9	0.11667	2.86832e-03	5.69792e-04	6.33607e-06
1	0.15853	5.45098e-03	1.31098e-03	2.90151e-05

Table 4: The results of the approximate solutions of Example 3 and h = 0.1.

x_n	Runge-Kutta method	Adomian decomposition method	Proposed method
0	0	0	0
0.1	1.77737e-02	1.77742e-02	1.78076e-02
0.2	4.19657e-02	4.19670e-02	4.21365e-02
0.3	7.36730e-02	7.36813e-02	7.40067e-02
0.4	1.13439e-01	1.13495e-01	1.13882e-01
0.5	1.60989e-01	1.61261e-01	1.61540e-01
0.6	2.15187e-01	2.16202e-01	2.15944e-01
0.7	2.74266e-01	2.77198e-01	2.75111e-01
0.8	3.36210e-01	3.42288e-01	3.35980e-01
0.9	3.99101e-01	4.04491e-01	3.95286e-01
1	4.61346e-01	4.37632e-01	4.44430e-01

Example 4. Consider the following differential equation:

$$y'(x) - 2y(x) + y^2(x) = g(x) \tag{44}$$

$$g(x) = \begin{cases} (x-1)^2, & 0 \le x \le \dfrac{1}{2} \\ x^2 - 2, & \dfrac{1}{2} < x \le 1. \end{cases} \tag{45}$$

It is solved as follows with the proposed method under boundary conditions.

$$y(0) = 0. \tag{46}$$

(a) For $0 \le x \le 1/2$, $P_0(x) = -2$, $P_1(x) = 1$, $Q_{00}(x) = 1$.
For $g_1(x) = (x-1)^2$, $N = 2$ and the ordering points are $x_0 = 0$, $x_1 = 1/4$, $x_2 = 1/2$ for the interval $0 \le x \le 1/2$.
The solution is investigated in the following form:

$$y(x) \cong \sum_{n=0}^{1} a_n x^n. \tag{47}$$

Using our proposed method, the fundamental matrix equation is obtained as

$$P_1 X B + P_0 X A + Q_{00} X_{00}^* \overline{A} = G. \tag{48}$$

Here,

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} X(0) \\ X\left(\frac{1}{4}\right) \\ X\left(\frac{1}{2}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{16} \\ 1 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

$$Q_{00} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

$$X_{00}^{*} = \begin{bmatrix} X(0)\,\overline{X}(0) \\ X\left(\frac{1}{4}\right)\overline{X}\left(\frac{1}{4}\right) \\ X\left(\frac{1}{2}\right)\overline{X}\left(\frac{1}{2}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \end{bmatrix}$$

$$\overline{X}(x) = \operatorname{diag}\left[X(x) \;\; X(x) \;\; X(x)\right]$$

$$A = \left[a_0 \;\; a_1 \;\; a_2\right]^T,$$

$$\overline{A} = \left[a_0 A \;\; a_1 A \;\; a_2 A\right]^T$$

$$G_1(x) = \left[g_1(0) \;\; g_1\left(\frac{1}{4}\right) \;\; g_1\left(\frac{1}{2}\right)\right]^T = \left[1 \;\; \frac{9}{16} \;\; \frac{1}{4}\right]^T. \tag{49}$$

The following fundamental matrix equation is obtained as

$$\begin{bmatrix} -2 & 1 & 0 \\ -2 & 0 & \frac{3}{8} \\ -2 & 0 & \frac{1}{2} \end{bmatrix} A + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \end{bmatrix} \overline{A} = \begin{bmatrix} 1 \\ \frac{9}{16} \\ \frac{1}{4} \end{bmatrix}. \tag{50}$$

$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} A + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overline{A} = 0$ is obtained for the boundary condition $y(0) = 0$ and using the proposed method, the desired matrix equation

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & \frac{1}{2} \end{bmatrix} A + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \end{bmatrix} \overline{A} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{4} \end{bmatrix} \tag{51}$$

is constructed.

Thus, Taylor's coefficients are found as $a_0 = 0$, $a_1 = 1$, $a_2 = 0$ and for $0 \le x \le 1/2$, the solution

$$y(x) = x, \quad 0 \le x \le \frac{1}{2} \tag{52}$$

is obtained.

(b) For $g_1(x) = x^2 - 2$, $N = 2$ and the ordering points are $x_0 = 1/2$, $x_1 = 3/4$, $x_2 = 1$ for the interval $1/2 < x \le 1$. $y(1/2) = 1/2$ is obtained for $x_0 = 1/2$ by considering $y(x) = x$. Thus, the following matrix equation is written as follows:

$$\begin{bmatrix} -2 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{4} \\ -2 & -1 & 0 \end{bmatrix} A + \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \overline{A} = \begin{bmatrix} -\frac{7}{4} \\ \frac{1}{2} \\ -1 \end{bmatrix}. \tag{53}$$

Thus, by calculating Taylor's coefficients $a_0 = 1$, $a_1 = -1$, $a_2 = 0$ for $1/2 < x \le 1$ the solution

$$y(x) = 1 - x, \quad \frac{1}{2} < x \le 1 \tag{54}$$

is obtained.

The solution of (44) under condition (46)

$$y(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1 - x, & \frac{1}{2} < x \le 1 \end{cases} \tag{55}$$

overlaps with the exact solution of the problem.

5. Conclusion

Nonlinear differential equations used in engineering, physics, mathematics or in many modelling problems are usually difficult to solve analytically. In this study, a Taylor matrix method has been given to solve a class of nonlinear differential equations in order to find a consistent approximate solution. A significant advantage of the proposed method, the Taylor coefficients of the solution can be found obviously by developing computer programs. The results related to examples have been shown in Tables 1–4 and Figures 1 and 2. As seen from tables and figures, when the value of N is increased, the numerical results rapidly improve. Residual error analysis has been also developed for the accuracy of solutions. The proposed method can be improved with new strategies to solve other high order nonlinear differential equations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

The abstract that has the same title was presented in the ICAAMM 2017 (International Conference on Applied Analysis and Mathematical Modelling).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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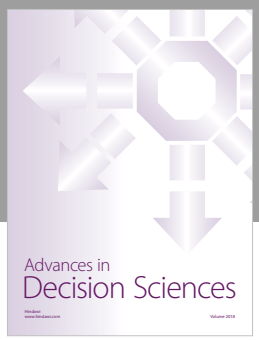
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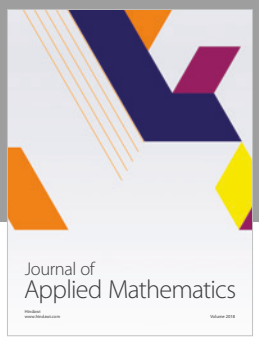
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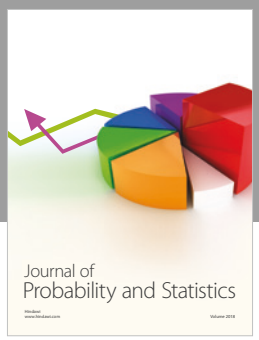
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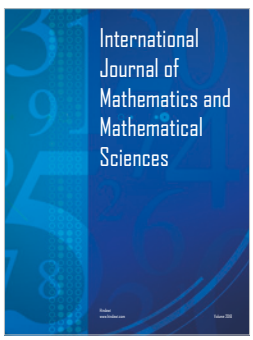








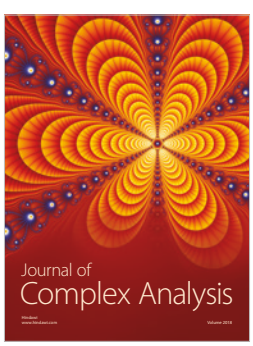


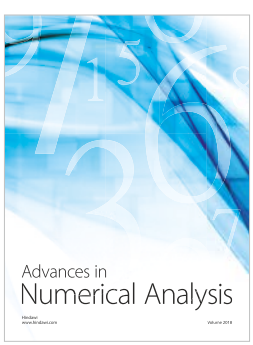


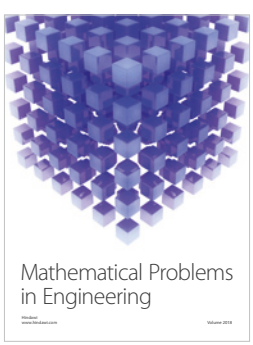




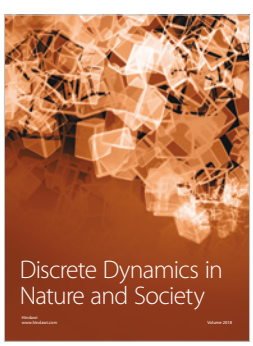




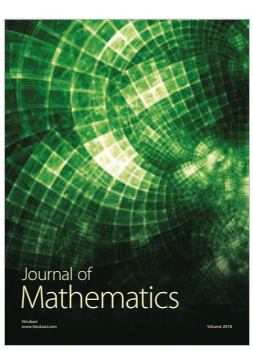




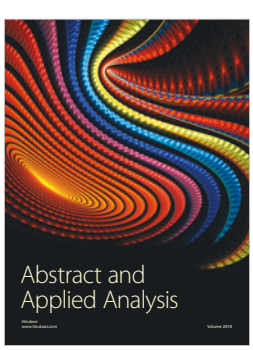


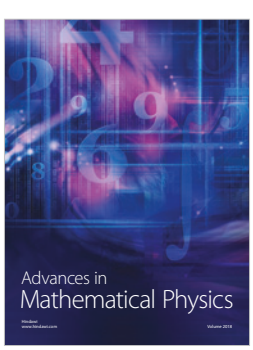












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