Solve the following system by using Smith-Normal Form:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + h_1(t) \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + h_2(t) \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + h_n(t) \end{aligned}\right\}$$

$$\frac{dX}{dt} = A.X + H \qquad D = \frac{d}{dt}$$

$$DX = A.X + H$$

$$\underbrace{(DI - A)}_{\text{Polynomial matrix: } B}.X = H$$

$$B.X = H$$

Smith-Normal form of B is N. That is;

$$P.B.Q = N.$$

Let the solution of the system be

$$X = QY.$$

If so, $X = QY$ satisfies the equation.

$$B \cdot X = H$$

$$B \cdot Q \cdot Y = H$$

$$\underbrace{P \cdot B \cdot Q}_{N} \cdot Y = P \cdot H$$

$$\boxed{N \cdot Y = P \cdot H}.$$

* Find Y from this equality.

At the end, by using Y, the solution of the system is obtained.

$$\underline{X = QY}$$

Example:
$$\left.\begin{aligned} \frac{dx}{dt} + x + y &= 1 \\ \frac{dy}{dt} - x + 3y &= -1 \end{aligned}\right\}$$
solve this system by using Smith-Normal form.

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(D+1)x + y = 1$$
$$-x + (D+3)y = -1$$

$$\underbrace{\begin{bmatrix} D+1 & 1 \\ -1 & D+3 \end{bmatrix}}_{DI - A = B} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{B \cdot X = H}$$

$$B = \begin{bmatrix} D+1 & 1 \\ -1 & D+3 \end{bmatrix}$$

Find the Smith-Normal form of B.

$$\underset{C_{12}}{\begin{bmatrix} D+1 & 1 \\ -1 & D+3 \end{bmatrix}} \sim \underset{C_{21}(-D-1)}{\begin{bmatrix} 1 & D+1 \\ D+3 & -1 \end{bmatrix}} \sim \underset{R_{21}(-D-3)}{\begin{bmatrix} 1 & 0 \\ D+3 & -D^2-4D-4 \end{bmatrix}} \sim$$

$$\sim \underset{R_2(-1)}{\begin{bmatrix} 1 & 0 \\ 0 & -D^2-4D-4 \end{bmatrix}} \sim \begin{bmatrix} 1 & 0 \\ 0 & D^2+4D+4 \end{bmatrix} = N$$

$$P \rightarrow \underset{R_{21}(-D-3)}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \sim \underset{R_2(-1)}{\begin{bmatrix} 1 & 0 \\ -D-3 & 1 \end{bmatrix}} \sim \begin{bmatrix} 1 & 0 \\ D+3 & -1 \end{bmatrix} = P$$

$$Q \rightarrow \underset{C_{12}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \sim \underset{C_{21}(-D-1)}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \sim \begin{bmatrix} 0 & 1 \\ 1 & -D-1 \end{bmatrix} = Q$$

$$NY = PH$$

$$\begin{bmatrix} 1 & 0 \\ 0 & D^2+4D+4 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ D+3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ (D^2+4D+4)y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow y_1 = 1$$

$$(D^2+4D+4)y_2 = 4$$

$$y_2'' + 4y_2' + 4y_2 = 4$$

$$\left.\begin{aligned} y_H &= c_1 e^{-2t} + c_2 . t . e^{-2t} \\ y_p &= 1 \end{aligned}\right\}$$

$$y_2 = y_H + y_p$$

$$\Rightarrow y_2 = c_1 e^{-2t} + c_2 t e^{-2t} + 1$$

$$X = Q \cdot Y$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -D-1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ c_1 e^{-2t} + c_2 t \cdot e^{-2t} + 1 \end{bmatrix}$$

$$\left. \begin{aligned} x &= c_1 e^{-2t} + c_2 t \cdot e^{-2t} + 1 \\ y &= (c_1 - c_2) e^{-2t} + c_2 t e^{-2t} \end{aligned} \right\} \text{ is found.}$$

Example:

$$\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{dx}{dt} + x + y = e^t$$

$$\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{dx}{dt} \qquad = e^{-t}$$

Solve the system by using Smith-Normal form.

$$B \cdot X = H$$

$$(D^2+D+1)x + (D^2+1)y = e^t$$

$$(D^2+D)x + D^2 y \qquad = e^{-t}$$

$$B = \begin{bmatrix} D^2+D+1 & D^2+1 \\ D^2+D & D^2 \end{bmatrix} \qquad H = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\xrightarrow{R_{12}(-1)} \sim \underset{C_{21}(-1)}{\begin{bmatrix} 1 & 1 \\ D^2+D & D^2 \end{bmatrix}} \sim \underset{H_{21}(-D^2-D)}{\begin{bmatrix} 1 & 0 \\ D^2+D & -D \end{bmatrix}} \sim \underset{C_2(-1)}{\begin{bmatrix} 1 & 0 \\ 0 & -D \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} = N$$

$$P = \begin{bmatrix} 1 & -1 \\ -D^2-D & D^2+D+1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$B \cdot X' = H$$

$$N \cdot Y = P \cdot H \qquad X = Q \cdot Y$$

$$\begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -D^2-D & D^2+D+1 \end{bmatrix} \cdot \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$$

$$y_1 = e^t - e^{-t}$$

$$Dy_2 = -e^t - e^t + e^{-t} - e^{-t} + e^{-t}$$

$$y_2' = -2e^t + e^{-t}$$

$$y_2 = -2e^t - e^{-t} + c_1$$

$$X = Q \cdot Y$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} e^t - e^{-t} \\ -2e^t - e^{-t} + c_1 \end{bmatrix}$$

$$\left.\begin{aligned} x &= -e^t - 2e^{-t} + c_1 \\ y &= 2e^t + e^{-t} - c_1 \end{aligned}\right\} \text{ is found.}$$