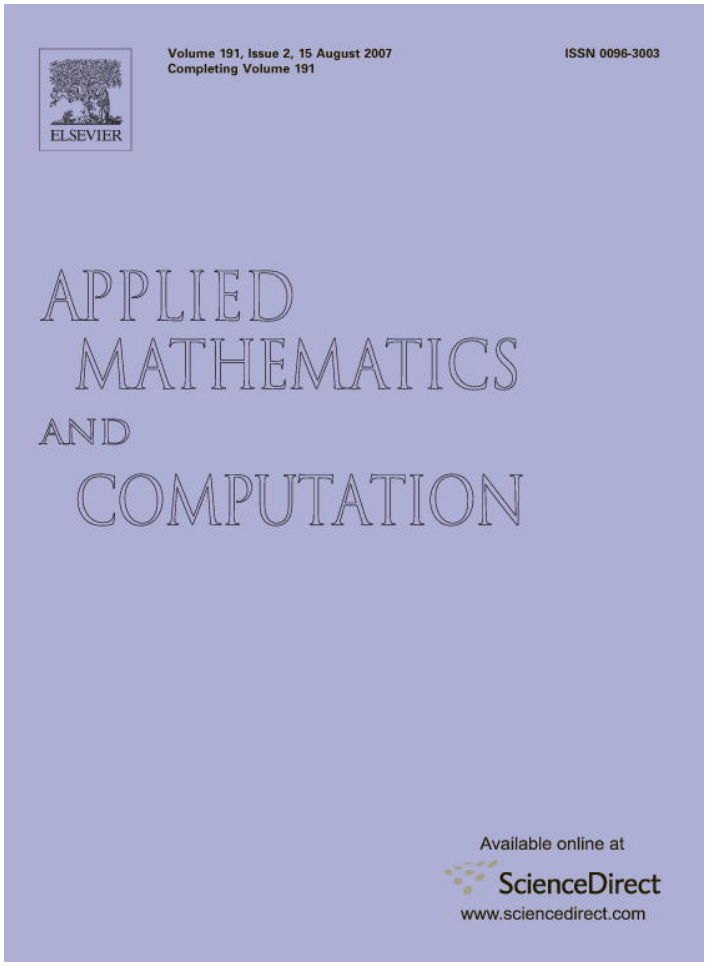
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Computation of eigenvalues and eigenfunctions of a discontinuous boundary value problem with retarded argument

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Abstract

In this study a discontinuous boundary value problem with retarded argument and with transmission conditions at the point of discontinuity are investigated. In the special case that our problem is continuous (i.e. when $\delta = 1$ in below) the obtained results coincidewith corresponding results in [S.B. Norkin, On boundary problem of Sturm–Liouville type for second-order differential equation with retarded argument, Izv. Vysś. Ućebn. Zaved. Matematika 6 (7) (1958) 203–214].
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1. Introduction

The theory of differential equations with retarded arguments is one of the actual branch of the theory of ordinary differential equations. Particularly, there has been increasing interest in spectral analysis of boundary value problems for differential equations with retarded arguments. There is quite substantial literature concerning such problems. Here we mention the results of [1–8], and the corresponding references are cited. Also various physical applications of such problems can be found in [7]. Boundary value problems with transmission conditions arise, as a rule, in the theory of heat and mass transfer in varied assortment of physical transfer problems.

In this paper we investigated eigenvalues and eigenfunctions of the boundary value problem for the differential equation

$$u''(x) + q(x)u(x - \Delta(x)) + \lambda u(x) = 0 \tag{1.1}$$

on $[0, \frac{\pi}{2}) \bigcup (\frac{\pi}{2}, \pi]$, with boundary conditions

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$$u(0)\cos\alpha + u'(0)\sin\alpha = 0, \tag{1.2}$$

$$u(\pi)\cos\beta + u'(\pi)\sin\beta = 0 \tag{1.3}$$

and transmission conditions

$$u\left(\frac{\pi}{2}-0\right) - \delta u\left(\frac{\pi}{2}+0\right) = 0, \tag{1.4}$$

$$u'\left(\frac{\pi}{2}-0\right) - \delta u'\left(\frac{\pi}{2}+0\right) = 0, \tag{1.5}$$

where the real-valued function $q(x)$ is continuous in $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$ and has a finite limit $q(\frac{\pi}{2}\pm 0) = \lim_{x\to\frac{\pi}{2}\pm 0} q(x)$, the real-valued function $\Delta(x) \geqslant 0$ continuous in $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$ and has a finite limit $\Delta(\frac{\pi}{2}\pm 0) = \lim_{x\to\frac{\pi}{2}\pm 0}\Delta(x), x-\Delta(x) \geqslant 0$, if $x\in[0,\frac{\pi}{2}), x-\Delta(x)\geqslant\frac{\pi}{2}$, if $x\in(\frac{\pi}{2},\pi]$; λ is a real eigenparameter and $\alpha,\beta,\delta\neq 0$ are arbitrary real numbers. Let $\omega_1(x,\lambda)$ be a solution of Eq. (1.1) on $[0,\frac{\pi}{2}]$, satisfying the initial conditions

$$\omega_1(0,\lambda) = \sin\alpha, \quad \omega_1'(0,\lambda) = -\cos\alpha. \tag{1.6}$$

The initial conditions (1.6) define a unique solution of Eq. (1.1) on $[0,\frac{\pi}{2}]$ (cf. [7, p. 12, Theorem 1.2.1]).

After defining above solution we shall define the solution $\omega_2(x,\lambda)$ of Eq. (1.1) on $[\frac{\pi}{2},\pi]$ by means of the solution $\omega_1(x,\lambda)$ by the initial conditions

$$\omega_2\left(\frac{\pi}{2},\lambda\right) = \delta^{-1}\omega_1\left(\frac{\pi}{2},\lambda\right), \quad \omega_2'\left(\frac{\pi}{2},\lambda\right) = \delta^{-1}\omega_1'\left(\frac{\pi}{2},\lambda\right). \tag{1.7}$$

The initial conditions (1.7) are defined as a unique solution of Eq. (1.1) on $[\frac{\pi}{2},\pi]$.

Consequently, the function $\omega(x,\lambda)$ defined on $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$ by the equality

$$\omega(x,\lambda) = \begin{cases} \omega_1(x,\lambda), & x\in\left[0,\frac{\pi}{2}\right), \\ \omega_2(x,\lambda), & x\in\left(\frac{\pi}{2},\pi\right] \end{cases}$$

is a such solution of the Eq. (1.1) on $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$, which satisfies one of the boundary conditions (namely (1.2)) and both transmission conditions (1.4) and (1.5).

Lemma 1.1. *Let $\omega(x,\lambda)$ be a solution of Eq. (1.1) and $\lambda > 0$. Then the following integral equations hold*:

$$\omega_1(x,\lambda) = \sin\alpha\cos sx - \frac{\cos\alpha}{s}\sin sx - \frac{1}{s}\int_0^x q(\tau)\sin s(x-\tau)\omega_1(\tau-\Delta(\tau),\lambda)\,d\tau \quad (s=\sqrt{\lambda},\lambda>0), \tag{1.8}$$

$$\omega_2(x,\lambda) = \frac{1}{\delta}\omega_1\left(\frac{\pi}{2},\lambda\right)\cos s\left(x-\frac{\pi}{2}\right) + \frac{\omega_1'\left(\frac{\pi}{2},\lambda\right)}{s\delta}\sin s\left(x-\frac{\pi}{2}\right) - \frac{1}{s}\int_{\frac{\pi}{2}}^x q(\tau)\sin s(x-\tau)\omega_2(\tau-\Delta(\tau),\lambda)\,d\tau \quad \left(s=\sqrt{\lambda},\lambda>0\right). \tag{1.9}$$

Proof. To prove this, it is enough to substitute $-s^2\omega_1(\tau,\lambda) - \omega_1''(\tau,\lambda)$ and $-s^2\omega_2(\tau,\lambda) - \omega_2''(\tau,\lambda)$ instead of $q(\tau)\omega_1(\tau-\Delta(\tau),\lambda)$ and $q(\tau)\omega_2(\tau-\Delta(\tau),\lambda)$ in the integrals in (1.8) and (1.9) respectively and integrate by parts twice. □

Theorem 1.1. *The problem (1.1)–(1.5) can have only simple eigenvalues.*

Proof. Let $\tilde{\lambda}$ be an eigenvalue of the problem (1.1)–(1.5) and

$$\tilde{u}(x,\tilde{\lambda}) = \begin{cases} \tilde{u}_1(x,\tilde{\lambda}), & x\in\left[0,\frac{\pi}{2}\right), \\ \tilde{u}_2(x,\tilde{\lambda}), & x\in\left(\frac{\pi}{2},\pi\right] \end{cases}$$

be a corresponding eigenfunction. Then from (1.2) and (1.6) it follows that the determinant

$$W[\tilde{u}_1(0,\tilde{\lambda}),\omega_1(0,\tilde{\lambda})] = \begin{vmatrix} \tilde{u}_1(0,\lambda) & \sin\alpha \\ \tilde{u}_1'(0,\lambda) & \cos\alpha \end{vmatrix} = 0,$$

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and by Theorem 2.2.2 in [7] the functions $\tilde{u}_1(x,\tilde{\lambda})$and $\omega_1(x,\tilde{\lambda})$ are linearly dependent on $[0,\frac{\pi}{2}]$. We can also prove that the functions $\tilde{u}_2(x,\tilde{\lambda})$ and $\omega_2(x,\tilde{\lambda})$ are linearly dependent on $[\frac{\pi}{2},\pi]$. Hence

$$\tilde{u}_i(x,\tilde{\lambda}) = K_i\omega_i(x,\tilde{\lambda}) \quad (i=1,2) \tag{1.10}$$

for some $K_1 \neq 0$ and $K_2 \neq 0$. We must show that $K_1 = K_2$. Suppose that $K_1 \neq K_2$. From the equalities (1.4) and (1.10), we have

$$\begin{aligned}\tilde{u}\left(\frac{\pi}{2}-0,\tilde{\lambda}\right) - \delta\tilde{u}\left(\frac{\pi}{2}+0,\tilde{\lambda}\right) &= \tilde{u}_1\left(\frac{\pi}{2},\tilde{\lambda}\right) - \delta\tilde{u}_2\left(\frac{\pi}{2},\tilde{\lambda}\right) = K_1\omega_1\left(\frac{\pi}{2},\tilde{\lambda}\right) - \delta K_2\omega_2\left(\frac{\pi}{2},\tilde{\lambda}\right)\\ &= K_1\delta\omega_2\left(\frac{\pi}{2},\tilde{\lambda}\right) - K_2\delta\omega_2\left(\frac{\pi}{2},\tilde{\lambda}\right) = \delta(K_1-K_2)\omega_2\left(\frac{\pi}{2},\tilde{\lambda}\right) = 0.\end{aligned}$$

Since $\delta(K_1 - K_2) \neq 0$ it follows that

$$\omega_2\left(\frac{\pi}{2},\tilde{\lambda}\right) = 0. \tag{1.11}$$

By the same procedure from equality (1.5) we can derive that

$$\omega_2'\left(\frac{\pi}{2},\tilde{\lambda}\right) = 0. \tag{1.12}$$

From the fact that $\omega_2(x,\tilde{\lambda})$ is a solution of the differential Eq. (1.1) on $[\frac{\pi}{2},\pi]$ and satisfies the initial conditions (1.11) and (1.12) it follows that $\omega_2(x,\tilde{\lambda}) = 0$ identically on $[\frac{\pi}{2},\pi]$ (cf. [7, p. 12, Theorem 1.2.1]).

By using (1.7), (1.11) and (1.12) we may also find

$$\omega_1\left(\frac{\pi}{2},\tilde{\lambda}\right) = \omega_1'\left(\frac{\pi}{2},\tilde{\lambda}\right) = 0.$$

From the latter discussions of $\omega_2(x,\tilde{\lambda})$ it follows that $\omega_1(x,\tilde{\lambda}) = 0$ identically on $[0,\frac{\pi}{2}]$. Hence $\omega(x,\tilde{\lambda}) = 0$ identically on $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$. But this contradicts (1.6), thus completing the proof. □

2. An existence theorem

The function $\omega(x,\lambda)$ defined in section 1 is a nontrivial solution of Eq. (1.1) satisfying boundary condition (1.2) and transmission conditions (1.4) and (1.5). Putting $\omega(x,\lambda)$ into (1.3), we get the characteristic equation

$$F(\lambda) \equiv \omega(\pi,\lambda)\cos\beta + \omega'(\pi,\lambda)\sin\beta = 0. \tag{2.1}$$

By Theorem 1.1 the set of eigenvalues of boundary value problem (1.1)–(1.5) coincides with the set of real roots of Eq. (2.1).

Let $q_1 = \int_0^{\frac{\pi}{2}} |q(\tau)|\,d\tau$ and $q_2 = \int_{\frac{\pi}{2}}^{\pi} q(\tau)\,d\tau$.

Lemma 2.1

(1) *Let $\lambda \geqslant 4q_1^2$. Then for the solution $\omega_1(x,\lambda)$ of Eq. (1.8), the following inequality holds*:

$$|\,\omega_1(x,\lambda)| \leqslant \frac{1}{q_1}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}, \quad x \in \left[0,\frac{\pi}{2}\right]. \tag{2.2}$$

(2) *Let $\lambda \geqslant \max\{4q_1^2, 4q_2^2\}$. Then for the solution $\omega_2(x,\lambda)$ of Eq. (1.9), the following inequality holds*:

$$|\omega_2(x,\lambda)| \leqslant \frac{4}{|\delta|q_1}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}, \quad x \in \left[\frac{\pi}{2},\pi\right] \tag{2.3}$$

Proof. Let $B_{1\lambda} = \max_{[0,\frac{\pi}{2}]}|\omega_1(x,\lambda)|$. Then from (1.8), it follows that, for every $\lambda > 0$, the following inequality holds:

$$B_{1\lambda} \leqslant \sqrt{\sin^2\alpha + \frac{\cos^2\alpha}{s^2}} + \frac{1}{s}B_{1\lambda}q_1.$$

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If $s \geqslant 2q_1$, then we get (2.2). Differentiating (1.8) with respect to x, we have

$$\omega_1'(x,\lambda) = -s\sin\alpha\sin sx - \cos\alpha\cos sx - \int_0^x q(\tau)\cos s(x-\tau)\omega_1(\tau - \Delta(\tau),\lambda)\,d\tau. \tag{2.4}$$

From (2.4) and (2.2), it follows that, for $s \geqslant 2q_1$, the following inequality holds:

$$|\omega_1'(x,\lambda)| \leqslant \sqrt{s^2\sin^2\alpha + \cos^2\alpha} + \sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}.$$

Hence

$$\frac{|\omega_1'(x,\lambda)|}{s} \leqslant \frac{1}{q_1}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha}, \tag{2.5}$$

where $s > 2q_1$ and $x \in [0, \frac{\pi}{2}]$.

Let $B_{2\lambda} = \max_{[\frac{\pi}{2},\pi]}|\omega_2(x,\lambda)|$. Then, from (1.9), (2.2) and (2.5) it follows that, for $s \geqslant 2q_1$, the following inequality holds:

$$B_{2\lambda} \leqslant \frac{2}{|\delta|q_1}\sqrt{4q_1^2\sin^2\alpha + \cos^2\alpha} + \frac{1}{s}q_2 B_{2\lambda}.$$

Hence if $s \geqslant \max\{4q_1^2, 4q_2^2\}$ then we get (2.3). □

Theorem 2.1. *The problem (1.1)–(1.5) has an infinite set of positive eigenvalues.*

Proof. Differentiating (1.9) with respect to x, we get

$$\omega_2'(x,\lambda) = -\frac{s}{\delta}\omega_1\left(\frac{\pi}{2},\lambda\right)\sin s\left(x-\frac{\pi}{2}\right) + \frac{1}{\delta}\omega_1'\left(\frac{\pi}{2},\lambda\right)\cos s\left(x-\frac{\pi}{2}\right) - \int_{\frac{\pi}{2}}^x q(\tau)\cos s(x-\tau)\omega_2(\tau-\Delta(\tau),\lambda)\,d\tau. \tag{2.6}$$

From (1.8), (1.9), (2.1), (2.4) and (2.6), we get

$$\begin{aligned}
&\left[\frac{1}{\delta}\left(\sin\alpha\cos s\frac{\pi}{2} - \frac{\cos\alpha}{s}\cos s\frac{\pi}{2} - \frac{1}{s}\int_0^{\frac{\pi}{2}} q(\tau)\sin s\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)\,d\tau\right)\cos s\frac{\pi}{2}\right.\\
&\quad - \frac{1}{s\delta}\left(s\sin\alpha\sin s\frac{\pi}{2} + \cos\alpha\cos s\frac{\pi}{2} + \int_0^{\frac{\pi}{2}} q(\tau)\cos s\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)\,d\tau\right)\sin s\frac{\pi}{2}\\
&\quad \left. - \frac{1}{s}\int_{\frac{\pi}{2}}^{\pi} q(\tau)\sin s(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)\,d\tau\right]\cos\beta + \left[-\frac{s}{\delta}\left(\sin\alpha\cos s\frac{\pi}{2} + \frac{\cos\alpha}{s}\sin s\frac{\pi}{2}\right.\right.\\
&\quad + \frac{1}{s}\int_0^{\frac{\pi}{2}} q(\tau)\sin s\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)\,d\tau\Big)\sin s\frac{\pi}{2} - \frac{1}{\delta}\left(s\sin\alpha\sin s\frac{\pi}{2} + \cos\alpha\cos s\frac{\pi}{2}\right.\\
&\quad \left.\left. + \int_0^{\frac{\pi}{2}} q(\tau)\cos s\left(\frac{\pi}{2}-\tau\right)\omega_1(\tau-\Delta(\tau),\lambda)\,d\tau\right)\cos s\frac{\pi}{2} - \int_{\frac{\pi}{2}}^{\pi} q(\tau)\cos s(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)\,d\tau\right]\sin\beta = 0.
\end{aligned} \tag{2.7}$$

There are four possible cases:

1. $\sin\alpha \neq 0$, $\sin\beta \neq 0$;
2. $\sin\alpha \neq 0$, $\sin\beta = 0$;
3. $\sin\alpha = 0$, $\sin\beta \neq 0$;
4. $\sin\alpha = 0$, $\sin\beta = 0$.

We shall consider only case 1.

Let λ be sufficiently large. Then, by (2.2) and (2.3), the Eq. (2.7) may be rewritten in the form

$$s\sin s\pi + O(1) = 0. \tag{2.8}$$

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Obviously, for large s Eq. (2.8) has an infinite set of roots. The other cases may be considered analogically. Thus the theorem is proved. □

3. Asymptotic Formulas for Eigenvalues and Eigenfunctions

Now we begin to study asymptotic properties of eigenvalues and eigenfunctions. Below, we shall consider only case 1. In the following we shall assume that s is sufficiently large. From (1.8) and (2.2), we get

$$\omega_1(x,\lambda) = \mathrm{O}(1) \tag{3.1}$$

on $[0,\frac{\pi}{2}]$. From (1.9) and (2.3), we get

$$\omega_2(x,\lambda) = \mathrm{O}(1) \tag{3.2}$$

on $[\frac{\pi}{2},\pi]$.

The existence and continuity of the derivatives $\omega'_{1s}(x,\lambda)$ for $0 \leqslant x \leqslant \frac{\pi}{2}, |\lambda| < \infty$, and $\omega'_{2s}(x,\lambda)$ for $\frac{\pi}{2} \leqslant x \leqslant \pi, |\lambda| < \infty$, follows from Theorem 1.4.1 in [7].

Lemma 3.1. *In case* 1

$$\omega'_{1s}(x,\lambda) = \mathrm{O}(1), \quad x \in \left[0,\frac{\pi}{2}\right] \quad \textit{and} \tag{3.3}$$

$$\omega'_{2s}(x,\lambda) = \mathrm{O}(1), \quad x \in \left[\frac{\pi}{2},\pi\right] \tag{3.4}$$

are hold.

Proof. By differentiation of (1.8) with respect to s, we get, by (3.1)

$$\omega'_{1s}(x,\lambda) = -\frac{1}{s}\int_0^x q(\tau)\sin s(x-\tau)\omega'_{1s}(\tau-\Delta(\tau),\lambda)\,\mathrm{d}\tau + K(x,\lambda), \quad (|K(x,\lambda)| \leqslant K_0). \tag{3.5}$$

Let $C_\lambda = \max_{[0,\frac{\pi}{2}]}|\omega'_{1s}(x,\lambda)|$. The existence of C_λ follows from the continuity of derivation for $x \in [0,\frac{\pi}{2}]$. From (3.5)

$$C_\lambda \leqslant \frac{1}{s}q_1 C_\lambda + K_0.$$

Now let $s \geqslant 2q_1$. Then $C_\lambda \leqslant 2K_0$ and the validity of the asymptotic formula (3.3) follows. Formula (3.4) may be proved analogically.

Let n be a natural number. We shall say that the number λ is situated near the number n^2 if $|n-\sqrt{\lambda}| < \frac{1}{4}$. □

Theorem 3.1. *Let n be a natural number. For each sufficiently large n, in case* 1, *there is exactly one eigenvalue of the problem* (1.1)–(1.5) *near* n^2.

Proof. We consider the expression which is denoted by O(1) in the Eq. (2.8)

$$\frac{\delta}{\sin\alpha\sin\beta}\left\{-\frac{\sin(\alpha-\beta)}{\delta}\cos s\pi + \frac{\cos\alpha\cos\beta}{s\delta}\sin s\pi + \int_0^{\frac{\pi}{2}}\left[\frac{\cos\beta}{s\delta}\sin s(\pi-\tau) + \frac{\sin\beta}{\delta}\cos s(\pi-\tau)\right]q(\tau)\omega_1(\tau-\Delta(\tau),\lambda)\,\mathrm{d}\tau\right.$$
$$\left. + \int_{\frac{\pi}{2}}^{\pi}\left[\frac{\cos\beta}{s}\sin s(\pi-\tau) + \sin\beta\cos s(\pi-\tau)\right]q(\tau)\omega_2(\tau-\Delta(\tau),\lambda)\,\mathrm{d}\tau\right\}.$$

If formulas (3.1)–(3.4) are taken into consideration, it can be shown by differentiation with respect to s that for large s this expression has bounded derivative.

It is obvious that for large s the roots of Eq. (2.8) are situated close to entire numbers. We shall show that, for large n, only one root (2.8) lies near to each n.

We consider the function $F(s) = s\sin s\pi + \mathrm{O}(1)$. Its derivative, which has the form $F'(s) = \sin s\pi + s\pi\cos s\pi + \mathrm{O}(1)$, does not vanish for s close to n for sufficiently large n. Thus our assertion follows by Rolle's Theorem. □

Eq. (2.8) gives us unable to get asymptotic formulas for eigenvalues of the problem (1.1)–(1.5).

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Let n be sufficiently large. In what follows we shall denote by $\lambda_n = s_n^2$ the eigenvalue of the problem (1.1)–(1.5) situated near n^2. We set $s_n = n + \delta_n$. Then from (2.8) it follows that $\delta_n = O(\frac{1}{n})$. Consequently

$$s_n = n + O\left(\frac{1}{n}\right). \tag{3.6}$$

The formula (3.6) make it possible to obtain asymptotic expressions for eigenfunction of the problem (1.1)–(1.5).

From (1.8), (2.4) and (3.1), we get

$$\omega_1(x,\lambda) = \sin\alpha\cos sx + O\left(\frac{1}{s}\right) \quad \text{and} \tag{3.7}$$

$$\omega_1'(x,\lambda) = -s\sin\alpha\sin sx + O(1). \tag{3.8}$$

From (1.9), (3.2), (3.7) and (3.8), we get

$$\omega_2(x,\lambda) = \frac{\sin\alpha}{\delta}\cos s\frac{\pi}{2}\cos s\left(x-\frac{\pi}{2}\right) - \frac{\sin\alpha}{\delta}\sin s\frac{\pi}{2}\sin s\left(x-\frac{\pi}{2}\right) + O\left(\frac{1}{s}\right) = \frac{\sin\alpha}{\delta}\cos sx + O\left(\frac{1}{s}\right). \tag{3.9}$$

By putting (3.6) in the (3.7) and (3.9), we derive that

$$u_{1n}(x) = \omega_1(x,\lambda_n) = \sin\alpha\cos nx + O\left(\frac{1}{n}\right) \quad \text{and}$$

$$u_{2n}(x) = \omega_2(x,\lambda_n) = \frac{\sin\alpha}{\delta}\cos nx + O\left(\frac{1}{n}\right).$$

Hence the eigenfunctions $u_n(x)$ have the following asymptotic representation:

$$u_n(x) = \begin{cases} \sin\alpha\cos nx + O(\frac{1}{n}) & \text{for } x \in [0,\frac{\pi}{2}), \\ \frac{\sin\alpha}{\delta}\cos nx + O(\frac{1}{n}) & \text{for } x \in (\frac{\pi}{2},\pi]. \end{cases}$$

Under some additional conditions the more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

(a) The derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$ and have finite limits $q'(\frac{\pi}{2}\pm 0) = \lim_{x\to\frac{\pi}{2}\pm 0} q'(x)$ and $\Delta''(\frac{\pi}{2}\pm 0) = \lim_{x\to\frac{\pi}{2}\pm 0}\Delta''(x)$, respectively.
(b) $\Delta'(x) \leqslant 1$ in $[0,\frac{\pi}{2})\bigcup(\frac{\pi}{2},\pi]$, $\Delta(0) = 0$ and $\lim_{x\to\frac{\pi}{2}+0}\Delta(x) = 0$.

It is easy to see that, using (b)

$$x - \Delta(x) \geqslant 0, \quad x \in \left[0,\frac{\pi}{2}\right) \tag{3.10}$$

and

$$x - \Delta(x) \geqslant \frac{\pi}{2}, \quad x \in \left(\frac{\pi}{2},\pi\right] \tag{3.11}$$

are obtained.

By (3.7), (3.9), (3.10) and (3.11) we have

$$\omega_1(\tau - \Delta(\tau),\lambda) = \sin\alpha\cos s(\tau-\Delta(\tau)) + O\left(\frac{1}{s}\right) \quad \text{and} \tag{3.12}$$

$$\omega_2(\tau - \Delta(\tau),\lambda) = \frac{\sin\alpha}{\delta}\cos s(\tau-\Delta(\tau)) + O\left(\frac{1}{s}\right) \tag{3.13}$$

on $[0,\frac{\pi}{2})$ and $(\frac{\pi}{2},\pi]$ respectively.

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Putting these expressions into (2.7), we have

$$
\begin{aligned}
&-\frac{s}{\delta}\sin\alpha\sin\beta\sin s\pi+\frac{\sin(\alpha-\beta)}{\delta}\cos s\pi-\frac{\cos\alpha\cos\beta}{s\delta}\sin s\pi\\
&\quad-\frac{\cos\beta}{s\delta}\int_0^{\frac{\pi}{2}}q(\tau)\sin s(\pi-\tau)\left[\sin\alpha\cos s(\tau-\Delta(\tau))+\mathrm{O}\left(\frac{1}{s}\right)\right]\mathrm{d}\tau\\
&\quad-\frac{\cos\beta}{s}\int_{\frac{\pi}{2}}^{\pi}q(\tau)\sin s(\pi-\tau)\left[\frac{\sin\alpha}{\delta}\cos s(\tau-\Delta(\tau))+\mathrm{O}\left(\frac{1}{s}\right)\right]\mathrm{d}\tau\\
&\quad-\frac{\sin\beta}{\delta}\int_0^{\frac{\pi}{2}}q(\tau)\cos s(\pi-\tau)\left[\sin\alpha\cos s(\tau-\Delta(\tau))+\mathrm{O}\left(\frac{1}{s}\right)\right]\mathrm{d}\tau\\
&\quad-\sin\beta\int_{\frac{\pi}{2}}^{\pi}q(\tau)\cos s(\pi-\tau)\left[\frac{\sin\alpha}{\delta}\cos s(\tau-\Delta(\tau))+\mathrm{O}\left(\frac{1}{s}\right)\right]\mathrm{d}\tau\\
&=-\frac{s}{\delta}\sin\alpha\sin\beta\sin s\pi+\frac{\sin(\alpha-\beta)}{\delta}\cos s\pi\\
&\quad-\frac{\sin\alpha\sin\beta}{\delta}\left[\cos s\pi\int_0^{\pi}q(\tau)\cos s\tau\cos s(\tau-\Delta(\tau))\,\mathrm{d}\tau+\sin s\pi\int_0^{\pi}q(\tau)\sin s\tau\cos s(\tau-\Delta(\tau))\,\mathrm{d}\tau\right]+\mathrm{O}\left(\frac{1}{s}\right)\\
&=-\frac{s}{\delta}\sin\alpha\sin\beta\sin s\pi+\frac{\sin(\alpha-\beta)}{\delta}\cos s\pi\\
&\quad-\frac{\sin\alpha\sin\beta}{\delta}\left[\cos s\pi\int_0^{\pi}q(\tau)\frac{1}{2}[\cos s\Delta(\tau)+\cos s(2\tau-\Delta(\tau))]\,\mathrm{d}\tau\right.\\
&\quad\left.+\sin s\pi\int_0^{\frac{\pi}{2}}q(\tau)\frac{1}{2}[\sin s\Delta(\tau)\sin s(2\tau-\Delta(\tau))]\,\mathrm{d}\tau=0.\right.
\end{aligned}
\tag{3.14}
$$

Let we define

$$
\left.\begin{aligned}
A(x,s,\Delta(\tau))&=\tfrac{1}{2}\textstyle\int_0^x q(\tau)\sin s\Delta(\tau)\,\mathrm{d}\tau,\\
B(x,s,\Delta(\tau))&=\tfrac{1}{2}\textstyle\int_0^x q(\tau)\cos s\Delta(\tau)\,\mathrm{d}\tau.
\end{aligned}\right\}
\tag{3.15}
$$

It is obviously that these functions are bounded for $0\leqslant x\leqslant\pi, 0<s<\infty$.

Under the conditions (a) and (b) the following formulas:

$$
\left.\begin{aligned}
\textstyle\int_0^x q(\tau)\cos s(2\tau-\Delta(\tau))\,\mathrm{d}\tau&=\mathrm{O}\left(\tfrac{1}{s}\right),\\
\textstyle\int_0^x q(\tau)\sin s(2\tau-\Delta(\tau))\,\mathrm{d}\tau&=\mathrm{O}\left(\tfrac{1}{s}\right)
\end{aligned}\right\}
\tag{3.16}
$$

can be proved by the same technique in Lemma 3.3.3 in [7]. From (3.15), (3.16) and (3.14), we have

$$
\sin s\pi[s\sin\alpha\sin\beta+\sin\alpha\sin\beta A(\pi,s,\Delta(\tau))]-\cos s\pi[\sin(\alpha-\beta)-\sin\alpha\sin\beta B(\pi,s,\Delta(\tau))]+\mathrm{O}\left(\frac{1}{s}\right)=0.
$$

Hence

$$
\tan s\pi=\frac{1}{s}(\cot\beta-\cot\alpha-B(\pi,s,\Delta(\tau)))+\mathrm{O}\left(\frac{1}{s^2}\right).
$$

Again, if we take $s_n=n+\delta_n$, then from (3.6),

$$
\tan(n+\delta_n)\pi=\tan\delta_n\pi=\frac{1}{n}(\cot\beta-\cot\alpha-B(\pi,n,\Delta(\tau)))+\mathrm{O}\left(\frac{1}{n^2}\right).
$$

Hence for large n,

$$
\delta_n=\frac{1}{n\pi}(\cot\beta-\cot\alpha-B(\pi,n,\Delta(\tau)))+\mathrm{O}\left(\frac{1}{n^2}\right)
$$

and finally

$$
s_n=n+\frac{1}{n\pi}(\cot\beta\cot\alpha-B(\pi,n,\Delta(\tau)))+\mathrm{O}\left(\frac{1}{n^2}\right).
\tag{3.17}
$$

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Thus, we have proven the following theorem

Theorem 3.2. *If conditions* (a) *and* (b) *are satisfied then, the positive eigenvalues* $\lambda_n = s_n^2$ *of the problem* (1.1)–(1.5) *have the* (3.17) *asymptotic representation for* $n \to \infty$.

We now may obtain a sharper asymptotic formula for the eigenfunctions. From (1.8) and (3.12)

$$\omega_1(x,\lambda) = \sin\alpha\cos sx - \frac{\cos\alpha}{s}\sin sx - \frac{\sin\alpha}{s}\int_0^x q(\tau)\sin s(x-\tau)\cos(\tau - \Delta(\tau))\,d\tau + O\left(\frac{1}{s^2}\right).$$

Thus, from (3.15) and (3.16)

$$\omega_1(x,\lambda) = \sin\alpha\cos sx\left[1+\frac{1}{s}A(x,s,\Delta(\tau))\right] - \frac{\sin sx}{s}[\cos\alpha + \sin\alpha B(x,s,\Delta(\tau))] + O\left(\frac{1}{s^2}\right). \tag{3.18}$$

Replacing s by s_n and using (3.17), we have

$$\begin{aligned} u_{1n}(x) = \omega_1(x,\lambda_n) = \sin\alpha\Bigg\{&\cos nx\left[1+\frac{1}{n}A(x,n,\Delta(\tau))\right] - \frac{\sin nx}{n\pi} \\ &\times[(\cot\beta - \cot\alpha - B(\pi,n,\Delta(\tau)))x + (\cot\alpha + B(x,n,\Delta(\tau)))\pi]\Bigg\} + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{3.19}$$

From (2.4), (3.12) and (3.15), we have

$$\begin{aligned} \frac{\omega_1'(x,\lambda)}{s} &= -\sin\alpha\sin sx - \frac{\cos\alpha}{s}\cos sx - \frac{\sin\alpha}{s}\int_0^x q(\tau)\frac{1}{2}[\cos s(x-\tau) + \cos(sx - (2\tau - \Delta(\tau)))]\,d\tau + O\left(\frac{1}{s^2}\right) \\ &= -\sin\alpha\sin sx\left(1+\frac{1}{s}A(x,s,\Delta(\tau))\right) + \frac{\cos sx}{s}(\cos\alpha + \sin\alpha B(x,s,\Delta(\tau))) + O\left(\frac{1}{s^2}\right), \quad x \in \left[0,\frac{\pi}{2}\right]. \end{aligned} \tag{3.20}$$

From (1.9), (3.13), (3.16), (3.18) and (3.20), we have

$$\begin{aligned} \omega_2(x,\lambda) = {}& \frac{1}{\delta}\left\{\sin\alpha\cos s\frac{\pi}{2}\left[1+\frac{1}{s}A\left(\frac{\pi}{2},s,\Delta(\tau)\right)\right] - \frac{\sin s\frac{\pi}{2}}{s}\left[\cos\alpha + \sin\alpha B\left(\frac{\pi}{2},s,\Delta(\tau)\right)\right] + O\left(\frac{1}{s^2}\right)\right\}\cos s\left(x-\frac{\pi}{2}\right) \\ &- \frac{1}{\delta}\left\{\sin\alpha\sin s\frac{\pi}{2}\left[1+\frac{1}{s}A\left(\frac{\pi}{2},s,\Delta(\tau)\right)\right] - \frac{\cos s\frac{\pi}{2}}{s}\left[\cos\alpha + \sin\alpha B\left(\frac{\pi}{2},s,\Delta(\tau)\right)\right] + O\left(\frac{1}{s^2}\right)\right\}\sin s\left(x-\frac{\pi}{2}\right) \\ &- \frac{1}{s}\int_{\frac{\pi}{2}}^x q(\tau)\sin s(x-\tau)\left[\frac{\sin\alpha}{\delta}\cos s(\tau - \Delta(\tau)) + O\left(\frac{1}{s}\right)\right]d\tau \\ = {}& \frac{\sin\alpha}{\delta}\cos sx\left[1+\frac{1}{s}A\left(\frac{\pi}{2},s,\Delta(\tau)\right)\right] - \frac{\sin sx}{\delta s}\left[\cos\alpha + \sin\alpha B\left(\frac{\pi}{2},s,\Delta(\tau)\right)\right] \\ &- \frac{\sin\alpha}{\delta s}\int_{\frac{\pi}{2}}^x q(\tau)\frac{1}{2}[\sin s(x-\Delta(\tau)) + \sin s(x-(2\tau-\Delta(\tau)))]\,d\tau + O\left(\frac{1}{s^2}\right) \\ = {}& \frac{\sin\alpha}{\delta}\cos sx\left[1+\frac{1}{s}A\left(\frac{\pi}{2},s,\Delta(\tau)\right) + \frac{1}{2s}\int_{\frac{\pi}{2}}^x q(\tau)\sin s\Delta(\tau)\,d\tau\right] \\ &- \frac{\sin sx}{\delta s}\left[\cos\alpha + \sin\alpha B\left(\frac{\pi}{2},s,\Delta(\tau)\right) + \sin\alpha\frac{1}{2}\int_{\frac{\pi}{2}}^x q(\tau)\cos s\Delta(\tau)\,d\tau\right] \\ = {}& \frac{\sin\alpha}{\delta}\cos sx\left[1+\frac{1}{s}A(x,s,\Delta(\tau))\right] - \frac{\sin sx}{\delta s}[\cos\alpha + \sin\alpha B(x,s,\Delta(\tau))] + O\left(\frac{1}{s^2}\right), \quad x \in \left[\frac{\pi}{2},\pi\right]. \end{aligned}$$

Now, replacing s by s_n and using (3.17), we have

$$\begin{aligned} u_{2n}(x) = \omega_2(x,\lambda_n) = \frac{\sin\alpha}{\delta}\Bigg\{&\cos nx\left[1+\frac{1}{n}A(x,n,\Delta(\tau))\right] - \frac{\sin nx}{n\pi} \\ &\times[(\cot\beta - \cot\alpha - B(\pi,n,\Delta(\tau)))x + (\cot\alpha + B(x,n,\Delta(\tau)))\pi]\Bigg\} + O\left(\frac{1}{n^2}\right). \end{aligned} \tag{3.21}$$

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Thus, we have proven the following theorem.

Theorem 3.3. *If conditions* (a) *and* (b) *are satisfied then, the eigenfunctions* $u_n(x)$ *of the problem* (1.1)–(1.5) *have the following asymptotic representation for* $n \to \infty$:

$$u_n(x) = \begin{cases} u_{1n}(x) & \text{for } x \in \left[0, \frac{\pi}{2}\right), \\ u_{2n}(x) & \text{for } x \in \left(\frac{\pi}{2}, \pi\right] \end{cases}$$

where $u_{1n}(x)$ *and* $u_{2n}(x)$ *defined as in* (3.19) *and* (3.21) *respectively.*

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