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LINEAR CONTROL SYSTEMS ON HOMOGENEOUS SPACES

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A thesis submitted by Okan DUMAN in partial fulfillment of the requirements for the degree of **DOCTOR OF PHILOSOPHY** is approved by the committee on 19.04.2024 in Department of Mathematics, Program of Mathematics.

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Okan DUMAN

Signature

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Dedicated to my grandmother, the eternal voice of reason who always asks,
"Are you still studying, when are you going to get a real job?"

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LIST OF SYMBOLS

Σ_G	A Linear Control System on Lie Group G
$\times$	Cartesian Product
$\overline{A}$	Closure of A
$\mathcal{C}$	Control Set
$\mathcal{U}$	Control Space of Control Parameters
$\mathcal{D}$	Derivation of Lie Algebra
$\in$	Element of
$=$	Equal
$\dim G$	Dimension of G
$\cap$	Intersection
int A	Interior of A
$\mathbb{H}$	Heisenberg Group
$\simeq$	Isomorphism
$\mathfrak{g}$	Lie Algebra of G
$\mathcal{O}^-$	Negative Orbit
$\Leftarrow$	Necessary Condition
$\notin$	Not Element of
$\neq$	Not Equal
$\nsubseteq$	Not Subset or Equal
$\mathcal{O}^+$	Positive Orbit
$Aut(G)$	Set of Automorphisms of G
$\subseteq$	Subset or Equal
$\Rightarrow$	Sufficient Condition
$\cup$	Union

LIST OF ABBREVIATIONS

a.e.	Almost Every
iff	If and Only If
LARC	Lie Algebra Rank Condition
LCS	Linear Control System
2D	Two-Dimensional
ODEs	Ordinary Differential Equations
w.r.t.	With Respect to

LIST OF FIGURES

ABSTRACT

Linear Control Systems on Homogeneous Spaces

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Doctor of Philosophy Thesis

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The thesis focuses primarily on the study of linear control systems on homogeneous spaces, in particular in the context of the 3D Heisenberg Lie group $\mathbb{H}$ and its closed subgroups. The main motivation stems from a result of P. Jouan [1], which shows an interesting connection between control affine systems on manifolds and linear control systems (abbreviated as LCSs) on Lie groups or homogeneous spaces. The goal is to provide a comprehensive characterization of all possible LCSs on homogeneous spaces of $\mathbb{H}$ and to investigate their dynamical properties. The comprehensive analysis covers various dimensions of closed subgroups, providing a thorough understanding of LCSs on homogeneous spaces of $\mathbb{H}$. The first chapters lay the foundation by introducing basic concepts related to linear vector fields, LCSs on Lie groups, and the classification of closed subgroups of $\mathbb{H}$. Building on this, subsequent chapters deal with the projection of LCSs onto homogeneous spaces, considering invariance criteria for subgroups under the flows of systems. The core analysis revolves around controllability and control sets, with detailed investigations of non-normal subgroups of different dimensions. A special attention is given to the more complex structures and controllability issues of certain cases. Each case is studied in detail, step by step, and the geometric descriptions obtained are given, with emphasis on their topological and geometric properties.

In the initial chapters, we provide an overview of fundamental concepts in theory, aiming to enhance comprehension of the subsequent sections of the thesis. Chapters 3 and 4 focus on constructing the necessary results for explicitly characterizing all possible LCSs on the homogeneous spaces of the Heisenberg group. Chapter 5 deals with the dynamics of these systems and provides a detailed analysis of their controllability issues. In the final chapter, the results obtained so far are reviewed and the implications of the findings are also presented, along with the consideration of new problems to be addressed in future research.

Keywords: Linear control systems, Heisenberg group, Homogeneous spaces.

ÖZET

Homojen Uzaylarda Lineer Kontrol Sistemleri

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Matematik Anabilim Dalı
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Bu tez, temel olarak 3D Heisenberg Lie grubu $\mathbb{H}$ ve onun kapalı alt grupları özelinde, homojen uzaylar üzerindeki doğrusal kontrol sistemlerinin incelenmesine odaklanmaktadır. Ana motivasyonu, P. Jouan'ın manifoldlar üzerindeki kontrol afin sistemleri ile Lie grupları veya homojen uzaylar üzerindeki doğrusal kontrol sistemleri (kısaca LCSs) arasında oldukça ilginç bir bağlantı olduğunu gösteren bir sonucu oluşturmaktadır. Tezdeki hedef, $\mathbb{H}$ ile oluşturulan homojen uzaylar üzerindeki mümkün olan tüm doğrusal kontrol sistemlerinin kapsamlı bir karakterizasyonunu sağlamak ve dinamik özelliklerini araştırmaktır. Bu detaylı analiz, çeşitli boyutlardaki kapalı alt grupları kapsamakta ve $\mathbb{H}$ Lie grubunun homojen uzayları üzerindeki LCS'lerin tüm yönleriyle anlaşılmasına olanak sağlamaktadır. İlk bölümlerde lineer vektör alanları, Lie grupları üzerindeki LCS'ler tanıtılarak bir temel oluşturulmakta ve $\mathbb{H}$ grubunun kapalı alt gruplarının sınıflandırılması yapılmaktadır. Tüm bunlara dayanarak, sonraki bölümler, sistemlerin akışları (flow) altında alt gruplar için invaryantlık kriterleri göz önüne alınacak şekilde, LCS'lerinin homojen uzaylar üzerine projeksiyonunu ele almaktadır. Bu analizlerin odak noktası, farklı boyutlardaki normal olmayan alt grupların detaylı incelemeleri ile kontrol edilebilirlik ve kontrol kümeleri etrafında yoğunlaşmaktadır. Belirli durumlarda ortaya çıkan çok daha karmaşık yapılar ve bunların kontrol edilebilirlik problemlerine özel bir önem verilmektedir. Ortaya çıkan her bir durum, adım adım ayrıntılı olarak incelenip, elde edilen geometrik tasvirler; topolojik ve geometrik özelliklerine vurgu yapılarak açıklanmaktadır.

İlk bölümlerde, tezin sonraki bölümlerinin daha iyi anlaşılmasını sağlamak amacıyla teorideki temel kavramlara genel bir bakış sunulmaktadır. 3 ve 4’üncü bölümler, Heisenberg grubunun homojen uzayları üzerindeki olası tüm LCS’lerini açık bir şekilde karakterize etmek için gerekli sonuçları inşa etmeye odaklanmaktadır. 5’inci bölüm, bu sistemlerin dinamiklerini ele almakta ve bunların kontrol edilebilirlik problemlerinin ayrıntılı bir analizini sunmaktadır. Son bölümde, şu ana kadar elde edilen sonuçlar gözden geçirilmekte ve bulguların sonuçları, gelecekteki araştırmalarda ele alınacak yeni problemlerin değerlendirilmesiyle birlikte sunulmaktadır.

Anahtar Kelimeler: Lineer kontrol sistemleri, Heisenberg grubu, Homojen uzaylar.

1
INTRODUCTION

A dynamical system, in a broad sense, encompasses anything that changes or evolves. This term applies not only to technical systems, such as vehicles, electrical circuits, or power plants, but also to the behavior of structures like skyscrapers or bridges when they are subjected to forces such as strong winds or earthquakes. The principles of dynamical systems are not limited to engineering, but also have applications in various fields, including economics, biology, and social sciences. Typically, dynamical systems are described as interacting with their environment through inputs and outputs. When analyzing such systems, one can observe the monitored outputs for specified inputs. By responding to these outputs with appropriate inputs, the system can be steered toward a desired state. This process is known as control and is widely applicable in domains dealing with dynamical systems.

Mathematical control theory, a branch of applied mathematics, studies the fundamental principles, theories, and challenges involved in the analysis and design of control systems. These systems, considered as dynamical systems, follow ODEs on finite-dimensional smooth manifolds. Thus, a control system is essentially a family of ODEs, where the family is parameterized by control parameters. The laws governing their behavior are not completely predetermined, but are shaped by these control parameters in the associated system. All differential equations in this family are defined on the same manifold, called the state space of the control system. There is flexibility in the selection of allowable values for the control parameters, allowing us to select any dynamical system from the family, and we can freely change these values over time. Once a control is fixed, our control system becomes a non-autonomous ODE. A solution to these equations is uniquely identified by the initial condition and is referred to as an admissible trajectory of the control system w.r.t. the specific control. Consequently, this trajectory represents a curve in the state space. The initial condition (or state) serves as the starting point for the trajectory. In particular, different controls typically yield different trajectories starting from the same fixed state. Together, these trajectories form the reachable set associated with the given initial state.

The central focus of the control theory is to deal with the reachability issue, which involves precisely determining the states that can be reached from a given initial state. After having confirmed the possibility of reaching a certain state, the goal is to reach it in the most efficient way. To clarify, we aim to answer the following question, which is called controllability problem: "Given an initial state of the system, is it possible to reach any arbitrary state via admissible trajectories in positive time?". Or, alternatively, are there certain regions in the state space in which the controllability is guaranteed? Since the analysis of this problem is not a trivial task, branches of mathematics such as geometry, algebra, and topology are heavily used. For this reason, control theory is often used together with Lie theory, and thus some control problems are modeled on Lie groups or their coset manifold spaces.

1.1 Background and Motivation

The origin of mathematical control theory can be traced back to 1868 when J. C. Maxwell described the mechanical governor used to control the speed of a steam engine. He articulated the device's functionality through mathematical principles. Advancing into the present day, the 1950s saw the emergence of modern control theory. R. E. Kalman played a crucial role in its development, particularly through his contributions to filtering, the algebraic analysis of linear systems, and linear quadratic control. Similarly in the 1960s, R. E. Bellman and L. S. Pontryagin introduced powerful techniques for obtaining laws for optimal control within the related context. Later, this field broadened in the 1970s thanks to the research of R. Brockett, followed by V. Jurdjevic and H. J. Sussmann, making it an even more attractive area of research. To this day, the study of the control systems has a rich history, with a particular focus on linear control systems (abrev. LCSs) on $\mathbb{R}^n$, which plays a crucial role in understanding various real-world applications (see, for example, [2–5]). In this context, we briefly discuss LCSs on Euclidean spaces, as it serves to better understand the main topic of this thesis, namely LCSs on homogeneous spaces of a Lie group, and has recently received a lot of attention. An LCS on Euclidean space $\mathbb{R}^n$ is defined by the following family of parametrized ODEs:

$$\dot{q}(t) = Pq(t) + \omega_1(t)b^1 + \omega_2(t)b^2 + \cdots + \omega_m(t)b^m$$

where $P \in \mathbb{R}^{n\times n}, b^j \in \mathbb{R}^n, \omega \in \mathcal{U}$, and $\mathcal{U}$ means the control space of the control parameters $\omega = (\omega_1, \omega_2, \ldots \omega_m)$. It follows that the drift $\dot{q}(t) = Pq(t)$ is controlled by m engines $b^j \in \mathbb{R}^n$ through the different component of the integrable function $\omega = (\omega_1, \omega_2, \dots \omega_m) : [0, T_\omega] \to \Omega \subseteq \mathbb{R}^m$, where Ω is topologically closed with $0 \in \text{int}(\Omega)$. The system is called unrestricted if $\Omega = \mathbb{R}^m$ and restricted if Ω is bounded. This system in its matrix form can be considered as

$$\dot{q} = Pq + B\omega, \quad q \in \mathbb{R}^n$$

where P is $n \times n$ and B is $n \times m$ matrix, and $\omega = (\omega_1, \dots, \omega_m)$ is an admissible control. L. Markus, in [5], studied this system in the context of matrix groups. Subsequently, V. Ayala and J. Tirao, in [6], extended the study to a connected Lie group. More precisely, let G be a connected Lie group, and $\mathfrak{g}$ be its Lie algebra identified with the vector space of all left-invariant vector fields. An LCS on G is defined by a family of ODEs:

$$\Sigma_G : \quad \dot{q}(t) = \mathcal{Y}(q(t)) + \sum_{j=1}^{m} \omega_j(t) B^j(q(t))$$

where the dynamics of Σ_G is governed by the drift vector field $\mathcal{Y}$ which belongs to the normalizer of $\mathfrak{g}$ in the Lie algebra of all smooth vector fields on G, and the control vectors $B^1, B^2, \cdots, B^m$ are left-invariant vector fields. The control function $\omega = (\omega_1, \cdots, \omega_m) : [0, \infty) \to \Omega \subseteq \mathbb{R}^m$, with $0 \in \text{int}(\Omega)$, belongs to the set of admissible controls. By admissible controls we mean a set which is $\mathcal{U}_{bd} = \{\omega \in \mathcal{L}^\infty(\mathbb{R}, \mathbb{R}^m) : \omega(t) \in \Omega \text{ for a.e. } t\}$ or the set of all piecewise constant functions $\mathcal{U}$ defined on $\mathbb{R}$ that take values in Ω.

In the absence of a comprehensive global theory with general hypotheses, several authors have studied LCSs on various state spaces, including nilpotent, solvable, simple, semi-simple, compact/non-compact, abelian Lie groups and the direct and semi-direct product between them, as well as flag manifolds [7–13]. There are several reasons for the importance of LCSs on Lie groups. First, LCSs on Lie groups are more natural and inherently preserve the Lie structure after the invariant ones. Second, they can serve as approximations to certain nonlinear systems. Additionally, P. Jouan demonstrated in [1] that any affine control system on a connected manifold whose dynamics generates a finite-dimensional Lie algebra is diffeomorphic to an LCS on a Lie group or a homogeneous space. This highlights the significance of understanding the topological and geometric behavior of the system Σ_G. Since equivalent systems have the same topological, dynamical, and algebraic properties, it becomes possible to obtain information about any arbitrary system Σ_M that satisfies Jouan's condition by either a linear system Σ_G or a homogeneous system $\Sigma_{G\setminus H}$. This is one of the main reasons why it is necessary to classify LCSs Σ_G on different classes of Lie groups.

1.2 Outline and Structure

In the thesis, the focus will be on the study of LCSs on homogeneous spaces. Here, we consider the 3D Heisenberg Lie group $\mathbb{H}$ together with its closed subgroups (discrete subgroups included and normal subgroups excluded [1]) to form its homogeneous spaces and classify on such state spaces all possible linear control systems, which is not a trivial task. The main motivation for this thesis comes from the above-mentioned result of P. Jouan, which emphasizes a quite interesting connection between a control affine system on a manifold and an LCS either on a Lie group or on a homogeneous space. More precisely, a control affine system on a manifold is equivalent by mean of a diffeomorphism to an LCS on a Lie group or a homogeneous space if and only if the vector fields that describe the system are complete and generate a finite dimensional Lie algebra. It follows that one might find in some suitable context a control system on a manifold that is equivalent to a linear control system on a homogeneous space of $\mathbb{H}$. Hence, we find it convenient to give in this study complete characterization of all possible linear systems on homogeneous spaces of $\mathbb{H}$ and deal with dynamical properties of such systems as a concrete case. To have such linear systems on homogeneous spaces of the Heisenberg group we have to determine explicitly the conditions that guarantee well-defined induced dynamics on various quotient spaces. Since this requires a certain invariance criteria of subgroups of $\mathbb{H}$ under the flow of the drift (i.e., a linear vector field) of the original dynamics (See [1, Proposition 4.1]) we start with listing these conditions first to obtain the induced or projectable drift and control vectors (i.e., left-invariant vector fields) on the corresponding homogeneous spaces.

One of the key insights to understanding the dynamical properties of a control system is to study control sets in both topological and/or algebraic sense. These sets provide the maximum regions in the state space where approximate controllability occurs. It should be noted that determining controllability property, characterizing eventual topological properties of control sets of all induced LCSs becomes highly non-trivial job. For example, even in the case of low-dimensional groups, the properties of control sets for such dynamics on Lie groups and homogeneous spaces might differ significantly (See [7, 8]). For this reason, we are going to perform a detailed analysis of all possible LCSs on the homogeneous spaces that are obtained for each closed subgroup. Moreover, using these homogeneous spaces as the state

[1] We exclude normal subgroups of $\mathbb{H}$ since otherwise the corresponding homogeneous spaces receive a Lie group structure and linear control systems on such state spaces has been already studied in a series of papers. See, [6, 8–10, 14].

spaces, we are able to fully characterize the control sets and controllability of one-input LCSs.

The thesis is divided into six chapters:

In Chapter 2, we mention some generalities in control setting to facilitate a better understanding of the rest of the thesis.

In Chapter 3, in the first part, we give the definition, properties and essential facts related to a linear vector field, as well as insights into LCSs on Lie groups. Then we outline the conditions necessary to project an LCS onto a homogeneous space. Again here, we fix the format on which the whole exposition is based on. More precisely, rather than the group of upper triangular matrices with only 1s in the main diagonal we prefer to interpret the Heisenberg group $\mathbb{H}$ as the cartesian product $\mathbb{R}^2 \times \mathbb{R}$ and express all the necessary arguments such as the group multiplication, invariant and linear vector fields and their Lie brackets, etc to be in accordance with this format. This gives us the form of a typical LCS on $\mathbb{H}$. Together with that, we are able to obtain any closed subgroup of $\mathbb{H}$ with dimensions 0,1 and 2 up to isomorphism. Moreover, we focus on a certain invariance criteria of subgroups (that is, discrete and non-normal subgroups) of $\mathbb{H}$ under the flow of a linear vector field. By using the classification of closed subgroups in this chapter, we are able to obtain in Theorem 3.13 the conditions a linear vector field should satisfy in order to achieve the desired invariance condition of the subgroups under consideration. Such results allow us to show how many types of LCSs exist on the homogeneous spaces of $\mathbb{H}$ depending on the dimension of the manifold $L\backslash\mathbb{H}$.

In Chapter 4, we define what we mean by an LCS on homogeneous spaces of $\mathbb{H}$ and list, up to equivalence, all possible such systems.

In Chapter 5, a detailed analysis of the controllability issue and the control sets of each of the determined dynamics is conducted. This chapter is divided into two main parts. The classification of both normal and non-normal closed subgroups of $\mathbb{H}$ is presented in Theorem 3.11 and Remark 3.12. The main emphasis in this chapter is on studying the structures of homogeneous spaces by non-normal subgroups, since the treatment of the other case has already been handled in recent literature by a number of papers. These non-normal subgroups of dimension one for $\mathbb{H}$ are as follows: (i) $L = \mathbb{R}\mathbf{e}_1 \times \mathbb{Z}$ and (ii) $L = \mathbb{R}\mathbf{e}_1 \times \{0\}$. The first part of this chapter deals with the controllability issue of LCSs on the corresponding homogeneous spaces $L\backslash\mathbb{H} \simeq \mathbb{R} \times \mathbb{T}$ and $L\backslash\mathbb{H} \simeq \mathbb{R} \times \mathbb{R}$. In particular, any induced control system Σ on $L\backslash\mathbb{H}$ is equivalent to one of the systems indicated in the diagram (4.3). After investigating the controllability of such a system on the homogeneous space

$\mathbb{R} \times \mathbb{T}$, our subsequent focus in this part is on the controllability of the induced control systems on the other homogeneous space $\mathbb{R} \times \mathbb{R}$. We have to emphasize that the latter case presents a much more intricate and complex structure, which makes this part both challenging and interesting enough. This means that the controllability of these LCSs will be studied here in a very detailed manner by means of a rather technical case-by-case analysis. Finally, in this part, the structures of the control sets for the considered systems are explicitly presented for each case and its sub-case, whenever they exist. In the second part of this chapter, we consider the zero dimensional (discrete) non-normal closed subgroup $L = \mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}$ of $\mathbb{H}$ and associated homogeneous space $L\backslash\mathbb{H} \simeq \mathbb{T} \times \mathbb{R} \times \mathbb{T}$.

In the final chapter, Chapter 6, we discuss the results we have obtained so far and what new problems our findings will provide a solution for in the future.

2
BASIC KNOWLEDGE OF GEOMETRIC CONTROL THEORY

This chapter is dedicated to the introduction of fundamental concepts and notations in control theory, which are essential for the development of this thesis. First, it presents initial definitions and results concerning Lie groups and homogeneous manifolds. Then, the basic properties of a control system on differentiable manifolds and their control sets are elucidated.

2.1 Lie Groups and Their Homogeneous Manifolds

In this section, basic definitions and some results about Lie groups and homogeneous spaces will be given. For further insights into this topic and additional details beyond what is given here, refer to [15] and [16].

Definition 2.1. A Lie group is a smooth manifold G endowed with a group structure such that the map $\rho : (s, t) \mapsto st^{-1}$ from $G \times G$ to G is smooth.

Definition 2.2. Let S be an abstract subgroup of a Lie group G. If S is an immersed submanifold and the product $\rho|_{S\times S} \to S$ is smooth, then it is called a Lie subgroup.

As we will detail below, note that if S is a closed subgroup of a Lie group G, this implies that S is not only a submanifold but also a Lie subgroup of G. This means that S inherits the induced topology of G. Certainly, it is indeed possible to have a Lie subgroup that is not a closed subset. An illustrative example is the map $\xi : \mathbb{R} \to S^1 \times S^1$ defined by $s \mapsto (e^{2\pi is}, e^{2\pi is\alpha})$, where α is irrational (i.e., the line of irrational slope). This map is a one-to-one homomorphism and an immersion. Despite being dense in the torus, its image is not an embedding.

In the Lie theory, there are two significant maps known as translations. Let us briefly discuss them.

Definition 2.3. The left translation $L_p : G \to G$ is defined as $q \mapsto pq$, and the right translation $R_p : G \to G$ is defined as $q \mapsto qp$ for all $p \in G$.

Can be seen that these maps are diffeomorphisms, which provide a way to navigate within a Lie group. Specifically, any $p \in G$ can be brought to the identity element $\mathrm{e} \in G$ via $L_{p^{-1}}$, and $(\mathrm{d}L_{p^{-1}})_p : T_pG \to T_{\mathrm{e}}G$ becomes a vector space isomorphism.

Definition 2.4. A vector field B on a Lie group G is called left-invariant if $B \circ L_p = (dL_p)B$ for all $p \in G$.

It is worth noting that if B is a left-invariant vector field, then $B_p = (dL_p)_{\mathrm{e}}(B_{\mathrm{e}})$ holds for all $p \in G$. In other words, the value of the vector field at p is entirely determined by its value at the identity element. Moreover, the collection $\mathfrak{g}$ of all left-invariant vector fields on G forms a real vector space, and this vector space can be identified with the tangent space of G at the identity w.r.t. an isomorphism, denoted as $\mathfrak{g} \simeq T_{\mathrm{e}}G$.

Remark 2.1. In the realm of differentiable manifolds, using a tangent vector ν to "transport a point in the direction of ν" in a coordinate-independent manner is not feasible. This limitation is due to the fact that there is not a unique curve on the manifold among the numerous curves that share ν as a tangent. On a Lie group, however, this becomes achievable, since the existence of left-invariant vector fields ensures a unique flow in the direction of ν.

Definition 2.5. A one-parameter subgroup of G is represented by a smooth homomorphism $\xi : \mathbb{R} \to G$.

For a given left-invariant vector field B, a unique one-parameter subgroup denoted ξ_B is defined, satisfying $\xi_B(0) = \mathrm{e}$ and $\dot{\xi}_B(\tau) = B$ for all τ. This $\xi_B(\tau)$ essentially is said to be the local flow and is commonly referred to as the flow of B. Now we are ready to define the exponential map. The exponential map $\exp : \mathfrak{g} \to G$ is defined as $\exp(B) = \phi_B(1)$, where B is a left-invariant vector field and ξ_B is the associated one-parameter subgroup. We have $\xi_B(\tau) = \xi_{\tau B}(1) = \exp(\tau B)$, since changing the parameterization corresponds to changing the tangent vectors.

In the following, we want to briefly mention a special class of Lie groups and their properties (for details see [15, 17]), which will play an important role in the thesis.

Definition 2.6. A Lie algebra is called nilpotent if there exists an integer $\alpha \geq 1$ such that all brackets of $[B_1, [B_2, [B_3, \cdots, [B_\alpha, B_{\alpha+1}] \cdots]]]$ are zero for all $B_1, \cdots B_{\alpha+1} \in \mathfrak{g}$.

Let us write an alternative definition: Define $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\{[B_1, B_2] : B_1, B_2 \in \mathfrak{g}\}$, and iteratively, $\mathfrak{g}^{s+1} = \mathrm{span}\,[\mathfrak{g}, \mathfrak{g}^s]$. This creates a descending sequence of ideals in $\mathfrak{g}$ such that $\mathfrak{g} \supset \mathfrak{g}^2 \supset \ldots \supset \mathfrak{g}^\alpha \supset \mathfrak{g}^{\alpha+1} \supset \ldots$. If at some point $\mathfrak{g}^{\alpha+1} = \{\mathbf{0}\}$, then $\mathfrak{g}$ is called nilpotent. The smallest integer α at which this vanishing occurs is called the step of the Lie algebra.

Definition 2.7. A Lie group is said to be nilpotent if it is connected and its Lie algebra is nilpotent.

This definition means that we have a criterion for checking the nilpotency of connected Lie groups as follows: A connected Lie group G is nilpotent iff its Lie algebra $\mathfrak{g}$ is nilpotent. Then we have a strong connection with Lie groups and their Lie algebras by the following.

Proposition 2.2. Suppose that G is a connected and simply connected nilpotent Lie group and $\mathfrak{g}$ is its Lie algebra. The exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism.

In this manner, the relation between $\exp(B_1)\exp(B_2)$ and $\exp(B_1 + B_2)$ for $B_1, B_2 \in \mathfrak{g}$ is considered as follows:

Definition 2.8. The Baker-Campbell-Hausdorff formula (BCH) is given by

$$\exp(B_1)\exp(B_2) = \exp(C(B_1, B_2))$$

where $C(B_1, B_2)$ is a series depending on $B_1, B_2 \in \mathfrak{g}$ and its brackets. The first few terms are determined by

$$C(B_1, B_2) = B_1 + B_2 + \frac{1}{2}[B_1, B_2] + \frac{1}{12}[[B_1, B_2], B_2] - \frac{1}{12}[[B_1, B_2], B_1] + \cdots$$

The subsequent terms in the series depend on the brackets containing four or more elements. This series converges for sufficiently small of B_1 and B_2. In particular, if the group G is nilpotent, this series is finite for all $B_1, B_2 \in \mathfrak{g}$.

Example 2.3. Let us take the Heisenberg Lie group of dimension three

$$H = \left\{ \begin{pmatrix} 1 & \upsilon_1 & \upsilon_3 \\ 0 & 1 & \upsilon_2 \\ 0 & 0 & 1 \end{pmatrix} : \upsilon_1, \upsilon_2, \upsilon_3 \in \mathbb{R} \right\}.$$

The map $\xi : H \to \mathbb{R}^3$ given by $\xi(P) = (\upsilon_1, \upsilon_2, \upsilon_3)$ is a diffeomorphism of H and $\mathbb{R}^3$ for $P \in H$. By the matrix multiplication, it is seen that the group operation $*$ on

H can be defined by

$$(\upsilon_1, \upsilon_2, \upsilon_3) * (\rho_1, \rho_2, \rho_3) = (\upsilon_1 + \rho_1, \upsilon_2 + \rho_2, \upsilon_3 + \rho_3 + \upsilon_1\rho_2)$$

and it is different than the usual vector addition in $\mathbb{R}^3$, where the identity is $(0, 0, 0)$ and the inverse of $(\upsilon_1, \upsilon_2, \upsilon_3)$ is $(-\upsilon_1, -\upsilon_2, \upsilon_1\upsilon_2 - \upsilon_3)$. That is why this correspondence is not an isomorphism in algebraic sense. So, related to the basis vectors of Euclidean space of dimension 3 we can define three different type of curves. We first define the curve $\sigma : \mathbb{R} \to \mathbb{R}^3$ for $e_1 = (1, 0, 0)$ such that $\sigma(t) = (t, 0, 0)$. In fact, σ is a curve through the origin of $\mathbb{R}^3$ and $\dot{\sigma}(0) = e_1$. Now, take $\phi = \xi^{-1} \circ \sigma$ to define a curve in H passing through the identity as follows

$$\phi(t) = \xi^{-1}(\sigma(t)) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H, \quad \forall t \in \mathbb{R}.$$

So, calculating the differential of this curve at the identity we will have one of the generators of the Lie algebra $\mathfrak{h}$ of H. That is,

$$\left(\frac{d}{dt}\right)_{t=0} \phi(t) = \left(\frac{d}{dt}\right)_{t=0} \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = B_1 \in \mathfrak{h}.$$

Similarly, depending on e_2 and e_3, we respectively have

$$B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

And finally the Lie algebra $\mathfrak{h}$ of H can be written in the terms of its generators as follows

$$\mathfrak{h} = \operatorname{Span}\{B_1, B_2, B_3\}.$$

Since $[B_1, B_2] = B_3$ is only one non-null Lie bracket, $\mathfrak{h} = \operatorname{Span}_{\mathcal{L}.A}\{B_1, B_2\}$. In fact, B_3 belongs to $\mathcal{Z}(\mathfrak{h})$, the center of $\mathfrak{h}$.

Note that we have the following

$$\exp(B_1)\exp(B_2) = \exp\left(B_1 + B_2 + \frac{1}{2}[B_1, B_2]\right)$$

since $[B_1, [B_1, B_2]] = [B_2, [B_1, B_2]] = 0$, i.e., Heisenberg group is said to be 2-step nilpotent.

Example 2.4. Let G be a Lie group. The connected component of the identity G_0 is an open subgroup and, as a consequence of the embedding property of open submanifolds, it is a Lie subgroup.

Example 2.5. For a Lie group G, any one-parameter subgroup

$$\{\exp(\tau B) : B \in \mathfrak{g}, \tau \in \mathbb{R}\}$$

is a Lie subgroup. In fact, when the curve $\tau \mapsto \exp(\tau B)$ is closed, an injective immersion $\mathbb{S}^1 \to G$ is established. Conversely, if the curve is not closed, the one-parameter group defines an injective immersion $\mathbb{R} \to G$. In either case, the map $\tau \mapsto \exp(\tau X)$ represents an injective differentiable homomorphism, leading to the conclusion that its image forms a Lie subgroup.

The closed subgroup Theorem, established by Cartan, guarantees that every closed subgroup S of a Lie group G is indeed a Lie subgroup. In other words, S has a differentiable manifold structure, which makes it a Lie subgroup. This theorem is a fundamental result in Lie group theory, which finds extensive applications in various contexts:

Theorem 2.6. *Every closed subgroup S of a Lie group G is a Lie subgroup. In particular, S has an embedded manifold structure, which establishes it as a Lie subgroup. The associated Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is given as*

$$\mathfrak{s} = \{B \in \mathfrak{g} : \forall \tau \in \mathbb{R}, \quad \exp \tau B \in S\}.$$

It is appropriate to highlight a key result about coset manifold spaces that will be used frequently in this thesis. Specifically, when S is a closed subgroup, the quotient topology on $S \backslash G$ is Hausdorff. Furthermore, the space $S \backslash G$ of left (or right) cosets is referred to as a homogeneous space. The following theorem elucidates the properties of the corresponding quotient differentiable structure:

Theorem 2.7. *Consider a Lie group G with a closed subgroup $S \subset G$. In such a case there exists a differentiable structure on $S \backslash G$ which is associated with the quotient topology, satisfying:*

(1) $\dim(S \backslash G) = \dim G - \dim S$

(2) The canonical projection $\pi : G \to S \backslash G$ is a submersion.

(3) Any map $\varphi : S\backslash G \to M$ with the smooth manifold M is differentiable iff $\varphi \circ \pi$ is differentiable.

2.2 Control Systems and Control Sets

We will begin by defining what we mean by a control system, followed by an introduction to controllability and control sets, along with their topological, geometric, and algebraic properties. See [2] and [18] for more details.

Definition 2.9. Let M be a finite dimensional smooth manifold and let $\mathbb{R}^m$ denote the m-dimensional Euclidean space. Given a compact convex subset $\Omega \subset \mathbb{R}^m$ satisfying $0 \in \text{int}\,\Omega$, we mean by a control-affine system evolving on M the following (parametrized) family of ODEs

$$\Sigma_M : \quad \dot{p}(t) = f_0(p(t)) + \sum_{j=1}^{m} \omega_j(t) f_j(p(t)),$$

where $f_0, f_1, \ldots, f_m$ are smooth vector fields defined on M and the control parameter $\omega = (\omega_1, \ldots, \omega_m)$ belongs to the set $\mathcal{U}$ of the piecewise constant functions such that $\omega(t) \in \Omega$. Additionally, f_0 is called the *drift vector field* and $f_1, \ldots, f_m$ the *control vector fields*.

First, in our analysis we will assume that all vector fields $f_0, f_1, \ldots, f_m$ are complete, which means that the solutions of the corresponding differential equations exist for all real numbers. For an initial state $p \in M$ and $\omega \in \mathcal{U}$ there exists a unique solution $\phi(t, p, \omega)$ of Σ_M defined on an open interval containing $t = 0$, satisfying $\phi(0, p, \omega) = p$. It is important to remark that, in the general case, $t \mapsto \phi(t, p, \omega)$ is the unique locally absolutely continuous curve that satisfy to the associated differential equation almost everywhere in the Carathéodory sense. However, since we consider that all vector fields in the system Σ_M are complete, the solutions of this system are defined in the whole real line. Therefore, it is possible to obtain a map

$$\phi : \mathbb{R} \times M \times \mathcal{U} \to M, \qquad (t, p, \omega) \mapsto \phi(t, p, \omega)$$

satisfying the *cocycle property*

$$\phi(t + \tau, p, \omega) = \phi\big(t, \phi(\tau, p, \omega), \Theta_\tau \omega\big)$$

for all $t, \tau \in \mathbb{R}$, $p \in M$, $\omega \in \mathcal{U}$. Here, the *shift flow* $\Theta : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ is defined by

$$(\Theta_t \omega)(\tau) := \omega(t + \tau).$$

Moreover, it follows that for all $\tau_1, \tau_2 \geq 0$ and $\omega_1, \omega_2 \in \mathcal{U}$

$$\phi\big(\tau_2, \phi(\tau_1, p, \omega_1), \omega_2\big) = \phi(\tau_1 + \tau_2, p, \omega)$$

where

$$\omega(\tau) = \begin{cases} \omega_1(\tau), & \tau \in [0, \tau_1] \\ \omega_2(\tau - \tau_1), & \tau \in (\tau_1, \tau_2 + \tau_1]. \end{cases}$$

We note that if we consider Ω to be convex and $\mathcal{U} \subset \mathcal{L}^\infty(\mathbb{R}, \mathbb{R}^m)$, it follows that $\mathcal{U}$ is compact and can be endowed with a metrizable structure in the weak*-topology of $\mathcal{L}^\infty(\mathbb{R}, \mathbb{R}^m)$. Furthermore, it is obtained that both the maps ϕ and Θ are continuous with regard to this topology. See [2, 19] for detailed exposition.

Definition 2.10. For any $p \in M$ and $\tau > 0$ the following sets

$$\mathcal{O}^+_{\leq\tau}(p) := \{q \in M : \exists \omega \in \mathcal{U},\ t \in [0, \tau] \text{ with } \phi(t, p, \omega) = q\}$$
$$\mathcal{O}^-_{\leq\tau}(p) := \{q \in M : \exists \omega \in \mathcal{U},\ t \in [0, \tau] \text{ with } \phi(t, q, \omega) = p\}$$

are the *set of reachable points* from p up to time τ and the *set of controllable points* to p within time τ, respectively. Therefore, the sets referred to as the reachable and the controllable set (also known as the *positive orbit* and *negative orbit*) of p are, respectively as follows

$$\mathcal{O}^+(p) := \bigcup_{\tau > 0} \mathcal{O}^+_{\leq\tau}(p)$$
$$\mathcal{O}^-(p) := \bigcup_{\tau > 0} \mathcal{O}^-_{\leq\tau}(p).$$

Definition 2.11. We say that the control system Σ_M

(1) is *locally accessible at* p if the sets $\mathcal{O}^+_{\leq\tau}(p)$ and $\mathcal{O}^-_{\leq\tau}(p)$ have non-empty interior for all $\tau > 0$

(2) is *locally accessible* if (1) holds at every point $p \in M$

(3) satisfies the *Lie algebra rank condition* (abrev. LARC) if $\mathcal{L}(p) = T_pM$ for all $p \in M$, where $\mathcal{L}$ denotes the smallest Lie algebra of vector fields containing Σ_M

(4) is said to be *controllable* if $M = \mathcal{O}^+(p)$ for all $p \in M$.

Remark 2.8. A particular consequence of the LARC is that, when it holds, the sets $\mathcal{O}^+_{\leq\tau}(p)$ and $\mathcal{O}^-_{\leq\tau}(p)$ have non-empty interior for all $\tau > 0$ and $p \in M$ (see [18]).

Next we introduce the concept of control sets encountered [2].

Definition 2.12. A non-empty set $\mathcal{C} \subset M$ is a control set of Σ_M if it is maximal, w.r.t. the set inclusion, with the following properties:

(1) $\forall p \in \mathcal{C} \quad \exists \omega \in \mathcal{U}$ such that $\phi\left(\mathbb{R}^+, p, \omega\right) \subset \mathcal{C}$

(2) It holds that $\mathcal{C} \subset \overline{\mathcal{O}^+(p)}$ for all $p \in \mathcal{C}$.

According to [2, Proposition 3.2.4], any subset $\mathcal{C}$ of M with a nonempty interior that is maximal with respect to the item (2) of Definition 2.12 is identified as a control set. To understand the dynamics of a control system, it is essential to capture the topological, geometric, and/or algebraic properties of its control sets. For instance, they allow us to obtain many dynamical properties of the system, such as equilibrium and recurrence points, periodic and bounded orbits, etc. Furthermore, the exact controllability is satisfied in its interior, which means that points can be steered into each other by a solution of the system in positive time. More precisely, a control set $\mathcal{C}$ is called invariant in positive time if for any $\tau > 0$ and $\omega \in \mathcal{U}$ we have that $\phi(\tau, \mathcal{C}, \omega) \subset \mathcal{C}$. Analogously, $\mathcal{C}$ is invariant in negative time if for any $\tau > 0$ and $\omega \in \mathcal{U}$ it holds that $\phi(-\tau, \mathcal{C}, \omega) \subset \mathcal{C}$.

The upcoming theorem provides a concise summary of essential traits associated with control sets having non-empty interior, see [2, Theorem 3.1.5].

Theorem 2.9. *Let $\mathcal{C}$ be a control set of Σ_M with non-empty interior and assume that Σ_M is locally accessible. The followings are satisfied*

(1) $\mathcal{C}$ *is connected and* $\overline{\operatorname{int} \mathcal{C}} = \overline{\mathcal{C}}$.

(2) *For any $x \in \mathcal{C}$ and $y \in \operatorname{int} \mathcal{C}$ it holds that*

$$\operatorname{int} \mathcal{C} \subset \mathcal{O}^+(x) \quad \textit{and} \quad \mathcal{C} = \overline{\mathcal{O}^+(y)} \cap \mathcal{O}^-(y).$$

(3) *Suppose that $\phi(\tau, x, \omega)$ is a periodic trajectory, i.e., $\phi(\tau + \rho, x, \omega) = \phi(\tau, x, \omega)$ for some $\rho > 0$ and all $\tau \in \mathbb{R}$. From here, if $x \in \operatorname{int} \mathcal{C}$ then $\phi(\tau, x, \omega) \in \operatorname{int} \mathcal{C}$ for all $\tau \in \mathbb{R}$.*

(4) $\mathcal{C} = \overline{\mathcal{C}}$ *iff* $\mathcal{C}$ *is invariant in positive time iff* $\mathcal{C} = \overline{\mathcal{O}^+(x)}$ *for any* $x \in \mathcal{C}$.

(5) $\mathcal{C} = \operatorname{int} \mathcal{C}$ *iff* $\mathcal{C}$ *is invariant in negative time iff* $\mathcal{C} = \mathcal{O}^-(x)$ *for any* $x \in \mathcal{C}$.

3
LINEAR CONTROL SYSTEMS

In this chapter we will first give the definition, properties and fundamental facts about linear vector fields and concerning linear control systems on the Lie groups. In Section 3.1, we present the conditions for projecting an LCS onto homogeneous space. In Section 3.2, we introduce 3D Heisenberg Lie group and its Lie algebra. Then we obtain explicitly form of linear and invariant vector fields on it and determine the flow of a linear vector field, form of automorphisms and derivations. Then, we obtain the form of an LCS on $\mathbb{H}$. In Section 3.3, we characterize all closed subgroups L of $\mathbb{H}$ and those invariant w.r.t. the flow of a linear vector field. In particular, we consider separately normal and non-normal closed subgroups to distinguish the corresponding homogeneous spaces with manifold or Lie group structure. Such results will allow us to demonstrate the various LCSs that exist on the homogeneous spaces of $\mathbb{H}$, according to the dimension of the coset manifold $L\backslash\mathbb{H}$.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and e stands for the identity element of G.

Definition 3.1. The normalizer of a Lie algebra $\mathfrak{g}$ is defined by

$$norm_{\nu(\mathrm{G})}(\mathfrak{g}) := \{\mathcal{Y} \in \nu(\mathrm{G}) : \forall B \in \mathfrak{g} \quad [\mathcal{Y}, B] \in \mathfrak{g}\}$$

where $\nu(\mathrm{G})$ represents the set of all smooth vector fields on G.

A vector field $\mathcal{Y}$ on G is called *linear* if it is an element of the set $norm_{\nu(\mathrm{G})}(\mathfrak{g})$ and satisfies $\mathcal{Y}(\mathrm{e}) = 0$. The following result establishes conditions under which a vector field on G is considered linear, providing equivalence between these conditions (see [1, Theorem 3.1]).

Theorem 3.1. *Let us consider a vector field $\mathcal{Y}$ on a connected Lie group* G*. The following statements are equivalent:*

(1) *$\mathcal{Y}$ is linear*

(2) *The flow $\{\xi_\tau\}_{\tau\in\mathbb{R}}$ of $\mathcal{Y}$ is a 1-parameter subgroup of* $\mathrm{Aut}(\mathrm{G})$*, the group of all automorphisms of* G

(3) *$\mathcal{Y}$ satisfies*

$$\forall q, p \in \mathrm{G} \quad \mathcal{Y}(qp) = (dL_q)_p\mathcal{Y}(p) + (dR_p)_q\mathcal{Y}(q)$$

where L_q and R_p represent the left and right translations on G*, and $(dL_q)_p, (dR_p)_q$ their derivatives at the points p and q, respectively.*

Remark 3.2. It should be noted that a linear vector field on a connected Lie group is complete. Indeed, let $\mathcal{Y}$ be a linear vector field. It is known that $\mathcal{Y}(\mathrm{e}) = 0$ by the definition. It follows that $\{\xi_\tau\}_{\tau\in\mathbb{R}}$ is well-defined in a neighborhood N_τ of e. Since our Lie group is connected, N_τ generates all of the group. Therefore, for any $q \in \mathrm{G}$ there exist $q_1, \ldots, q_n$ such that $q = q_1 \cdots q_n$. From (2) of Theorem 3.1 we get that

$$\xi_\tau(q) = \xi_\tau(q_1 \cdots q_n) = \xi_\tau(q_1) \cdots \xi_\tau(q_n)$$

is well-defined and hence $\mathcal{Y}$ is complete.

Remark 3.3. Suppose that $\{\xi_\tau\}_{\tau\in\mathbb{R}}$ denote the 1-parameter subgroup of $\mathrm{Aut}(\mathrm{G})$ generated by the linear vector field $\mathcal{Y}$. If we take any vector field B we have that

$$[\mathcal{Y}, B](\mathrm{e}) = \frac{d}{d\tau}\bigg|_{\tau=0} (d\xi_{-\tau})_{\xi_\tau(\mathrm{e})} B(\xi_\tau(\mathrm{e})) = \frac{d}{d\tau}\bigg|_{\tau=0} (d\xi_{-\tau})_e B(\mathrm{e})$$

since $\xi_\tau(\mathrm{e}) = \mathrm{e}$ for all $\tau \in \mathbb{R}$. Moreover, if B is a left invariant vector field, we can write the following

$$\begin{aligned}[\mathcal{Y}, B](g) =& \frac{d}{d\tau}\bigg|_{\tau=0} (d\xi_{-\tau})_{\xi_\tau(g)} B(\xi_\tau(g)) = \frac{d}{d\tau}\bigg|_{\tau=0} (d\xi_{-\tau})_{\xi_\tau(g)} (dL_{\xi_\tau(g)})_{\mathrm{e}} B(\mathrm{e}) \\ =& \frac{d}{d\tau}\bigg|_{\tau=0} (dL_g)_{\mathrm{e}} (d\xi_{-\tau})_{\mathrm{e}} B(\mathrm{e}) = (dL_g)_{\mathrm{e}} [\mathcal{Y}, B](\mathrm{e})\end{aligned}$$

for all $g \in \mathrm{G}$ and we used that $\xi_{-\tau} \circ L_{\xi_\tau(g)} = L_g \circ \xi_{-\tau}$ in the above equality (as a straightforward outcome of (2) in Theorem 3.1). Thus, for a given linear vector field $\mathcal{Y}$, one can always associate to such a vector field a derivation $\mathcal{D} = -ad(\mathcal{Y})$ of the corresponding Lie algebra $\mathfrak{g}$ of G. Recall that a Lie algebra derivation $\mathcal{D}$ of $\mathfrak{g}$ is a linear map on $\mathfrak{g}$ satisfying the Leibnitz rule, that is, $\mathcal{D}[B_1, B_2] = [\mathcal{D}B_1, B_2] +$

$[B_1, \mathcal{D}B_2]$ for every $B_1, B_2 \in \mathfrak{g}$. Although the converse does not occur in general we have for a connected and simply connected Lie group G that given a derivation $\mathcal{D}$ of the Lie algebra $\mathfrak{g}$ of G, there exists a linear vector field associated to $\mathcal{D}$ through the formula $(d\xi_\tau)_{\mathrm{e}} = \mathrm{e}^{\tau\mathcal{D}}$, $\forall \tau \in \mathbb{R}$, where ξ_τ stands for the flow. In particular, from the following diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{(d\xi_\tau)_{\mathrm{e}}} & \mathfrak{g} \\ {\scriptstyle \exp}\big\downarrow & \circlearrowleft & \big\downarrow{\scriptstyle \exp} \\ G & \xrightarrow{\xi_\tau} & G \end{array}$$

we have that $\xi_\tau(\exp B) = \exp(\mathrm{e}^{\tau\mathcal{D}}B)$ for every $B \in \mathfrak{g}$ and $\tau \in \mathbb{R}$.

Now, let us introduce an LCS, which will be the primary focus of this thesis:

Definition 3.2. An LCS on G is the family of ODEs

$$\Sigma_{\mathrm{G}}: \quad \dot{q}(t) = \mathcal{Y}(q(t)) + \sum_{j=1}^{m} \omega_j(t) B_j(q(t)),$$

where $\mathcal{Y}$ represents a linear vector field, while $B_1, \ldots, B_m$ denote left invariant vector fields, and $\omega = (\omega_1, \ldots, \omega_m) \in \mathcal{U}$ are control functions, defined analogously to those in the system Σ_M.

Let us take any $\omega \in \mathcal{U}$ and $\tau \in \mathbb{R}$ and denote $\phi(\tau, \mathrm{e}, \omega)$ as the solution of Σ_{G} with the initial condition at the identity element $\mathrm{e} \in \mathrm{G}$. Consequently, for any element $q \in \mathrm{G}$, the solutions to Σ_{G} starting at q are determined accordingly

$$\phi(\tau, q, \omega) = \phi(\tau, \mathrm{e}, \omega) \cdot \xi_\tau(q) = L_{\phi(\tau,\mathrm{e},\omega)}(\xi_\tau(q))$$

where $\{\xi_\tau\}_{\tau \in \mathbb{R}}$ is the flow of $\mathcal{Y}$.

Remark 3.4. Any linear vector field defines a derivation, but the converse is true only for simply connected Lie groups. Furthermore, in the case where G is a connected and simply connected nilpotent Lie group, the exponential map serves as a diffeomorphism. This implies that, especially for a given derivation, it becomes possible to explicitly compute the drift $\mathcal{X}$ through the above diagram using the logarithmic map $\log(p) = Y$ where $p \in G$.

3.1 Projections on Homogeneous Spaces

This section concentrates on equivalent systems and explores how an LCS can be projected onto a homogeneous space. It follows that equivalent systems preserve controllability, topological properties of control sets and positive (or negative) orbits. Since in the sequel we also consider control-affine systems on a connected Lie group (and hence its corresponding homogeneous space) we find it convenient to provide some basic definitions and facts about these issues.

Below, we introduce conjugations between control-affine systems. Such concepts help us simplify their dynamical analysis by changing the coordinates of the manifold M. Assume that $\widetilde{M}$ is another smooth manifold and following is a control-affine system on $\widetilde{M}$

$$\Sigma_{\widetilde{M}}: \quad \dot{y}(t) = g_0(y(t)) + \sum_{j=1}^{m} \omega_j(t) g_j(y(t)), \quad \omega \in \mathcal{U}.$$

Definition 3.3. If $\mu : M \to \widetilde{M}$ is a smooth map, we say that a vector field X on M and a vector field $\widetilde{X}$ on $\widetilde{M}$ are μ-conjugated (sometimes said μ-related) if $d\mu \circ X = \widetilde{X} \circ \mu$. In particular, we say that Σ_M and $\Sigma_{\widetilde{M}}$ are μ-conjugated if $d\mu \circ f_j = g_j \circ \mu$ for each $j \in \{0, 1, 2, \ldots, m\}$. In case μ is a diffeomorphism, Σ_M and $\Sigma_{\widetilde{M}}$ are called equivalent systems.

Next, a connection between control sets of conjugated systems is established by the following result.

Proposition 3.5. Suppose that Σ_M is μ-conjugated to the system $\Sigma_{\widetilde{M}}$. Then followings are satisfied:

(1) Suppose that $\mathcal{C}_{\Sigma_M}$ is a control set of Σ_M. Then there exists a control set $\mathcal{C}_{\Sigma_{\widetilde{M}}}$ of $\Sigma_{\widetilde{M}}$ such that $\mu(\mathcal{C}_{\Sigma_M}) \subset \mathcal{C}_{\Sigma_{\widetilde{M}}}$,

(2) Suppose that $\mu^{-1}(\{q_0\}) \subset \operatorname{int} \mathcal{C}_{\Sigma_M}$ is satisfied for some $q_0 \in \operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}}$. Then $\mathcal{C}_{\Sigma_M} = \mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$.

Proof. Pick any $p \in \mathcal{C}_{\Sigma_M}$. By the first item of the Definition 2.12, there is a control $\omega \in \mathcal{U}$ such that $\phi_{\Sigma_M}(\mathbb{R}^+, p, \omega) \subset \mathcal{C}_{\Sigma_M}$. We know that Σ_M and $\Sigma_{\widetilde{M}}$ are μ-conjugated systems, then their solution curves are also conjugates, i.e.,

$$\underbrace{\mu\left(\phi_{\Sigma_M}\left(\mathbb{R}^+, p, \omega\right)\right)}_{\subset\, \mu(\mathcal{C}_{\Sigma_M})} = \phi_{\Sigma_{\widetilde{M}}}\left(\mathbb{R}^+, \mu(p), \omega\right).$$

Moreover, by the second item of the Definition 2.12, we have that $\mathcal{C}_{\Sigma_M} \subset \overline{\mathcal{O}^+_{\Sigma_M}(p)}$. Now, using the smoothness and being a conjugation of μ, we get that

$$\mu(\mathcal{C}_{\Sigma_M}) \subset \mu\left(\overline{\mathcal{O}^+_{\Sigma_M}(p)}\right) \subset \overline{\mathcal{O}^+_{\Sigma_{\widetilde{M}}}(\mu(p))}.$$

Finally, we obtain that items (1) and (2) of the Definition 2.12 are satisfied by the set $\mu(\mathcal{C}_{\Sigma_M})$. Thus, we can find that there is a control set $\mathcal{C}_{\Sigma_{\widetilde{M}}}$ of $\Sigma_{\widetilde{M}}$ such that $\mu(\mathcal{C}_{\Sigma_M}) \subset \mathcal{C}_{\Sigma_{\widetilde{M}}}$ by the maximality.

(2) Here we need to show under the hypothesis that the set $\mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$ is the control set for the system Σ_M. Taking any element $p \in \mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$, we can find a control $\omega \in \mathcal{U}$ such that $\phi_{\Sigma_{\widetilde{M}}}\left(\mathbb{R}^+, \mu(p), \omega\right) \subset \mathcal{C}_{\Sigma_{\widetilde{M}}}$. Similarly in the previous item, we have that by the conjugation of curves

$$\mu\left(\phi_{\Sigma_M}\left(\mathbb{R}^+, p, \omega\right)\right) = \phi_{\Sigma_{\widetilde{M}}}\left(\mathbb{R}^+, \mu(p), \omega\right) \subset \mathcal{C}_{\Sigma_{\widetilde{M}}},$$

which implies that $\phi_{\Sigma_M}\left(\mathbb{R}^+, p, \omega\right) \subset \mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$. Thus, the first item of Definition 2.12 is shown. Now let us show the second one. For this, let us start by showing the statement that controllability is satisfied in $\mu^{-1}\left(\operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$. Taking any elements $p_1, p_2 \in \mu^{-1}\left(\operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$, we have that $\mu(p_1), \mu(p_2) \in \operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}}$. Since it is also known to be $q_0 \in \operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}}$ with these, there are controls $\omega_1, \omega_2 \in \mathcal{U}$ and $\tau_1, \tau_2 > 0$ by the controllability that we can reach q_0 via the following ways: $q_0 = \phi_{\Sigma_{\widetilde{M}}}\left(\tau_1, \mu(p_1), \omega_1\right)$ and $q_0 = \phi_{\Sigma_{\widetilde{M}}}\left(-\tau_2, \mu(p_2), \omega_2\right)$. Hence, we obtain that $\phi_{\Sigma_M}\left(\tau_1, p_1, \omega_1\right) \in \mu^{-1}(q_0)$ and $\phi_{\Sigma_M}\left(-\tau_2, p_2, \omega_2\right) \in \mu^{-1}(q_0)$. On the other hand, we know from our assumption that $\mu^{-1}(\{q_0\}) \subset \operatorname{int} \mathcal{C}_{\Sigma_M}$ and the controllability is satisfied in $\operatorname{int} \mathcal{C}_{\Sigma_M}$, then we have that there is a control $\omega_3 \in \mathcal{U}$ and $\tau_3 > 0$ such that

$$\phi_{\Sigma_M}\left(\tau_3, \phi_{\Sigma_M}\left(\tau_1, p_1, \omega_1\right), \omega_3\right) = \phi_{\Sigma_M}\left(-\tau_2, p_2, \omega_2\right),$$

which gives us the following

$$\phi_{\Sigma_M}\left(\tau_2, \phi_{\Sigma_M}\left(\tau_3, \phi_{\Sigma_M}\left(\tau_1, p_1, \omega_1\right), \omega_3\right), \Theta_{-\tau_2}\omega_2\right) = p_2.$$

Therefore, we conclude that controllability is satisfied in the set $\mu^{-1}\left(\operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$. In addition, let us show that this set satisfies the second item of the Definition 2.12. By the (1) of Theorem 2.9 and the smoothness of μ, we get that

$$\overline{\mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right)} = \overline{\mu^{-1}\left(\operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}}\right)}.$$

Due to the expression $\phi_{\Sigma_M}(\mathbb{R}^+, p, \omega) \subset \mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$ obtained above, the following is determined for all $p \in \mu^{-1}\left(\operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$

$$\mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right) \subset \overline{\mathcal{O}^+_{\Sigma_M}(p)}.$$

In addition to this, we have $\mu(p) \in \mathcal{C}_{\Sigma_{\widetilde{M}}}$ and

$$\operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}} \subset \overline{\mathcal{O}^+_{\Sigma_{\widetilde{M}}}(\mu(p))},$$

which implies that there exists $\omega_* \in \mathcal{U}$ and $\tau_* > 0$ such that

$$\mu\left(\phi_{\Sigma_M}(\tau_*, p, \omega_*)\right) = \phi_{\Sigma_{\widetilde{M}}}(\tau_*, \mu(p), \omega_*) \in \operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}}.$$

From here, we obtain $\phi_{\Sigma_M}(\tau_*, p, \omega_*) \in \mu^{-1}\left(\operatorname{int} \mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$, which gives us

$$\mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right) \subset \overline{\mathcal{O}^+_{\Sigma_M}(\phi_{\Sigma_M}(\tau_*, p, \omega_*))} \subset \overline{\mathcal{O}^+_{\Sigma_M}(p)}.$$

With all this, we conclude that $\mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$ satisfies the second item of Definition 2.12. Finally, from the last subset expression we know that the set $\mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right)$ must be contained in a control set of Σ_M and the intersection $\mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right) \cap \mathcal{C}_{\Sigma_M} \neq \emptyset$, we obtain $\mu^{-1}\left(\mathcal{C}_{\Sigma_{\widetilde{M}}}\right) \subset \mathcal{C}_{\Sigma_M}$. So, the desired equality is shown, and this complete the proof. ■

Now, consider S as a topologically closed subgroup of G. The space $S\backslash\mathrm{G}$, consisting of left cosets of S, has a manifold structure, and we represent π as the canonical projection from G to the homogeneous space $S\backslash\mathrm{G}$. Given a left-invariant vector field B in the Lie algebra $\mathfrak{g}$, the projection $d\pi \circ B$ onto $S\backslash\mathrm{G}$ is well-defined and recognized as an invariant vector field on $S\backslash\mathrm{G}$. It is worth noting that $d\pi \circ \mathfrak{g} = \{d\pi \circ B : B \in \mathfrak{g}\}$, the set of such vector fields, forms a Lie algebra, and $d\pi$ serves as a Lie algebra morphism from $\mathfrak{g}$ to $d\pi \circ \mathfrak{g}$. Therefore, in order to project an LCS onto a homogeneous space, the question arises as to how the linear vector field can be projected. By [1, Proposition 4], the conditions for projecting a linear vector field onto a homogeneous space are demonstrated in the next proposition:

Proposition 3.6. Assume that S is a closed subgroup of G with the homogeneous space $S\backslash G$. Then the followings are satisfied:

(1) A vector field F defined on the homogeneous space $S\backslash\mathrm{G}$ is said to be linear if it is π-conjugated to a linear vector field $\mathcal{Y}$ defined on G, i.e., $d\pi \circ \mathcal{Y} = F \circ \pi$.

(2) A vector field $\mathcal{Y}$ on G is said to be π-conjugated to a vector field on $S\backslash\mathcal{Y}$ iff S remains invariant under $\mathcal{Y}$, which can be stated as $\xi_\tau(S) \subseteq S$ for all $\tau \in \mathbb{R}$ w.r.t. the flow of $\mathcal{Y}$.

Remark 3.7. If S is a discrete subgroup, the condition of invariance stated above is met if and only if S is a subset of the set of singularities of $\mathcal{Y}$. On the other hand, if S is a connected subgroup, this is equivalent to the condition that its Lie algebra, denoted as $\mathfrak{s}$, remains invariant under the derivation $\mathcal{D}$ associated with the vector field $\mathcal{Y}$.

3.2 Linear Control Systems on the Heisenberg Group $\mathbb{H}$

In this section, we take a different approach to defining the Heisenberg group, emphasizing its vector space structure. Specifically, rather than considering it as the group of upper triangular matrices with only 1s on the main diagonal, we choose to interpret the Heisenberg group $\mathbb{H}$ as the cartesian product $\mathbb{R}^2 \times \mathbb{R}$. This perspective allows us to express various essential arguments, including group multiplication, invariant and linear vector fields, and their Lie brackets, in alignment with this format. In doing so, we identify all possible closed subgroups of the Heisenberg Lie group and classify them as normal and non-normal. Finally, we concentrate on a certain invariance criteria of subgroups (that is, discrete and non normal subgroups) of $\mathbb{H}$ under the flow of a linear vector field.

Throughout the exposition, we let $\mathbb{H}$ denote the 3-dimensional Heisenberg (Lie) group and $\mathfrak{h}$ its Lie algebra. For simplicity, we will consider the Heisenberg group as $\mathbb{H} = \mathbb{R}^2 \times \mathbb{R}$, with product given by

$$(\mathbf{v}_1, z_1) * (\mathbf{v}_2, z_2) = \left(\mathbf{v}_1 + \mathbf{v}_2, z_1 + z_2 + \frac{1}{2}\langle \mathbf{v}_1, \theta \mathbf{v}_2 \rangle\right),$$

where $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $\mathbb{R}^2$ and θ stands for the counter-clockwise rotation of $\frac{\pi}{2}$-degrees, and $\mathbf{v_1}, \mathbf{v_2} \in \mathbb{R}^2$, $z_1, z_2 \in \mathbb{R}$.

The Lie algebra $\mathfrak{h} = \mathbb{R}^2 \times \mathbb{R}$ of $\mathbb{H}$ is equipped with the Lie bracket

$$\Big[(\zeta_1, \alpha_1), (\zeta_2, \alpha_2)\Big] = \Big(\mathbf{0}, \langle \zeta_1, \theta \zeta_2 \rangle\Big),$$

where $\zeta_1, \zeta_2 \in \mathbb{R}^2$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

One of the usefulness of defining the Heisenberg group and its associated algebra as previously, instead of the usual matrix version, is that for this setup the exponential map $\exp : \mathfrak{h} \to \mathbb{H}$ is reduced to the identity map on $\mathbb{H}$. In particular, every connected subgroup $L \subset \mathbb{H}$ is identified with its Lie subalgebra.

Given a subgroup $L \subset \mathbb{H}$, we denote by L_0 the connected component containing the identity element of $\mathbb{H}$ and simply call it *the identity component*, as usual. Note that the identity component L_0 is a closed normal subgroup of L and has the same Lie algebra as L. The other components are given by the cosets $g * L_0 = L_0 * g$ of L with $g \in L$.

Remark 3.8. We are assuming that $\mathbb{R}^n = M_{n\times 1}(\mathbb{R})$. Moreover, if $\eta \in \mathbb{R}^n$ then $\eta^\top$ stands for its transpose by the previous identification. Such considerations will be very useful later on. More precisely, if

$$\eta \in \mathbb{R}^2 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

then $\eta^\top = (a \quad b)$ for some $a, b \in \mathbb{R}$.

By our previous setup, it is not hard to see that a typical derivation $\mathcal{D}$ of the Lie algebra $\mathfrak{h}$ of $\mathbb{H}$ in its matrix form (w.r.t. the standard basis) is given by

$$\mathcal{D} = \begin{pmatrix} A & \mathbf{0} \\ \eta^\top & \mathrm{tr}A \end{pmatrix}$$

where $A \in \mathfrak{gl}(2)$ and $\eta \in \mathbb{R}^2$. Since Lie algebra derivations are closely connected with Lie algebra automorphisms we also find it useful to give explicit face of an automorphism. Therefore, let us determine the form of automorphisms of the Heisenberg group. By [15, Proposition 9.1], since $\mathbb{H}$ is a connected and simply connected group, we have $Aut(\mathbb{H}) = Aut(\mathfrak{h})$. Let $\psi \in Aut(\mathfrak{h})$ be any automorphism where

$$\psi = \begin{pmatrix} P & k_2 \\ k_1^\top & \lambda \end{pmatrix}, \qquad P \in \mathrm{Gl}(2) \text{ and } k_1, k_2 \in \mathbb{R}^2.$$

If we use the fact that $\{(\mathbf{e}_1, 0), (\mathbf{e}_2, 0), (\mathbf{0}, 1)\} \subset \mathbb{H} = \mathbb{R}^2 \times \mathbb{R}$ with the Lie bracket $\big[(\mathbf{e}_2, 0), (\mathbf{e}_1, 0)\big] = (\mathbf{0}, 1)$ where $\{\mathbf{e}_1, \mathbf{e}_2\}$ denote the canonical basis of $\mathbb{R}^2$, we find that the center of the Lie algebra $\mathfrak{h}$ is exactly $\mathrm{span} = \{(\mathbf{0}, 1)\}$, i.e., $\mathfrak{z}(\mathfrak{h}) = \{(\mathbf{0}, 1)\}$. We also know that any automorphism ψ always satisfies $\psi\big(\mathfrak{z}(\mathfrak{h})\big) = \mathfrak{z}(\mathfrak{h})$. Hence, we get

$$\psi\big((\mathbf{0}, 1)\big) = \begin{pmatrix} P & k_2 \\ k_1^\top & \lambda \end{pmatrix} \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \begin{pmatrix} k_2 \\ \lambda \end{pmatrix} \in \mathfrak{z}(\mathfrak{h})$$

and it implies that $k_2 = \mathbf{0}$. On the other hand, since ψ is a linear map, being an automorphism means that ψ is a Lie algebra isomorphism. Therefore, we have the following

$$\Big[\psi\big((\mathbf{e}_1,0)\big),\psi\big((\mathbf{e}_2,0)\big)\Big]=\psi\Big(\big[(\mathbf{e}_1,0),(\mathbf{e}_2,0)\big]\Big).$$

From here, we obtain that $\det P = \lambda$ by applying operations on the bracket. Finally, it is found that the form of any automorphism on $\mathbb{H}$ is as follows

$$\begin{pmatrix} P & \mathbf{0} \\ \eta^\top & \det P \end{pmatrix}$$

where $P \in \mathrm{Gl}(2)$ and $\eta \in \mathbb{R}^2$.

Now let us identify the left-invariant and linear vector fields on $\mathbb{H}$. Since the Lie algebra $\mathfrak{h}$ of $\mathbb{H}$ can be seen as the set of left-invariant vector fields on $\mathbb{H}$, we give below a usual expression of such a vector field which is notationally appropriate in the present context. Hence, if we pick a point $g = (\mathbf{v}, z) \in \mathbb{H}$, and an element $B = (\zeta, \alpha) \in \mathfrak{h}$, the left-invariant vector field on $\mathbb{H}$ is defined via the vector space structure by

$$\begin{aligned}(dL_g)B =&\frac{d}{d\tau}\bigg|_{\tau=0} L_g \circ \exp\tau B = \frac{d}{d\tau}\bigg|_{\tau=0}(g * \exp\tau B)\\ =&\frac{d}{d\tau}\bigg|_{\tau=0}\left(\mathbf{v}+\tau\zeta, z+\tau\alpha+\frac{\tau}{2}\langle \mathbf{v},\theta\zeta\rangle\right)\\ =&\left(\zeta,\alpha+\frac{1}{2}\langle \mathbf{v},\theta\zeta\rangle\right) = B(g)\end{aligned}$$

where the 1-parameter subgroup $\varrho(\tau) = \exp\tau B$ is a curve through e with $\varrho^{'}(0) = B$. It then follows that given a derivation $\mathcal{D}$ of $\mathfrak{h}$ as above, one might immediately associate to $\mathcal{D}$ the linear vector field $\mathcal{X}$ on $\mathbb{H}$ by $[B, \mathcal{X}] = \mathcal{D}B$ for every $B \in \mathfrak{h}$. Hence, at any state $g = (\mathbf{v}, z) \in \mathbb{H}$ we might write $\mathcal{X}$ through the matrix multiplication as follows:

$$\mathcal{X}(g) = \begin{pmatrix} A & \mathbf{0} \\ \eta^\top & \mathrm{tr}A \end{pmatrix}\begin{pmatrix} \mathbf{v} \\ z \end{pmatrix} = \Big(A\mathbf{v}\ ,\ \langle \eta, \mathbf{v}\rangle + z\,\mathrm{tr}A\Big).$$

Let $B \in \mathfrak{gl}(2)$ and let us define the following operator

$$\mathbf{\Lambda}^B : \mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad \mathbf{\Lambda}^B_t\eta = \int_0^t e^{sB^\top}\eta ds.$$

It then follows at once that using such an operator we get for $\mathcal{D}$ that

$$e^{t\mathcal{D}} = \begin{pmatrix} e^{tA} & \mathbf{0} \\ \left(e^{t\cdot\mathrm{tr}A}\mathbf{\Lambda}_t^{(A-\mathrm{tr}A\cdot I_2)}\eta\right)^\top & e^{t\cdot\mathrm{tr}A} \end{pmatrix},$$

where in the previous I_2 stands for the 2×2 identity matrix. As a consequence, the flow φ_t induced by $\mathcal{X}$ is given by

$$\varphi_t(\mathbf{v}, z) = \left(e^{tA}\mathbf{v}, \left\langle e^{t\cdot \mathrm{tr}A}\mathbf{\Lambda}_t^{(A-\mathrm{tr}A\cdot I_2)}\eta, \mathbf{v}\right\rangle + z e^{t\cdot \mathrm{tr}A}\right).$$

Remark 3.9. Since a linear vector field on $\mathbb{H}$ is determined by the $\eta \in \mathbb{R}^2$ and $A \in \mathfrak{gl}(2)$, we will usually write $\mathcal{X} = (\eta, A)$ to represent such a vector.

Before we mention LCSs on homogeneous spaces of $\mathbb{H}$ we give first a brief description of a linear control system on $\mathbb{H}$ since this is intimately related with that on the corresponding coset spaces of $\mathbb{H}$. Hence, let Ω be a compact subset of $\mathbb{R}^3$. By an LCS on $\mathbb{H}$ we understand a system of the form

$$\Sigma_{\mathbb{H}} : \dot{(\mathbf{v}, z)} = \mathcal{X}(\mathbf{v}, z) + \omega_1 B_1(\mathbf{v}, z) + \omega_2 B_2(\mathbf{v}, z) + \omega_3 B_3(\mathbf{v}, z)$$

where $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega$, $\mathcal{X}$ is a linear vector field and B_1, B_2, B_3 are left-invariant vector fields. In coordinates, $\Sigma_{\mathbb{H}}$ is defined by the family of ODEs as follows

$$\Sigma_{\mathbb{H}} : \begin{cases} \dot{\mathbf{v}} = A\mathbf{v} + \omega_1\zeta_1 + \omega_2\zeta_2 + \omega_3\zeta_3 \\ \dot{z} = \langle \eta, \mathbf{v}\rangle + z\mathrm{tr}A + \omega_1\alpha_1 + \omega_2\alpha_2 + \omega_3\alpha_3 + \frac{1}{2}\langle \mathbf{v}, \theta(\omega_1\zeta_1 + \omega_2\zeta_2 + \omega_3\zeta_3)\rangle \end{cases}$$

where $\omega \in \Omega$, $g = (\mathbf{v}, z) \in \mathbb{H}$, $B_i = (\zeta_i, \alpha_i) \in \mathfrak{h}$, $A \in \mathfrak{gl}(2)$ and $\eta \in \mathbb{R}^2$.

3.3 Invariant Subgroups of $\mathbb{H}$ by Linear Vector Fields

As we pointed out earlier, if a subgroup L of $\mathbb{H}$ is topologically closed, then the homogeneous space $L\backslash\mathbb{H}$ admits a manifold structure in such a way that the canonical projection $\pi : \mathbb{H} \to L\backslash\mathbb{H}$ is a submersion. Hence, we start with stating completely in this subsection all possible closed subgroups of $\mathbb{H}$. Nonetheless, we exclude the trivial cases where $L = \{(0, 0)\}$ and $L = \mathbb{H}$ and only focus on the non-trivial cases, namely, when the subgroup L of $\mathbb{H}$ is (i) non-trivial discrete, (ii) one-dimensional and (iii) two-dimensional.

Let us begin by finding, up to isomorphisms, the explicit form of one-dimensional and two-dimensional subalgebras.

Proposition 3.10. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ denote the canonical basis of $\mathbb{R}^2$. Then, up to isomorphisms, it holds that:

(1) There is a unique two-dimensional Lie subalgebra of $\mathfrak{h}$ which is

$$\mathfrak{l}_2 = \text{span}\{(\mathbf{e}_1, 0), (\mathbf{0}, 1)\}.$$

(2) There are only one-dimensional Lie subalgebras of $\mathfrak{h}$ that are

$$\mathfrak{l}_0 = \{\mathbf{0}\} \times \mathbb{R} \quad \text{or} \quad \mathfrak{l}_1 = \mathbb{R}\mathbf{e}_1 \times \{0\}.$$

Proof. (1) Let $\mathfrak{l}_2$ be a two-dimensional Lie subalgebra and $\beta = \{(\upsilon_1, \alpha_1), (\upsilon_2, \alpha_2)\}$ a basis of it. Then we have

$$\Big[(\upsilon_1, \alpha_1), (\upsilon_2, \alpha_2)\Big] = \Big(\mathbf{0}, \langle \upsilon_1, \theta\upsilon_2\rangle\Big).$$

Since β is a basis, then there exist $a_1, a_2 \in \mathbb{R}$ such that

$$\begin{aligned} a_1\upsilon_1 + a_2\upsilon_2 &= \mathbf{0} \\ a_1\alpha_1 + a_2\alpha_2 &= \langle \upsilon_1, \theta\upsilon_2\rangle. \end{aligned}$$

So, there are two possibilities here. The first one, if $\alpha_1 = \alpha_2 = 0$, then we get $\langle \upsilon_1, \theta\upsilon_2\rangle = 0$ and it means that $\upsilon_2 \in \mathbb{R}\upsilon_1$, which contradicts the fact that β is a basis. On the other hand, if $\alpha_1 \neq 0$ (or $\alpha_2 \neq 0$), then we have

$$\underbrace{-\alpha_2(\upsilon_1, \alpha_1) + \alpha_1(\upsilon_2, \alpha_2)}_{\in\mathfrak{l}_2} = (\underbrace{\alpha_1\upsilon_2 - \alpha_2\upsilon_1}_{\neq 0}, 0).$$

Therefore, we say that there exists an element $(\upsilon, 0)$ in $\mathfrak{l}_2$ with $\upsilon \neq 0$. Now, since $(\upsilon_1, \alpha_1) \in \mathfrak{l}_2$ with $\alpha_1 \neq 0$, then we can find an element in $\mathfrak{l}_2$, which is $(\frac{\upsilon_1}{\alpha_1}, \frac{\alpha_1}{\alpha_1}) := (w, 1) \in \mathfrak{l}_2$. Hence, it follows that $\{(\upsilon, 0), (w, 1)\}$ is a basis and

$$\Big[(\upsilon, 0), (w, 1)\Big] = \Big(\mathbf{0}, \langle \upsilon, \theta w\rangle\Big).$$

Similarly, by repeating the above steps we have that there exist $a, b \in \mathbb{R}$ such that

$$\begin{aligned} a\upsilon + bw &= \mathbf{0} \\ b &= \langle \upsilon, \theta w\rangle. \end{aligned}$$

If $b = 0$ then we find that $(\mathbf{0}, 1) \in \mathfrak{l}_2$ and again if $b \neq 0$ then $\big(\mathbf{0}, \langle \upsilon, \theta w\rangle\big) \neq (\mathbf{0}, 0)$. Finally, consider the $P \in \text{Gl}(2)$ satisfying $P\upsilon = \mathbf{e}_1$ where $\{(\upsilon, 0), (\mathbf{0}, 1)\}$ is a basis and define the following automorphism

$$\psi = \begin{pmatrix} P & \mathbf{0} \\ 0 & \det P \end{pmatrix}$$

with $\psi\big((v,0)\big) = (\mathbf{e}_1, 0)$ and $\psi\big((\mathbf{0},1)\big) = (\mathbf{0}, \det P)$. From this, we conclude that $\psi(\mathfrak{l}_2) = \operatorname{span}\{(\mathbf{e}_1,0),(\mathbf{0},1)\}$. So, up to automorphism, there is only one two-dimensional Lie subalgebra given by $\mathfrak{l}_2 = \operatorname{span}\{(\mathbf{e}_1,0),(\mathbf{0},1)\}$. Moreover, since $\mathfrak{l}_2$ is also an ideal, we have that $\mathfrak{h}\backslash\mathfrak{l}_2 = \mathbb{R}$.

(2) Let $\mathfrak{l}$ be a one-dimensional Lie subalgebra and $\widetilde{\beta} = \{(v,\alpha)\}$ a basis of it. If $v = \mathbf{0}$, then we obtain that $\mathfrak{l} = \{\mathbf{0}\} \times \mathbb{R}$ and call it $\mathfrak{l}_0$. Now let us consider the case where $v \neq \mathbf{0}$. If we determine the form of the automorphism as follows

$$\psi = \begin{pmatrix} P & \mathbf{0} \\ -\alpha(\det P)\frac{v}{\|v\|^2} & \det P \end{pmatrix}$$

where $P \in \mathrm{Gl}(2)$ with $\det P \neq 0$, we have that $\psi\big((v,\alpha)\big) \in \mathbb{R}\mathbf{e}_1 \times \{0\}$. Finally, we conclude that, up to automorphism, there is only one one-dimensional Lie subalgebra $\psi(\mathfrak{l}) = \operatorname{span}\{(\mathbf{e}_1,0)\} = \mathbb{R}\mathbf{e}_1 \times \{0\}$, calling it $\mathfrak{l}_1$ we complete the proof. ■

We emphasize that since the exponential map $\exp : \mathfrak{h} \to \mathbb{H}$ is the identity map (and hence, a global diffeomorphism) it follows from the previous proposition, up to isomorphisms, any subgroup $L \subset \mathbb{H}$ has the identity component as follows:

- If L is two-dimensional, the identity component of L is $L_0 = \mathfrak{l}_2$.
- If L is one-dimensional, then the identity component of L is determined as either $L_0 = \mathfrak{l}_0$ or $L_0 = \mathfrak{l}_1$.

The proposition below together with the Proposition 3.10 help us to construct homogeneous spaces we need for later references.

Theorem 3.11. *Up to isomorphisms, any closed subgroup $L \subset \mathbb{H}$ is given by:*

(1) $\dim L = 2$ *and* $L = (\mathbb{R} \times \mathbb{Z}p) \times \mathbb{R}$, *for* $p = 0, 1$*;*

(2) $\dim L = 1$ *and* $L = \mathbb{Z}^k \times \mathbb{R}$, *for* $k = 0, 1, 2$ *or* $L = \mathbb{R}\mathbf{e}_1 \times \mathbb{Z}p$, *for* $p = 0, 1$*;*

(3) $\dim L = 0$ *and* $L = \mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}p$ *for* $p = 0, 1$, $L = \{\mathbf{0}\} \times \mathbb{Z}$ *or* $L = \mathbb{Z}^2 \times \mathbb{Z}\frac{1}{p}$ *for* $p \in \mathbb{N}$.

Proof. (1) Let $L \subset \mathbb{H}$ be a closed subgroup with $\dim L = 2$ and assume w.l.o.g. that $L_0 = \mathbb{R}\mathbf{e}_1 \times \mathbb{R}$. The projection

$$\pi : \mathbb{H} \to \mathbb{R}, \qquad \pi(\mathbf{v}, z) = y,$$

where $\mathbf{v} = (x, y)$, is a group homomorphism with kernel given exactly by $\mathbb{R}\mathbf{e}_1 \times \mathbb{R}$. Hence we obtain that $L_0 \backslash \mathbb{H} = \mathbb{R}$ by the isomorphism Theorem. In particular, $\pi : \mathbb{H} \to \mathbb{R}$ takes L into a discrete subgroup of $\mathbb{R}$ and hence $\pi(L) = \mathbb{Z}a$, for some $a \geq 0$. Therefore,

$$L \subset \pi^{-1}(\mathbb{Z}a) = (\mathbb{R} \times \mathbb{Z}a) \times \mathbb{R}.$$

On the other hand, for any $g \in (\mathbb{R} \times \mathbb{Z}a) \times \mathbb{R}$ there exists $g_0 \in L$ such that $\pi(g) = \pi(g_0)$. Consequently,

$$g * g_0^{-1} \in \mathbb{R}\mathbf{e}_1 \times \mathbb{R} = L_0 \quad \Longrightarrow \quad g \in L_0 * g_0 \subset L \quad \Longrightarrow \quad L = (\mathbb{R} \times \mathbb{Z}a) \times \mathbb{R}.$$

If $a = 0$ the item is proved. If $a \neq 0$, the map

$$\phi : \mathbb{H} \to \mathbb{H}, \qquad \phi\big((x, y), z\big) = \big((x, a^{-1}y), z\big),$$

is an automorphism taking L to $(\mathbb{R} \times \mathbb{Z}) \times \mathbb{R}$, concluding the proof.

(2) Let us first assume that $L_0 = \{\mathbf{0}\} \times \mathbb{R}$. In this case, the homogeneous space $L_0 \backslash \mathbb{H}$ coincides with the Lie group $\mathbb{R}^2$ with canonical projection given by

$$\pi : \mathbb{H} \to \mathbb{R}^2, \qquad \pi(\mathbf{v}, z) = \mathbf{v}.$$

As previously, π takes L into a discrete subgroup of $\mathbb{R}^2$ implying that

$$\pi(L) = a\mathbb{Z} \times b\mathbb{Z},$$

for some $a, b \geq 0$. Thus, we conclude that $\pi(L) = \pi\left(a\mathbb{Z} \times b\mathbb{Z} \times \{0\}\right) = a\mathbb{Z} \times b\mathbb{Z}$ and hence,

$$a\mathbb{Z} \times b\mathbb{Z} \times \{0\} \subset L * (\{\mathbf{0}\} \times \mathbb{R}) = L$$

and

$$L \subset (a\mathbb{Z} \times b\mathbb{Z} \times \{0\}) * (\{\mathbf{0}\} \times \mathbb{R}) = a\mathbb{Z} \times b\mathbb{Z} \times \mathbb{R},$$

where the former equality follows from the fact that $Z(\mathbb{H}) = \{\mathbf{0}\} \times \mathbb{R}$ is the center of $\mathbb{H}$. As in the previous item, one can easily construct an isomorphism of $\mathbb{H}$ taking $a\mathbb{Z} \times b\mathbb{Z} \times \mathbb{R}$ onto $\mathbb{Z}^k \times \mathbb{R}$, where k depends on the numbers a and b. Therefore the equality desired follows.

Now assume that $L_0 = \mathbb{R}\mathbf{e}_1 \times \{0\}$. If $g = (\mathbf{v}, z) \in L$ then it follows that

$$g * L_0 = \left\{ \left(\mathbf{v} + t\mathbf{e}_1, z + \frac{t}{2} \langle \mathbf{v}, \mathbf{e}_2 \rangle \right) : t \in \mathbb{R} \right\}$$

is a line passing through the point $g = (\mathbf{v}, z)$ and parallel to the vector $(2\mathbf{e}_1, \langle \mathbf{v}, \mathbf{e}_2 \rangle)$. On the other hand,

$$\begin{aligned}
&\left(\mathbf{v} + t\mathbf{e}_1, z + \frac{t}{2} \langle \mathbf{v}, \mathbf{e}_2 \rangle \right) * \left(\mathbf{v} + s\mathbf{e}_1, z + \frac{s}{2} \langle \mathbf{v}, \mathbf{e}_2 \rangle \right) \\
&= \left(2\mathbf{v} + (t+s)\mathbf{e}_1, 2z + \frac{t+s}{2} \langle \mathbf{v}, \mathbf{e}_2 \rangle + \frac{1}{2} \langle \mathbf{v} + t\mathbf{e}_1, \theta(\mathbf{v} + s\mathbf{e}_1) \rangle \right) \\
&= \left(2\mathbf{v} + (t+s)\mathbf{e}_1, 2z + \frac{t+s}{2} \langle \mathbf{v}, \mathbf{e}_2 \rangle + \frac{1}{2} \Big(s \langle \mathbf{v}, \mathbf{e}_2 \rangle - t \langle \mathbf{v}, \mathbf{e}_2 \rangle \Big) \right) \\
&= \Big(2\mathbf{v} + (t+s)\mathbf{e}_1, 2z + s \langle \mathbf{v}, \mathbf{e}_2 \rangle \Big) \\
&= 2(\mathbf{v}, z) + t(\mathbf{e}_1, 0) + s(\mathbf{e}_1, \langle \mathbf{v}, \mathbf{e}_2 \rangle)
\end{aligned}$$

which shows that

$$(g * L_0)^2 = \{ 2(\mathbf{v}, z) + t(\mathbf{e}_1, 0) + s(\mathbf{e}_1, \langle \mathbf{v}, \mathbf{e}_2 \rangle) : t, s \in \mathbb{R} \}.$$

Note that $(g * L_0)^2$ is a plane if $\langle \mathbf{v}, \mathbf{e}_2 \rangle \neq 0$. Since L is a one dimensional subgroup, the fact that $(g * L_0)^2 \subset L$ implies that

$$(\mathbf{v}, z) \in L \quad \Longleftrightarrow \quad \langle \mathbf{v}, \mathbf{e}_2 \rangle = 0 \quad \Longleftrightarrow \quad \mathbf{v} \in \mathbb{R}\mathbf{e}_1$$

which is equivalent to say that $L \subset \mathbb{R}\mathbf{e}_1 \times \mathbb{R}$. On the other hand, since any $(\mathbf{v}, z) \in \mathbb{H}$ can be written as $(\mathbf{v}, z) = (\mathbf{v}, 0) * (\mathbf{0}, z)$, then

$$(\mathbf{v}, z) \in L \quad \Longrightarrow \quad (\mathbf{0}, z) = (\mathbf{v}, 0)^{-1} * (\mathbf{v}, z) \in L \quad \Longrightarrow \quad (\mathbf{0}, z) \in (\{\mathbf{0}\} \times \mathbb{R}) \cap L,$$

and this shows that

$$L = \mathbb{R}\mathbf{e}_1 \times \{0\} * ((\{\mathbf{0}\} \times \mathbb{R}) \cap L). \tag{3.1}$$

However, $(\{\mathbf{0}\} \times \mathbb{R}) \cap L$ in (3.1) is a discrete subgroup of the Lie group $\{\mathbf{0}\} \times \mathbb{R}$ and hence it happens that

$$(\{\mathbf{0}\} \times \mathbb{R}) \cap L = \{\mathbf{0}\} \times \mathbb{Z}a,$$

for some $a \geq 0$. Again, by the fact that $\{\mathbf{0}\} \times \mathbb{R}$ is the center of $\mathbb{H}$ we conclude that

$$L = \mathbb{R}\mathbf{e}_1 \times \{0\} * (\{\mathbf{0}\} \times \mathbb{Z}a) = \mathbb{R}\mathbf{e}_1 \times \mathbb{Z}a.$$

If $a = 0$ we get that $L = \mathbb{R}\mathbf{e}_1 \times \{0\}$ and for $a \neq 0$, the isomorphism

$$(\mathbf{v}, z) \mapsto \left(\frac{1}{\sqrt{a}}\mathbf{v}, \frac{1}{a}z\right),$$

takes L to the subgroup $\mathbb{R}\mathbf{e}_1 \times \mathbb{Z}$ as stated.

(3) As in the preceding case, by considering the subgroup $\{\mathbf{0}\} \times \mathbb{R}$ we have that $\pi(L)$ is a discrete subgroup of $(\{\mathbf{0}\} \times \mathbb{R})\backslash\mathbb{H} = \mathbb{R}^2$ and hence we obtain that

$$L \subset a\mathbb{Z} \times b\mathbb{Z} \times \mathbb{R}$$

for some $a, b \geq 0$. Moreover,

$$(\mathbf{v}, z) = (\mathbf{v}, 0) * (\mathbf{0}, z) \quad \Longrightarrow \quad (\mathbf{0}, z) \in (\{\mathbf{0}\} \times \mathbb{R}) \cap L,$$

which, as previously, allows us to conclude that

$$L = a\mathbb{Z} \times b\mathbb{Z} \times c\mathbb{Z}$$

where $a, b, c \geq 0$. If $a = b = 0$ then $c > 0$ and the automorphism

$$(\mathbf{v}, z) \mapsto \left(\frac{1}{\sqrt{c}}\mathbf{v}, \frac{1}{c}z\right),$$

takes L to the subgroup $\{\mathbf{0}\} \times \mathbb{Z}$. Analogously, if $a = 0$ and $b \neq 0$ or $b = 0$ and $a \neq 0$ and L is isomorphic to $\mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}p$ for $p = 0, 1$. On the other hand, if $ab \neq 0$, then

$$(a\mathbf{e}_1, 0), (b\mathbf{e}_2, 0) \in L \quad \Longrightarrow \quad \left(a\mathbf{e}_1 + b\mathbf{e}_2, -\frac{ab}{2}\right) \in L \quad \Longrightarrow \quad ab \in c\mathbb{Z},$$

and hence, L is isomorphic to $\mathbb{Z}^2 \times \mathbb{Z}\frac{1}{p}$ by the isomorphism

$$((x, y), z) \mapsto \left(\left(\frac{1}{a}x, \frac{1}{b}y\right), \frac{1}{ab}z\right)$$

where $p = \frac{ab}{c}$ for some $p \in \mathbb{N}$, concluding the proof. ■

Remark 3.12. An elemantary calculation shows that

$$(\mathbf{v}_1, z_1) * (\mathbf{v}_2, z_2) * (\mathbf{v}_1, z_1)^{-1} = \big(\mathbf{v}_2, z_2 + \langle \mathbf{v}_1, \theta \mathbf{v}_2 \rangle\big)$$

and hence, up to isomorphisms, the only normal subgroups of $\mathbb{H}$ are

(i) $(\mathbb{R} \times \mathbb{Z}p) \times \mathbb{R}$ for $p = 0, 1$

(ii) $\mathbb{Z}^k \times \mathbb{R}$ for $k = 0, 1, 2$

(iii) $\{\mathbf{0}\} \times \mathbb{Z}$.

As we have already said in Proposition 3.6, if $L \subset \mathbb{H}$ is a closed subgroup, then a linear vector field $\mathcal{X}$ is conjugated to a vector field on the homogeneous space $L\backslash\mathbb{H}$ if and only if L is invariant by the flow of $\mathcal{X}$. Therefore, our next step is to obtain conditions for a 1-parameter subgroup of automorphisms $\{\varphi_t\}_{t\in\mathbb{R}} \subset \mathrm{Aut}(\mathbb{H})$ to let a closed subgroup of $\mathbb{H}$ invariant.

Moreover, we will not take into account the normal subgroups of $\mathbb{H}$ mentioned in Remark 3.12 since otherwise the corresponding homogeneous spaces become Lie groups and the LCSs on such spaces has been already studied in a series of papers. See [6, 8–10, 14] for detailed exposition.

Theorem 3.13. *Assume that $\mathcal{X} = (\eta, A)$ is a linear vector field on $\mathbb{H}$ and $\{\varphi_t\}_{t\in\mathbb{R}}$ is its associated flow. The followings are met:*

(1) $\mathbb{Z}^2 \times \mathbb{Z}\frac{1}{p}, p \in \mathbb{N}$ *is φ_t-invariant if and only if* $\mathcal{D} \equiv 0$*;*

(2) $\mathbb{Z}\mathbf{e}_1 \times \{0\}$ *is φ_t-invariant if and only if*

$$A\mathbf{e}_1 = 0, \ \ A\mathbf{e}_2 = \beta\mathbf{e}_2 + \alpha\mathbf{e}_1 \ \ \textit{and} \ \ \eta \in \mathbb{R}\mathbf{e}_2, \ \ \textit{with} \ \ \alpha = 0 \ \ \textit{if} \ \ \eta \neq 0;$$

(3) $\mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}$ *is φ_t-invariant if and only*

$$A\mathbf{e}_1 = 0, \ \ A\mathbf{e}_2 = \alpha\mathbf{e}_1 \ \ \textit{and} \ \ \eta \in \mathbb{R}\mathbf{e}_2, \ \ \textit{with} \ \ \alpha = 0 \ \ \textit{if} \ \ \eta \neq 0;$$

(4) $\mathbb{R}\mathbf{e}_1 \times \{0\}$ *is φ_t-invariant if and only if*

$$A\mathbf{e}_1 = \lambda\mathbf{e}_1, \ \ A\mathbf{e}_2 = \beta\mathbf{e}_2 + \alpha\mathbf{e}_1 \ \ \textit{and} \ \ \eta \in \mathbb{R}\mathbf{e}_2, \ \ \textit{with} \ \ \alpha = 0 \ \ \textit{if} \ \ \eta \neq 0;$$

(5) $\mathbb{R}\mathbf{e}_1 \times \mathbb{Z}$ *is φ_t-invariant if and only if*

$$A\mathbf{e}_1 = \lambda\mathbf{e}_1, \ \ A\mathbf{e}_2 = -\lambda\mathbf{e}_2 + \alpha\mathbf{e}_1 \ \ \textit{and} \ \ \eta \in \mathbb{R}\mathbf{e}_2, \ \ \textit{with} \ \ \alpha = 0 \ \ \textit{if} \ \ \eta \neq 0.$$

Proof. (1) Pick a point $(\mathbf{v}, z) \in \mathbb{Z}^2 \times \mathbb{Z}\frac{1}{p}$, where $p \in \mathbb{N}$ and assume that

$$\varphi_t(\mathbf{v}, z) = \left(\mathrm{e}^{tA}\mathbf{v}\ ,\ \langle \mathrm{e}^{t\cdot\mathrm{tr}A}\mathbf{\Lambda}_t^{A-\mathrm{tr}A\cdot I_2}\eta, \mathbf{v}\rangle + z\mathrm{e}^{t\cdot\mathrm{tr}A}\right) \in \mathbb{Z}^2 \times \mathbb{Z}\frac{1}{p}$$

Then the following equations are obtained

$$\begin{cases} \mathrm{e}^{tA}\mathbf{v} \in \mathbb{Z}^2 \\ \langle \mathrm{e}^{t\cdot\mathrm{tr}A}\mathbf{\Lambda}_t^{A-\mathrm{tr}A\cdot I_2}\eta, \mathbf{v}\rangle + z\mathrm{e}^{t\cdot\mathrm{tr}A} \in \mathbb{Z}\frac{1}{p}. \end{cases} \tag{3.2}$$

It results from the first equation of (3.2) $A \equiv 0$ and η is orthogonal to the any vector $\mathbf{v} \in \mathbb{Z}^2$, implying that $\eta = 0$. Thus, $\mathcal{D} = 0$.

(2) Now, $\mathbb{Z}\mathbf{e}_1 \times \{0\}$ is φ_t invariant if and only if it holds that

$$\begin{cases} \mathrm{e}^{tA}\mathbf{e}_1 \in \mathbb{Z}\mathbf{e}_1 \\ \langle \mathbf{\Lambda}_t^{A-\mathrm{tr}A\cdot I_2}\eta, \mathbf{e}_1\rangle = 0. \end{cases} \tag{3.3}$$

The first equation of (3.3) implies that $t \mapsto \mathrm{e}^{tA}\mathbf{e}_1$ must be constant since it is a continuous curve and $\mathbb{Z}$ is a discrete subgroup. Now, if we take its derivative at $t = 0$ we immediately get $A\mathbf{e}_1 = 0$. On the other hand, from the second equation, we get that

$$0 = \frac{d}{dt}\bigg|_{t=0} \langle \mathbf{\Lambda}_t^{A-\mathrm{tr}A\cdot I_2}\eta, \mathbf{e}_1\rangle = \langle \mathrm{e}^{t\cdot(A-\mathrm{tr}A\cdot I_2)}\eta, \mathbf{e}_1\rangle\bigg|_{t=0} = \langle \eta, \mathbf{e}_1\rangle,$$

from which we conclude that the matrix A satisfies

$$A\mathbf{e}_1 = 0, \quad A\mathbf{e}_2 = \beta\mathbf{e}_2 + \alpha\mathbf{e}_1 \quad \text{and} \quad \eta \in \mathbb{R}\mathbf{e}_2 \ \text{ with } \ \alpha = 0 \ \text{ if } \ \eta \neq 0,$$

as stated.

(3) As previously, $\mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}$ is φ_t-invariant if and only if, for all $n, m \in \mathbb{Z}$,

$$\begin{cases} \mathrm{e}^{tA}\mathbf{e}_1 \in \mathbb{Z}\mathbf{e}_1 \\ n\langle \mathrm{e}^{t\cdot\mathrm{tr}A}\mathbf{\Lambda}_t^{A-\mathrm{tr}A\cdot I_2}\eta, \mathbf{e}_1\rangle + m\mathrm{e}^{t\cdot\mathrm{tr}A} \in \mathbb{Z}. \end{cases} \tag{3.4}$$

If we choose $n = 0$ and $m = 1$, we get from the second equation in (3.4) that $\mathrm{e}^{t\cdot\mathrm{tr}A} \in \mathbb{Z}$ for all $t \in \mathbb{R}$ which results $\mathrm{tr}A = 0$. Similarly, if we select $n = 1$ and $m = 0$ then

$$\mathrm{e}^{tA}\mathbf{e}_1 \in \mathbb{Z}\mathbf{e}_1 \quad \text{and} \quad \langle \mathbf{\Lambda}_t^{A}\eta, \mathbf{e}_1\rangle \in \mathbb{Z},$$

from which we get, like in the preceding item, that $A\mathbf{e}_1 = 0$ and $\eta \in \mathbb{R}\mathbf{e}_2$. Since $\mathrm{tr}A = 0$ we get already that

$$A\mathbf{e}_1 = \mathbf{0} \quad A\mathbf{e}_2 = \alpha\mathbf{e}_1 \quad \text{and} \quad \eta \in \mathbb{R}\mathbf{e}_2.$$

Now,

$$0 = \frac{d^2}{dt^2}\bigg|_{t=0} \langle \mathbf{\Lambda}_t^A \eta, \mathbf{e}_1 \rangle = \langle A\mathrm{e}^{tA}\eta, \mathbf{e}_1 \rangle \bigg|_{t=0} = \langle A\eta, \mathbf{e}_1 \rangle$$

which implies $A\mathbf{e}_2 = 0$ if $\eta \neq 0$. And this concludes the proof.

(4) Analogously as the previous cases, $\mathbb{R}\mathbf{e}_1 \times \{0\}$ is φ_t-invariant if and only if

$$\begin{cases} \mathrm{e}^{tA}\mathbf{e}_1 \in \mathbb{R}\mathbf{e}_1 \\ \langle \mathbf{\Lambda}_t^{A - \mathrm{tr}A \cdot I_2} \eta, \mathbf{e}_1 \rangle = 0, \end{cases} \tag{3.5}$$

which gives us $A\mathbf{e}_1 = \lambda\mathbf{e}_1$ and, by derivation of the second equation in (3.5) at $t = 0$, $\langle \eta, \mathbf{e}_1 \rangle = 0$. Consequently, if we write the matrix A in canonical form, we see that A is such that

$$A\mathbf{e}_1 = \lambda\mathbf{e}_1, \quad A\mathbf{e}_2 = \beta\mathbf{e}_2 + \alpha\mathbf{e}_1 \quad \text{and} \quad \eta \in \mathbb{R}\mathbf{e}_2 \text{ with } \alpha = 0 \text{ if } \eta \neq 0,$$

showing the assertion.

(5) The proof is similar as the items (3) and (4) and we will omit it. ■

4
LINEAR CONTROL SYSTEMS ON HOMOGENEOUS SPACES

Based on the information obtained in the previous chapter, in this chapter we explicitly classify the dynamics of all possible LCSs by projecting linear and invariant vector fields onto homogeneous spaces. For it, let $L \subset \mathbb{H}$ be a closed subgroup and denote by $\pi : \mathbb{H} \to L\backslash\mathbb{H}$ the standard canonical projection.

Definition 4.1. An LCS on the homogeneous space $L\backslash\mathbb{H}$ is the following control-affine system:

$$\Sigma_{L\backslash\mathbb{H}} : \quad \dot{P} = f_0(P) + \sum_{j=1}^{m} u_j f_j(P) \tag{4.1}$$

with $u \in \Omega, P \in L\backslash\mathbb{H}$ and $f_0, f_1, \ldots, f_m$ are vector fields on $L\backslash\mathbb{H}$ satisfying

$$d\pi \circ \mathcal{X} = f_0 \circ \pi \quad \text{ and } \quad d\pi \circ B_j = f_j \circ \pi, \;\; j = 1, \ldots, m$$

where $\mathcal{X}$ is a linear vector field and B_j are left-invariant vector fields and $m = 3 - \dim L$.

It follows from the Definition 3.3 that an LCS on a homogeneous space $L\backslash\mathbb{H}$ is π-conjugated to an LCS on $\mathbb{H}$. We also know that the vector field $d\pi \circ \mathcal{X}$ is well-defined on $L\backslash\mathbb{H}$ iff L is invariant under the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ of $\mathcal{X}$ which means $\varphi_t(L) = L$ for every $t \in \mathbb{R}$.

Now it is evident that to classify all potential LCSs on the homogeneous spaces of $\mathbb{H}$, the task is to identify the φ_t-invariant closed subgroups of $\mathbb{H}$. Let $\Sigma_{L\backslash\mathbb{H}}$ denote an LCS on $L\backslash\mathbb{H}$ as in (4.1) such that $\mathcal{X}$ and B_j are π-conjugated with the vector fields f_0 and $f_1, \ldots, f_m$ for $j = 1, \ldots, m$, respectively. Let $\psi \in \mathrm{Aut}(\mathbb{H})$ such that $\widehat{L} = \psi(L)$ is one of the subgroups in Theorem 3.13. Consider the vector fields $\widehat{\mathcal{X}}$

and $\hat{B}_j$ satisfying

$$d\psi \circ \widehat{\mathcal{X}} = \mathcal{X} \circ \psi \quad \text{and} \quad d\psi \circ \widehat{B}_j = B_j \circ \psi, \;\; j = 1, \ldots, m.$$

That L is invariant under the flow of $\mathcal{X}$ implies that $\widehat{L}$ is also invariant under the flow of $\widehat{\mathcal{X}}$. Therefore, we have well-defined vector fields $\widehat{f}_0$ and $\widehat{f}_1, \ldots, \widehat{f}_m$ on $\widehat{L}\backslash\mathbb{H}$ determined by the relations

$$\widehat{f}_0 \circ \widehat{\pi} = d\widehat{\pi} \circ \widehat{\mathcal{X}} \quad \text{and} \quad \widehat{f}_1 \circ \widehat{\pi} = d\widehat{\pi} \circ \widehat{B}, \;\; j = 1, \ldots, m,$$

where $\widehat{\pi} : \mathbb{H} \to \widehat{L}\backslash\mathbb{H}$ is the canonical projection. Since the map $\widehat{\psi} : L\backslash\mathbb{H} \to \widehat{L}\backslash\mathbb{H}$ defined by the relation $\widehat{\psi} \circ \pi = \widehat{\pi} \circ \psi$ is a diffeomorphism, the fact that

$$d\widehat{\psi} \circ f_0 = \widehat{f}_0 \circ \widehat{\psi} \quad \text{and} \quad d\widehat{\psi} \circ f_j = \widehat{f}_j \circ \widehat{\psi}, \;\; j = 1, \ldots m,$$

shows us that $\Sigma_{L\backslash\mathbb{H}}$ is equivalent to the LCS on $\widehat{L}\backslash\mathbb{H}$ given by

$$\Sigma_{\widehat{L}\backslash\mathbb{H}} : \quad \dot{Q} = \widehat{f}_0(Q) + \sum_{j=1}^{m} u_j \widehat{f}_j(Q).$$

As a result, we can assume that subgroup L is one of the subgroups determined in Theorem 3.13, and we just need to examine the following cases.

4.1 The Zero-Dimensional Subgroups

In this section we consider the homogeneous spaces of $\mathbb{H}$ by zero-dimensional subgroups. By Theorem 3.11 and Remark 3.12, up to isomorphisms, the only subgroups we have to consider are

$$\mathbb{Z}^2 \times \mathbb{Z}\frac{1}{p}, \;\; p \in \mathbb{N} \quad \text{and} \quad \mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}p, \;\; p = 0, 1.$$

However, by Theorem 3.13 the subgroup $\mathbb{Z}^2 \times \mathbb{Z}\frac{1}{p}, p \in \mathbb{N}$ is invariant by the flow of a linear vector field $\mathcal{X}$ if and only if $\mathcal{X} \equiv 0$. As a consequence, any induced LCS on the homogeneous space $\left(\mathbb{Z}^2 \times \mathbb{Z}\frac{1}{p}\right)\backslash\mathbb{H}$ is driftless left-invariant system and hence its dynamical behaviour is well-known (see for instance [20]).

Let us consider the case $L = \mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}p$ for $p = 0, 1$. If $(\mathbf{v}_1, z_1), (\mathbf{v}_2, z_2) \in \mathbb{H}$ are such that $L * (\mathbf{v}_1, z_1) = L * (\mathbf{v}_2, z_2)$, then by the definition there exists $(m, n) \in \mathbb{Z} \times \mathbb{Z}p$

such that

$$(m\mathbf{e}_1, n) * (\mathbf{v}_1, z_1) = (\mathbf{v}_2, z_2) \iff \left(m\mathbf{e}_1 + \mathbf{v}_1, z_1 + \frac{1}{2}\langle m\mathbf{e}_1, \theta\mathbf{v}_1\rangle + n\right) = (\mathbf{v}_2, z_2)$$
$$\iff \begin{cases} m\mathbf{e}_1 = \mathbf{v}_2 - \mathbf{v}_1 \\ z_2 = z_1 + \frac{m}{2}\langle \mathbf{e}_1, \theta\mathbf{v}_1\rangle + n \end{cases}$$

Using the first equation we obtain that

$$m = \langle \mathbf{v}_2, \mathbf{e}_1\rangle - \langle \mathbf{v}_1, \mathbf{e}_1\rangle, \quad \langle \mathbf{v}_2, \mathbf{e}_2\rangle = \langle \mathbf{v}_1, \mathbf{e}_2\rangle, \quad \text{and} \quad \langle \mathbf{e}_1, \theta\mathbf{v}_1\rangle = \langle \mathbf{e}_1, \theta\mathbf{v}_2\rangle.$$

Using now the second equation and the previous relations, give us that

$$\begin{aligned} z_2 = &\ z_1 + \frac{1}{2}\langle t\mathbf{e}_1, \theta\mathbf{v}_1\rangle + n = z_1 + \frac{1}{2}(\langle \mathbf{v}_2, \mathbf{e}_1\rangle - \langle \mathbf{v}_1, \mathbf{e}_1\rangle)\langle \mathbf{e}_1, \theta\mathbf{v}_1\rangle + n \\ = &\ z_1 - \frac{1}{2}\langle \mathbf{v}_1, \mathbf{e}_1\rangle\langle \mathbf{e}_1, \theta\mathbf{v}_1\rangle + \frac{1}{2}\langle \mathbf{v}_2, \mathbf{e}_1\rangle\langle \mathbf{e}_1, \theta\mathbf{v}_2\rangle + n \end{aligned}$$

and so

$$\left(z_2 + \frac{1}{2}\langle \mathbf{v}_2, \mathbf{e}_1\rangle\langle \mathbf{v}_2, \mathbf{e}_1\rangle\right) - \left(z_1 + \frac{1}{2}\langle \mathbf{v}_1, \mathbf{e}_1\rangle\langle \mathbf{v}_1, \mathbf{e}_2\rangle\right) = n.$$

Therefore, $L * (\mathbf{v}_1, z_1) = L * (\mathbf{v}_2, z_2)$ if and only if

$$[x_1] = [x_2], \quad y_1 = y_2 \quad \text{and} \quad \left[z_1 + \frac{1}{2}x_1y_1\right]_p = \left[z_2 + \frac{1}{2}x_2y_2\right]_p \tag{4.2}$$

where $\mathbf{v}_i = (x_i, y_i)$ with $i = 1, 2$. Note that by $[x]_1$ and $[x]_0$ we mean $[x]_1 := [x] = x + \mathbb{Z}$ and $[x]_0 := x$. Therefore, the homogeneous space $(\mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}p)\backslash\mathbb{H}$ is identified with $(\mathbb{T}\times\mathbb{R})\times\mathbb{T}^p$, where $\mathbb{T}^0 := \mathbb{R}$ and $\mathbb{T}^1 := \mathbb{T} = \mathbb{R}/\mathbb{Z}$. The canonical projection is given by

$$\pi_{0,p} : \mathbb{H} \to (\mathbb{T} \times \mathbb{R}) \times \mathbb{T}^p, \qquad \big((x, y), z\big) \mapsto \left(([x], y), \left[z + \frac{1}{2}xy\right]_p\right).$$

Remark 4.1. By considering the maps $f : \mathbb{H} \to \mathbb{H}$ and $h_p : \mathbb{H} \to (\mathbb{T} \times \mathbb{R}) \times \mathbb{T}^p$ defined, respectively, as

$$f\big((x, y), z\big) = \left((x, y), z + \frac{1}{2}xy\right) \quad \text{and} \quad h_p\big((x, y), z\big) = \big(([x], y), [z]_p\big),$$

and using that the differential of the canonical projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the identity map, it is easy to see that

$$h_p \circ f = \pi_{0,p} \quad \text{and} \quad d(\pi_{0,p}) = df \ \text{ for } \ p = 0, 1.$$

4.1.1 LCS's on the 3D Homogeneous Spaces

The forms of all possible LCSs in the spaces $(\mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}p)\backslash\mathbb{H}$ for $p = 0, 1$ are explicitly described in this section. As previously, by Theorem 3.13, if $\mathcal{X} = (\eta, A)$ is a linear vector field on $\mathbb{H}$ whose flow let $\mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}p$ for $p = 0, 1$ invariant then

$$\eta = \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & \alpha \\ 0 & (1-p)\beta \end{pmatrix}, \quad \text{with } \alpha = 0 \text{ if } \gamma \neq 0.$$

Therefore, in coordinates, $\mathcal{X}$ is given by the expression

$$\mathcal{X}\big((x, y), z\big) = \big((\alpha y, (1-p)\beta y), \gamma y + (1-p)\beta z\big).$$

Using Remark 4.1, we have the following

$$(d\pi_{0,p})_{((x,y),z)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}y & \frac{1}{2}x & 1 \end{pmatrix},$$

and it implies that

$$(d\pi_{0,p})_{((x,y),z)}\mathcal{X}\big((x, y), z\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}y & \frac{1}{2}x & 1 \end{pmatrix} \begin{pmatrix} \alpha y \\ (1-p)\beta y \\ \gamma y + (1-p)\beta z \end{pmatrix}$$

$$= \begin{pmatrix} \alpha y \\ (1-p)\beta y \\ (1-p)\beta \left(z + \frac{1}{2}xy\right) + \frac{1}{2}\alpha y^2 + \gamma y \end{pmatrix}$$

$$= \left(\alpha y, (1-p)\beta y, (1-p)\beta \left(z + \frac{1}{2}xy\right) + \frac{1}{2}\alpha y^2 + \gamma y\right)$$

and hence,

$$\widehat{\mathcal{X}}_{0,p}\big(([u], s), [t]_p\big) = \left(\alpha s, (1-p)\beta s, (1-p)\beta t + \frac{1}{2}\alpha s^2 + \gamma s\right),$$

with $\alpha = 0$ if $\gamma \neq 0$, is the general expression of a vector field on $(\mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}p)\backslash\mathbb{H}$ induced by a linear vector field on $\mathbb{H}$.

Now, let us consider a left-invariant vector field B. In coordinates, we have that

$$B\big((x, y), z\big) = \left(a, b, c + \frac{1}{2}(ay - bx)\right),$$

and hence,

$$(d\pi_{0,p})_{((x,y),z)}\, B\big((x,y),z\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}y & \frac{1}{2}x & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c+\frac{1}{2}(ay-bx) \end{pmatrix} = \begin{pmatrix} a \\ b \\ c+ay \end{pmatrix}$$
$$=\big(a,b,c+ay\big).$$

Therefore, the general expression of a vector field on $(\mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}p)\backslash\mathbb{H}$ induced by a left-invariant vector field on $\mathbb{H}$ is given by

$$\widehat{B}_{0,p}\big(([u],s),[t]_p\big) = (a,b,c+as)\,, \quad \big(([u],s),[t]_p\big) \in (\mathbb{T}\times\mathbb{R})\times\mathbb{T}^p.$$

As a consequence, we have the following expression for a general LCS on $(\mathbb{T}\times\mathbb{R})\times\mathbb{T}^p$ for $p=0,1$.

Proposition 4.2. An LCS on $(\mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}p)\backslash\mathbb{H} \simeq \mathbb{T}\times\mathbb{R}\times\mathbb{T}^p, p=0,1$ has the form

$$\Sigma_{0,p}: \begin{cases} \dot{[u]} = \alpha s + \omega_1 a_1 + \omega_2 a_2 + \omega_3 a_3 \\ \dot{s} = (1-p)\beta s + \omega_1 b_1 + \omega_2 b_2 + \omega_3 b_3 \\ \dot{[t]}_p = (1-p)\beta t + \frac{1}{2}\alpha s^2 + \gamma s + \omega_1 c_1 + \omega_2 c_2 + \omega_3 c_3 + (\omega_1 a_1 + \omega_2 a_2 + \omega_3 a_3)s \end{cases}$$

where $\omega \in \Omega$ with $a_i, b_i, c_i, \alpha, \gamma, \lambda \in \mathbb{R}, i=1,2,3$ and $\alpha = 0$ if $\gamma \neq 0$.

4.2 The One-Dimensional Subgroups

In this section, we analyze the homogeneous spaces of $\mathbb{H}$ by one-dimensional subgroups. Since we are interested in the case where the homogeneous space is not a Lie group, we have by Theorem 3.11 and Remark 3.12 that, up to isomorphisms, the only subgroups we have to consider are $\mathbb{R}\mathbf{e}_1 \times \mathbb{Z}p$, $p=0,1$. Let $(\mathbf{v}_1,z_1),(\mathbf{v}_2,z_2) \in \mathbb{H}$ and assume that $L*(\mathbf{v}_1,z_1) = L*(\mathbf{v}_2,z_2)$, where $L=\mathbb{R}\mathbf{e}_1\times\mathbb{Z}p$ for $p=0,1$. By the definition, there exists $(t,n)\in\mathbb{R}\times\mathbb{Z}$ such that

$$(t\mathbf{e}_1,n)*(\mathbf{v}_1,z_1) = (\mathbf{v}_2,z_2) \iff \left(t\mathbf{e}_1+\mathbf{v}_1, z_1+\frac{1}{2}\langle t\mathbf{e}_1,\mathbf{v}_1^*\rangle + n\right) = (\mathbf{v}_2,z_2)$$
$$\iff \begin{cases} t\mathbf{e}_1 = \mathbf{v}_2 - \mathbf{v}_1 \\ z_2 = z_1 + \frac{1}{2}\langle t\mathbf{e}_1,\mathbf{v}_1^*\rangle + n \end{cases}$$

Similar calculations as in the previous case, allows us to obtain that

$$L*((x_1,y_1),z_1) = L*((x_2,y_2),z_2) \iff$$
$$y_1 = y_2 \quad \text{and} \quad \left[z_1+\frac{1}{2}x_1y_1\right]_p = \left[z_2+\frac{1}{2}x_2y_2\right]_p$$

where $[x]_1 = x+\mathbb{Z}$ and $[x]_0 = x$. Therefore, the homogeneous space $(\mathbb{R}\mathbf{e}_1 \times \mathbb{Z}p)\backslash\mathbb{H}$ is identified with $\mathbb{R} \times \mathbb{T}^p$, where $\mathbb{T}^0 := \mathbb{R}$ and $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$.
The canonical projection is given by

$$\pi_{1,p} : \mathbb{H} \to \mathbb{R} \times \mathbb{T}^p, \qquad \big((x, y), z\big) \mapsto \left(y, \left[z + \frac{1}{2}xy\right]_p\right).$$

Remark 4.3. Similarly as in the zero-dimensional case, we can consider the maps $f : \mathbb{H} \to \mathbb{H}$ and $g_p : \mathbb{H} \to \mathbb{R} \times \mathbb{T}^p$ given by as

$$f\big((x, y), z\big) = \left((x, y), z + \frac{1}{2}xy\right) \quad \text{and} \quad g_p\big((x, y), z\big) = (y, [z]_p),$$

it is easy to see that $g_p \circ f = \pi_{1,p}$ for $p = 0, 1$. Moreover, since the differential of the canonical projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the identity, we get that $d(\pi_{1,p}) = \pi_2 \circ df$ for $p = 0, 1$, where

$$\pi_2 : \mathbb{R}^3 \to \mathbb{R}^2, \qquad \pi_2(x, y, z) = (y, z).$$

4.2.1 LCS's on the 2D Homogeneous Spaces

We have seen above that there are two different homogeneous spaces we are interested in. Now we need to find the forms of all possible LCSs on these spaces. Note that any induced control system Σ on $L\backslash\mathbb{H}$ will be equivalent to one of the systems in the following diagram (4.3).

$$\begin{array}{ccccc} & & \Sigma & & \\ & \swarrow^{(\mathbb{R}\mathbf{e}_1\times\{0\})\backslash\mathbb{H} \simeq \mathbb{R}\times\mathbb{R}} & & \searrow^{(\mathbb{R}\mathbf{e}_1\times\mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{R}\times\mathbb{T}} & \\ \Sigma_{1,0} & & & & \Sigma_{1,1} \end{array} \tag{4.3}$$

Let us begin by determining the form of an LCS defined on the homogeneous space $\mathbb{R}\times\mathbb{R}$, which we denote as $\Sigma_{1,0}$. Let $\mathcal{X}$ be a linear vector field on $\mathbb{H}$ and assume that $\mathbb{R}\mathbf{e}_1 \times \{0\}$ is invariant under the flow of $\mathcal{X} = (\eta, A)$. By Theorem 3.13 it follows that

$$\eta = \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}, \quad \text{with } \alpha = 0 \text{ if } \gamma \neq 0.$$

As a consequence, in coordinates, we get that

$$\mathcal{X}\big((x, y), z\big) = \big((\lambda x + \alpha y, \beta y), \gamma y + (\lambda + \beta)z\big).$$

Moreover, by a simple calculation, we obtain that

$$(df)_{((x,y),z)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}y & \frac{1}{2}x & 1 \end{pmatrix} \quad \Longrightarrow \quad (d\pi_{1,p})_{((x,y),z)} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}y & \frac{1}{2}x & 1 \end{pmatrix},$$

and consequently, using Remark 4.3 we have the following

$$\begin{aligned}(d\pi_{1,0})_{((x,y),z)}\,\mathcal{X}\big((x,y),z\big) &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}y & \frac{1}{2}x & 1 \end{pmatrix}\begin{pmatrix} \lambda x+\alpha y \\ \beta y \\ \gamma y+(\lambda+\beta)z \end{pmatrix} \\ &= \begin{pmatrix} \beta y \\ (\lambda+\beta)\left(z+\frac{1}{2}xy\right)+\frac{1}{2}\alpha y^2+\gamma y \end{pmatrix} \\ &= \left(\beta y, (\lambda+\beta)\left(z+\frac{1}{2}xy\right)+\frac{1}{2}\alpha y^2+\gamma y\right).\end{aligned}$$

Therefore, the general expression of a vector field on $(\mathbb{R}\mathrm{e}_1\times\{0\})\backslash\mathbb{H}$ induced by a linear vector field on $\mathbb{H}$ is given by

$$\widehat{\mathcal{X}}_{1,0}(s,t) = \big(\beta s, (\lambda+\beta)t+\alpha s^2+\gamma s\big), \quad \text{with} \quad \alpha = 0 \ \text{ if } \ \gamma\neq 0.$$

Analogous calculations, allow us to conclude that

$$\widehat{B}_{1,0}(s,t) = (b, c+as), \quad (s,t)\in\mathbb{R}\times\mathbb{R}.$$

Thus, for a general LCS on $\mathbb{R}\times\mathbb{R}$, we have the following expression.

Proposition 4.4. An LCS on $(\mathbb{R}\mathrm{e}_1\times\{0\})\backslash\mathbb{H}\simeq\mathbb{R}\times\mathbb{R}$ has the form

$$\Sigma_{1,0}: \begin{cases} \dot{s} = \beta s+\omega_1 b_1+\omega_2 b_2+\omega_3 b_3 \\ \dot{t} = (\lambda+\beta)t+\frac{1}{2}\alpha s^2+\gamma s+\omega_1 c_1+\omega_2 c_2+\omega_3 c_3+(\omega_1 a_1+\omega_2 a_2+\omega_3 a_3)s \end{cases}$$

where $\omega\in\Omega$ with $a_i, b_i, c_i, \alpha, \beta, \gamma, \lambda\in\mathbb{R}$, $i=1,2,3$ and $\alpha=0$ if $\gamma\neq 0$.

Let us now consider the other one-dimensional subgroup $\mathbb{R}\mathrm{e}_1\times\mathbb{Z}$ and detect the form of an LCS defined on the homogeneous space $\mathbb{R}\times\mathbb{T}$, which we denote as $\Sigma_{1,1}$. By Theorem 3.13, if $\mathbb{R}\mathrm{e}_1\times\mathbb{Z}$ is invariant by the flow of $\mathcal{X}=(\eta, A)$, we have that

$$\eta = \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \lambda & \alpha \\ 0 & -\lambda \end{pmatrix}$$

with $\alpha = 0$ if $\gamma \neq 0$. As a consequence, in coordinates, we have that

$$\mathcal{X}\big((x,y),z\big) = \big((\lambda x + \alpha y, -\lambda y), \gamma y\big).$$

By Remark 4.3, we get that

$$\begin{aligned}(d\pi_{1,1})_{((x,y),z)}\,\mathcal{X}\big((x,y),z\big) &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}y & \frac{1}{2}x & 1 \end{pmatrix}\begin{pmatrix} \lambda x + \alpha y \\ -\lambda y \\ \gamma y \end{pmatrix} \\ &= \begin{pmatrix} -\lambda y \\ \frac{1}{2}\alpha y^2 + \gamma y \end{pmatrix} \\ &= \left(-\lambda y, \frac{1}{2}\alpha y^2 + \gamma y\right),\end{aligned}$$

and hence,

$$\widehat{\mathcal{X}}_{1,1}(s,t) = \left(-\lambda s, \frac{1}{2}\alpha s^2 + \gamma s\right), \quad \text{with} \quad \alpha = 0 \text{ if } \quad \gamma \neq 0,$$

is the general expression of a vector field on $(\mathbb{R}\mathbf{e}_1 \times \mathbb{Z})\backslash\mathbb{H}$ induced by a linear vector field on $\mathbb{H}$. Analogous calculations, allow us to conclude that

$$\widehat{B}_{1,1}(s,[t]) = (b, c + as)\,, \quad (s,[t]) \in \mathbb{R} \times \mathbb{T},$$

is the general expression of a vector field on $(\mathbb{R}\mathbf{e}_1 \times \mathbb{Z})\backslash\mathbb{H}$ induced by a left-invariant vector field. As a consequence, we have the following expression for a general LCS on $\mathbb{R} \times \mathbb{T}$.

Proposition 4.5. An LCS on $(\mathbb{R}\mathbf{e}_1 \times \mathbb{Z})\backslash\mathbb{H} \simeq \mathbb{R} \times \mathbb{T}$ has the form

$$\Sigma_{1,1} : \begin{cases} \dot{s} = -\lambda s + \omega_1 b_1 + \omega_2 b_2 + \omega_3 b_3 \\ \dot{[t]} = \frac{1}{2}\alpha s^2 + \gamma s + \omega_1 c_1 + \omega_2 c_2 + \omega_3 c_3 + (\omega_1 a_1 + \omega_2 a_2 + \omega_3 a_3)s \end{cases}$$

where $\omega \in \Omega$ with $a_i, b_i, c_i, \alpha, \gamma, \lambda \in \mathbb{R}$, $i = 1, 2, 3$ and $\alpha = 0$ if $\gamma \neq 0$.

5
CONTROLLABILITY

The form of all possible LCSs on homogeneous spaces of the Heisenberg group $\mathbb{H}$ has been explicitly classified. A detailed analysis of the controllability issue and control sets of each of these dynamics will now be provided.

5.1 Controllability of LCS's on the 2D Homogeneous Spaces

In Theorem 3.11 and Remark 3.12, we have classified both normal and non-normal closed subgroups of dimension one of $\mathbb{H}$ and the main focus is given to the structure of homogeneous spaces through non-normal ones since the other case was already treated in the literature in a series of recent papers. These non-normal subgroups of $\mathbb{H}$ are (i) $L = \mathbb{R}\mathbf{e}_1 \times \mathbb{Z}$ and (ii) $L = \mathbb{R}\mathbf{e}_1 \times \{0\}$ and the structure of all LCSs on the corresponding homogeneous spaces $L\backslash\mathbb{H} \simeq \mathbb{R} \times \mathbb{T}$ and $L\backslash\mathbb{H} \simeq \mathbb{R} \times \mathbb{R}$ have been exposed. It is important to remember that any control system Σ on $L\backslash\mathbb{H}$ is equivalent to one of the systems shown in diagram (4.3) above. First, we investigate the controllability property of such a system on the homogeneous space $\mathbb{R} \times \mathbb{T}$. Our focus is then the consideration of the controllability of the induced control system on the other homogeneous space $\mathbb{R} \times \mathbb{R}$. The latter case, which we are going to deal with, has much more complex structures and appears to be quite complicated.

5.1.1 The Control Sets of $\Sigma_{1,1}$

To analyze the one-input LCSs on $\mathbb{R} \times \mathbb{T}$, we first prove the following proposition, which will be used in the proof of the related theorem.

Proposition 5.1. Assume that

$$\Sigma_{\mathbb{R}} : \ \dot{s} = -\lambda s + \omega b$$

is a control system on $\mathbb{R}$, where $b \neq 0$ and $\omega \in \Omega := [\omega_*, \omega^*]$. Then $\Sigma_{\mathbb{R}}$ admits only one control set $\mathcal{C}_{\Sigma_{\mathbb{R}}}$ satisfying:

$$\begin{cases} \mathcal{C}_{\Sigma_{\mathbb{R}}} = \frac{b}{\lambda}\Omega \quad \text{if} \quad \lambda > 0 \quad \text{or,} \\ \mathcal{C}_{\Sigma_{\mathbb{R}}} = \text{int}\,(\frac{b}{\lambda}\Omega) \quad \text{if} \quad \lambda < 0 \quad \text{or,} \\ \mathcal{C}_{\Sigma_{\mathbb{R}}} = \mathbb{R} \quad \text{if} \quad \lambda = 0. \end{cases}$$

Proof. The solutions of $\Sigma_{\mathbb{R}}$ are constructed by concatenations of the curves

$$\phi(\tau, s, \omega) = \mathrm{e}^{-\tau\lambda}\left(s - \frac{b}{\lambda}\omega\right) + \frac{b}{\lambda}\omega.$$

• Let λ be positive and assume that $b > 0$ since the case $b < 0$ is analoguous. Take any point $s_0 \in \frac{b}{\lambda}\Omega = \left[\frac{b}{\lambda}\omega_*, \frac{b}{\lambda}\omega^*\right]$, then we have

$$\begin{aligned} \phi(\tau, s_0, \omega) - \frac{b}{\lambda}\omega_* &= \mathrm{e}^{-\tau\lambda}\left(s_0 - \frac{b}{\lambda}\omega\right) + \frac{b}{\lambda}\omega - \frac{b}{\lambda}\omega_* \\ &\geq \mathrm{e}^{-\tau\lambda}\left(\frac{b}{\lambda}\omega_* - \frac{b}{\lambda}\omega\right) + \frac{b}{\lambda}\omega - \frac{b}{\lambda}\omega_* \\ &= \underbrace{\left(-\mathrm{e}^{-\tau\lambda} + 1\right)}_{>0}\underbrace{\left(\frac{b}{\lambda}\omega - \frac{b}{\lambda}\omega_*\right)}_{\geq 0} \geq 0. \end{aligned}$$

With similar calculations, we obtain the following

$$\begin{aligned} \phi(\tau, s_0, \omega) - \frac{b}{\lambda}\omega^* &= \mathrm{e}^{-\tau\lambda}\left(s_0 - \frac{b}{\lambda}\omega\right) + \frac{b}{\lambda}\omega - \frac{b}{\lambda}\omega^* \\ &\leq \mathrm{e}^{-\tau\lambda}\left(\frac{b}{\lambda}\omega^* - \frac{b}{\lambda}\omega\right) + \frac{b}{\lambda}\omega - \frac{b}{\lambda}w^* \\ &= \underbrace{\left(\mathrm{e}^{-\tau\lambda} - 1\right)}_{\geq 0}\underbrace{\left(\frac{b}{\lambda}\omega^* - \frac{b}{\lambda}\omega\right)}_{<0} \leq 0. \end{aligned}$$

Therefore, we find $\phi(\tau, s_0, \omega) \geq \frac{b}{\lambda}\omega_*$ and $\phi(\tau, s_0, \omega) \leq \frac{b}{\lambda}\omega^*$. It means that for all $\tau \geq 0$ and $\omega \in \Omega$ we have that

$$\phi\left(\tau, \frac{b}{\lambda}\Omega, \omega\right) \subset \frac{b}{\lambda}\Omega \implies \mathcal{O}^+(s_0) \subset \frac{b}{\lambda}\Omega \quad \text{for all} \quad s_0 \in \frac{b}{\lambda}\Omega.$$

Now let us take the points $s_0, s_1 \in \text{int}(\frac{b}{\lambda}\Omega)$ and suppose that $s_0 < s_1$. Since

$$\phi(\tau, s_0, \omega^*) = \mathrm{e}^{-\tau\lambda}\left(s_0 - \frac{b}{\lambda}\omega^*\right) + \frac{b}{\lambda}\omega^* \to \frac{b}{\lambda}\omega^* \quad \text{as} \quad \tau \to +\infty$$
$$\phi(\tau, s_1, \omega_*) = \mathrm{e}^{-\tau\lambda}\left(s_1 - \frac{b}{\lambda}\omega_*\right) + \frac{b}{\lambda}\omega_* \to \frac{b}{\lambda}\omega_* \quad \text{as} \quad \tau \to +\infty$$

then there exist $\tau_0, \tau_1 > 0$ such that

$$\phi(\tau_0, s_0, \omega^*) = s_1 \quad \text{and} \quad \phi(\tau_1, s_1, \omega_*) = s_0.$$

Hence, we have that $\mathcal{O}^+(s_0) = \text{int}\left(\frac{b}{\lambda}\Omega\right)$ for all $s_0 \in \text{int}\left(\frac{b}{\lambda}\Omega\right)$ and obtain the following by continuity

$$\mathcal{O}^+(s_0) = \frac{b}{\lambda}\Omega \quad \text{for all} \quad s_0 \in \frac{b}{\lambda}\Omega.$$

Consequently, it means that $\frac{b}{\lambda}\Omega$ is a control set of $\Sigma_{\mathbb{R}}$.

• Now let us show that $\Sigma_{\mathbb{R}}$ does not admit control sets in $\mathbb{R}\backslash\frac{b}{\lambda}\Omega = \left(\frac{b}{\lambda}\omega^*, +\infty\right) \cup \left(-\infty, \frac{b}{\lambda}\omega_*\right)$. For any $s_0 \in \mathbb{R}\backslash\frac{b}{\lambda}\Omega$, we obtain that

$$\phi(\tau, s_0, \omega) - s_0 = e^{-\tau\lambda}\left(s_0 - \frac{b}{\lambda}\omega\right) + \frac{b}{\lambda}\omega - s_0 = \left(e^{-\tau\lambda} - 1\right)\left(s_0 - \frac{b}{\lambda}w\right).$$

This allows us to achieve the following relations

$$s_0 > \frac{b}{\lambda}\omega^* \quad \Longrightarrow \quad \phi(\tau, s_0, \omega) \leq s_0 \quad \Longrightarrow \quad \mathcal{O}^+(s_0)\backslash\{s_0\} \subset (-\infty, s_0)$$

$$s_0 < \frac{b}{\lambda}\omega^* \quad \Longrightarrow \quad \phi(\tau, s_0, \omega) \geq s_0 \quad \Longrightarrow \quad \mathcal{O}^+(s_0)\backslash\{s_0\} \subset (s_0, +\infty)\,.$$

As a result, if taken as $s_0, s_1 \geq \frac{b}{\lambda}\omega^*$ with $s_0 < s_1$ then $\Sigma_{\mathbb{R}}$ has no trajectory starting at s_1 and approaching an arbitrary point s_0. It means that any control set of $\Sigma_{\mathbb{R}}$ contained in $\left(\frac{b}{\lambda}\omega^*, +\infty\right)$ cannot have two distinct points. Otherwise, it contradicts the condition of being a control set. Moreover, if $\{s_0\}$ is a control set of $\Sigma_{\mathbb{R}}$ contained in $\left(\frac{b}{\lambda}\omega^*, +\infty\right)$, we have the following by using the definition of control sets

$$\exists \omega \in \Omega \quad \forall \tau \in \mathbb{R} \quad \phi(\tau, s_0, \omega) = s_0 \quad \Longleftrightarrow \quad \left(\mathrm{e}^{-\tau\lambda} - 1\right)\left(s_0 - \frac{b}{\lambda}\omega\right) = 0,$$

which is definitely impossible. Hence, $\Sigma_{\mathbb{R}}$ does not admit control sets in $\left(\frac{b}{\lambda}\omega^*, +\infty\right)$. Similarly, $\Sigma_{\mathbb{R}}$ does not admit control sets in $\left(-\infty, \frac{b}{\lambda}\omega_*\right)$. Consequently, we show the uniqueness of $\frac{b}{\lambda}\Omega$. Analogously to the above steps, $\mathcal{C}_{\mathbb{R}} = \text{int}\left(\frac{b}{\lambda}\Omega\right)$ is obtained for the case $\lambda < 0$. Finally, let us prove the case $\lambda = 0$. The solutions of $\Sigma_{\mathbb{R}}$ are constructed by concatenations of the curves $\phi(\tau, s_0, \omega) = b\omega\tau + s_0$ since our dynamical system is $\dot{s} = \omega b$. Pick any arbitrary points s_1, s_2 with $s_1 < s_2$. In this case any $\omega_1, \omega_2 \in \Omega$ with $b\omega_1 > 0$ and $b\omega_2 < 0$, then we have

$$\tau_1 = \frac{s_2 - s_1}{b\omega_1} > 0 \quad \Longrightarrow \quad \phi(\tau_1, s_1, \omega_1) = b\omega_1 \frac{s_2 - s_1}{b\omega_1} + s_1 = s_2$$
$$\tau_2 = \frac{s_1 - s_2}{b\omega_2} > 0 \quad \Longrightarrow \quad \phi(\tau_2, s_2, \omega_2) = b\omega_2 \frac{s_1 - s_2}{b\omega_2} + s_2 = s_1$$

Therefore we achieve the controllability of $\Sigma_{\mathbb{R}}$ over the entire set of real numbers. ■

By Proposition 4.5, we have that a one-input linear control system $\Sigma_{1,1}$ on $\mathbb{R} \times \mathbb{T}$ has the form

$$\Sigma_{1,1}: \quad \begin{cases} \dot{s} = -\lambda s + \omega b \\ \dot{[t]} = \frac{1}{2}\alpha s^2 + \gamma s + \omega(c + as) \end{cases}$$

where $\omega \in \Omega$ with $a, b, c, \alpha, \gamma, \lambda \in \mathbb{R}$ and $\alpha = 0$ if $\gamma \neq 0$.
The following proposition characterizes the Lie algebra rank condition of the system $\Sigma_{1,1}$ which is indispensable for the controllability issue.

Proposition 5.2. The one-input LCS $\Sigma_{1,1}$ on $L\backslash\mathbb{H} \simeq \mathbb{R} \times \mathbb{T}$, satisfies the LARC if and only if

$$b(2a\lambda + b\alpha) \neq 0 \quad \text{or} \quad b(b\gamma + \lambda c) \neq 0.$$

Proof. Let us explicitly calculate

$$\text{span}_{\mathcal{LA}}\{\widehat{\mathcal{X}}_{1,1}, \widehat{B}_{1,1}\}(s, [t])$$

for all $(s, [t]) \in \mathbb{R} \times \mathbb{T}$, where

$$\widehat{\mathcal{X}}_{1,1}(s, [t]) = (-\lambda s, \frac{1}{2}\alpha s^2 + \gamma s) \quad \text{and} \quad \widehat{B}_{1,1}(s, [t]) = (b, c + as).$$

Firstly, looking at the Lie bracket of $\widehat{\mathcal{X}}_{1,1}$ and $\widehat{B}_{1,1}$ we have that

$$\begin{aligned} \left[\widehat{\mathcal{X}}_{1,1}, \widehat{B}_{1,1}\right] &= \Big(b\lambda, -a\lambda s - b(\alpha s + \gamma)\Big) \\ &= \lambda \widehat{B}_{1,1} - \Big\{ \underbrace{(0, 2\lambda a + b\alpha)}_{:=Z_1} s + \underbrace{(0, \lambda c + b\gamma)}_{:=Z_2} \Big\} \\ &= \lambda \widehat{B}_{1,1} - (sZ_1 + Z_2). \end{aligned}$$

Then let's consider the other brackets, respectively:

$$\begin{aligned} \left[sZ_1, \widehat{\mathcal{X}}_{1,1}\right] &= \lambda s Z_1 & \left[sZ_1, \widehat{B}_{1,1}\right] &= -bZ_1 \\ \left[Z_2, \widehat{\mathcal{X}}_{1,1}\right] &= 0 & \left[Z_2, \widehat{B}_{1,1}\right] &= 0 \\ \left[[\widehat{\mathcal{X}}_{1,1}, \widehat{B}_{1,1}], \widehat{\mathcal{X}}_{1,1}\right] &= -\lambda^2 \widehat{B}_{1,1} + 2\lambda Z_2 & \left[[\widehat{\mathcal{X}}_{1,1}, \widehat{B}_{1,1}], \widehat{B}_{1,1}\right] &= 2bZ_1. \end{aligned}$$

If it is continued in this process, we see that all brackets just depend on the vector fields $\widehat{\mathcal{X}}_{1,1}, \widehat{B}_{1,1}, Z_1$ and Z_2. Finally, we can obtain that LARC is satisfied if and only if $bZ \neq 0$ where $Z = sZ_1 + Z_2$. ■

The previous propositions will be used in our next result.

Theorem 5.3. *Under the LARC, the only control set of* $\Sigma_{1,1}$ *is*

$$\mathcal{C}_{\Sigma_{1,1}} = \mathcal{C}_{\Sigma_{\mathbb{R}}} \times \mathbb{T}$$

where $\Sigma_{\mathbb{R}}$ *is the LCS on* $\mathbb{R}$ *given by the first equation on* $\Sigma_{1,1}$.

Proof. Let us assume that $\lambda > 0$. For all $\mathbf{v} \in \mathbb{R} \times \mathbb{T}$ and $\omega \in \Omega$ we write the solution of the $\Sigma_{1,1}$ as

$$\phi(\tau, \mathbf{v}, w) = \Big(\phi_1(\tau, \mathbf{v}, w), \phi_2(\tau, \mathbf{v}, w)\Big),$$

and notice that ϕ_1 is actually the solution of the associated system $\Sigma_{\mathbb{R}}$. Since we are assuming the LARC, $b \neq 0$ and by Proposition 5.1, we have that the control set $\mathcal{C}_{\Sigma_{\mathbb{R}}} = \frac{b}{\lambda}\Omega$ is positively-invariant, implying that

$$\mathcal{O}^+(\mathbf{v}) \subset \mathcal{C}_{\Sigma_{\mathbb{R}}} \times \mathbb{T}$$

for all $\mathbf{v} \in \mathcal{C}_{\Sigma_{\mathbb{R}}} \times \mathbb{T}$. Let us consider the polynomial $p(\omega)$ given by

$$p(\omega) = \frac{b}{2\lambda^2}(b\alpha + 2a\lambda)\omega^2 + \frac{1}{\lambda}(b\gamma + c\lambda)\omega.$$

By the LARC, $p(\omega)$ is a nontrivial polynomial with, at most, two zeros in Ω. Consider now

$$\mathbf{v}_0, \mathbf{v}_1 \in \operatorname{int}(\mathcal{C}_{\Sigma_{\mathbb{R}}} \times \mathbb{T}) \quad \text{with} \quad \mathbf{v}_1 = \left(\frac{b}{\lambda}\omega_1, [t_1]\right), \quad \text{and} \quad p(\omega_1) \neq 0.$$

Since, by Proposition 5.1, controllability holds in $\operatorname{int}\mathcal{C}_{\Sigma_{\mathbb{R}}}$, there exists a positive time τ_0 and a control ω_0 such that

$$\phi_1(\tau_0, \mathbf{v}_0, \omega_0) = \frac{b}{\lambda}\omega_1 \quad \Longrightarrow \quad \phi(\tau_0, \mathbf{v}_0, \omega_0) = \Bigg(\frac{b}{\lambda}\omega_1, \underbrace{\phi_2(\tau_0, \mathbf{v}_0, \omega_0)}_{=[\hat{t}_1]\in\mathbb{T}}\Bigg) := \hat{\mathbf{v}}_1$$

On the other hand, since $\frac{b}{\lambda}\omega_1$ is a singularity of the first equation of the system $\Sigma_{1,1}$, we get that

$$\phi_1(\tau, \hat{\mathbf{v}}_1, \omega_1) = \frac{b}{\lambda}\omega_1, \quad \forall \tau \in \mathbb{R},$$

and hence, the second equation of $\Sigma_{1,1}$ is written as

$$\begin{aligned}\dot{[t]} &= \frac{1}{2}\alpha\Big(\frac{b}{\lambda}\omega_1\Big)^2 + \gamma\frac{b}{\lambda}\omega_1 + \omega_1\Big(c + a\frac{b}{\lambda}\omega_1\Big) \\ &= \frac{b}{2\lambda^2}\Big(b\alpha + 2a\lambda\Big)\omega_1^2 + \frac{1}{\lambda}\Big(b\gamma + c\lambda\Big)\omega_1 = \ p(\omega_1),\end{aligned}$$

implying that

$$\phi_2(\tau, \hat{\mathbf{v}}_1, \omega_1) = \left[\hat{t}_1 + \tau \cdot p(\omega_1)\right], \quad \forall \tau \in \mathbb{R}.$$

Therefore, the assumption $p(\omega_1) \neq 0$ implies the existence of $\tau_1 > 0$ such that $\phi_2(\tau_2, \hat{\mathbf{v}}_1, \omega_1) = [t_1]$, and so $\phi(\tau_1, \hat{\mathbf{v}}_1, \omega_1) = \mathbf{v}_1$. As a consequence, if $p(\omega) = 0$, we have that

$$(\mathcal{C}_{\Sigma_\mathbb{R}} \times \mathbb{T}) \setminus (\{0, \omega\} \times \mathbb{T}) \subset \mathcal{O}^+(\mathbf{v})$$

for all $\mathbf{v} \in (\mathcal{C}_{\Sigma_\mathbb{R}} \times \mathbb{T}) \setminus (\{0, \omega\} \times \mathbb{T})$.
Since $(\mathcal{C}_{\Sigma_\mathbb{R}} \times \mathbb{T}) \setminus (\{0, \omega\} \times \mathbb{T})$ is dense in $\mathcal{C}_{\Sigma_\mathbb{R}} \times \mathbb{T}$ we conclude that

$$\mathcal{C}_{\Sigma_\mathbb{R}} \times \mathbb{T} = \overline{\mathcal{O}^+(\mathbf{v})}$$

for all $\mathbf{v} \in \mathcal{C}_{\Sigma_\mathbb{R}} \times \mathbb{T}$. This shows that $\mathcal{C}_{\Sigma_{1,1}} = \mathcal{C}_{\Sigma_\mathbb{R}} \times \mathbb{T}$ is a control set of $\Sigma_{1,1}$ (See Figure 5.1b). Regarding the uniqueness of $\mathcal{C}_{\Sigma_{1,1}}$, it follows directly from the fact that $\mathcal{C}_{\Sigma_\mathbb{R}}$ is the only control set of the associated system $\Sigma_\mathbb{R}$. Since the case for $\lambda < 0$ is analogous to the previous one, let us now examine the situation where $\lambda = 0$. In this case, by Proposition 5.1, the control set $\mathcal{C}_{\Sigma_\mathbb{R}}$ of $\Sigma_\mathbb{R}$ is the whole real line and hence, we have to show that $\Sigma_{1,1}$ is controllable.
Similarly as the previous case, let us consider the polynomial

$$q(s) = \frac{1}{2}\alpha s^2 + \gamma s.$$

By the LARC, $q(s)$ is a nonzero polynomial. Moreover, the fact that $\gamma \neq 0$ implies $\alpha = 0$ which gives us $s = 0$ as the only root of q. Take any two points $\mathbf{v}_0 = (s_0, [t_0])$ and $\mathbf{v}_1 = (s_1, [t_1])$ in $\mathbb{R} \times \mathbb{T}$. Let us investigate the following cases for determining the trajectory from $\mathbf{v}_0$ to $\mathbf{v}_1$ (See Figure 5.1a).

(i) If $s_1 \neq 0$, the fact that $\Sigma_{\mathbb{R}}$ is controllable, assures the existence of ω_0 and τ_0 such that

$$\phi(\tau_0, \mathbf{v}_0, \omega_0) = (s_1, [\hat{t}_1]) := \hat{\mathbf{v}}_1.$$

By considering the control $\omega = 0$, we have that

$$\phi(\tau, \hat{\mathbf{v}}_1, 0) = \Big(s_1, \big[\hat{t}_1 + \tau \cdot q(s_1)\big]\Big).$$

As a consequence, there exists $\tau_1 > 0$ such that $\phi(\tau, \hat{\mathbf{v}}_1, 0) = (s_1, [t_1])$, showing, by concatenation, that we can reach $\mathbf{v}_1$ from $\mathbf{v}_0$, when $s_1 \neq 0$.

(ii) If $s_1 = 0$, take $\omega \neq 0$ and $\tau' > 0$. Then,

$$\mathbf{v}_1' := \phi_1(-\tau, \mathbf{v}_1, \omega) \quad \text{satisfies} \quad \phi(\tau, \mathbf{v}_1', \omega) = \mathbf{v}_1,$$

and the first component of $\mathbf{v}_1'$ is $s_1' = -\tau b\omega \neq 0$. By the previous item, there exists a trajectory connecting $\mathbf{v}_0$ and $\mathbf{v}_1'$ and hence, we can connect $\mathbf{v}_0$ to $\mathbf{v}_1$.

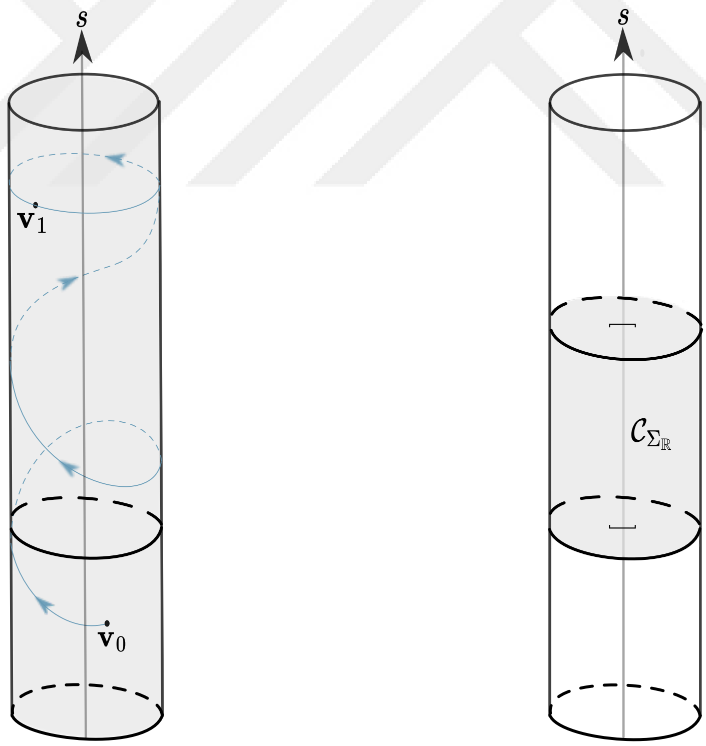


Figure 5.1 *(a) Trajectory connecting $\mathbf{v}_0$ and $\mathbf{v}_1$* *(b) Control set for nonzero λ.*

■

5.1.2 The Control Sets of $\Sigma_{1,0}$

Our current interest in this section is the controllability property of the induced control systems on the other homogeneous space $\mathbb{R}\times\mathbb{R}$. We have to emphasize that this case presents a much more intricate and complex structure, which makes this part both challenging and interesting enough. The controllability issue of these linear control systems in a very detailed manner will be studied here by means of a quite technical case-by-case analysis. Finally, in this subsection, the structures of the control sets for the systems under consideration are explicitly presented for each case and its sub-case whenever they exist.

By Proposition 4.4, we have that a one-input linear control system $\Sigma_{1,0}$ on $L\backslash\mathbb{H}\simeq\mathbb{R}\times\mathbb{R}$ has the form

$$\Sigma_{1,0}:\begin{cases}\dot{s}=\beta s+\omega b\\ \dot{t}=(\lambda+\beta)t+\frac{1}{2}\alpha s^2+\gamma s+\omega(c+as)\end{cases}\tag{5.1}$$

where $a,b,c,\alpha,\beta,\gamma\in\mathbb{R}$ with $\alpha=0$ if $\gamma\neq 0$.

The following proposition characterizes the LARC of the system $\Sigma_{1,0}$ which is indispensable for the controllability issue.

Proposition 5.4. The one-input LCS $\Sigma_{1,0}$ on $L\backslash\mathbb{H}\simeq\mathbb{R}\times\mathbb{R}$, satisfies the LARC if and only if

$$b\cdot\left((b\alpha+a(\lambda-\beta))^2+(b\gamma+c\lambda)^2\right)\neq 0.$$

Proof. Let us show that $\mathrm{span}_{\mathcal{LA}}\{\widehat{\mathcal{X}}_{1,0},\widehat{B}_{1,0}\}(s,t)=\mathbb{R}\times\mathbb{R}$ for all $(s,t)\in\mathbb{R}\times\mathbb{R}$. Firstly, looking at the Lie bracket of $\widehat{\mathcal{X}}_{1,0}$ and $\widehat{B}_{1,0}$ we have that

$$\begin{aligned}[\widehat{\mathcal{X}}_{1,0},\widehat{B}_{1,0}]=&\Big(-b\beta,(a\beta s-b(\alpha s+\gamma)-(\lambda+\beta)(c+as))\Big)\\ =&-\beta\widehat{B}_{1,0}-\Big\{\underbrace{(0,b\alpha+\lambda a-a\beta)}_{:=Z_1}s+\underbrace{(0,b\gamma+\lambda c)}_{:=Z_2}\Big\}\\ =&-\beta\widehat{B}_{1,0}-\{sZ_1+Z_2\}=-\beta\widehat{B}_{1,0}-Z.\end{aligned}$$

Then let us consider the other brackets, respectively:

$$\begin{aligned}[sZ_1,\widehat{\mathcal{X}}_{1,0}]&=\lambda sZ_1 & [sZ_1,\widehat{B}_{1,0}]&=-bZ_1,\\ [Z_2,\widehat{B}_{1,0}]&=0 & [Z_2,\widehat{\mathcal{X}}_{1,0}]&=(\lambda+\beta)Z_2,\\ [[\widehat{\mathcal{X}}_{1,0},\widehat{B}_{1,0}],\widehat{\mathcal{X}}_{1,0}]&=-\beta^2\widehat{B}_{1,0}-(\lambda+\beta)Z-\beta Z_2 & [[\widehat{\mathcal{X}}_{1,0},\widehat{B}_{1,0}],\widehat{B}_{1,0}]&=bZ_1.\end{aligned}$$

If it is continued in this process, we see that all brackets just depend on the vector fields $\widehat{\mathcal{X}}_{1,0},\widehat{B}_{1,0},Z_1,Z_2$ and Z. Finally, we can obtain that LARC is satisfied if and only if $bZ \neq 0$. Thus, the proof is complete. ■

In what follows, we define singular LCSs.

Definition 5.1. We say that $\Sigma_{1,0}$ on $L\backslash\mathbb{H} \simeq \mathbb{R}\times\mathbb{R}$ is singular if the associated vector field

$$\widehat{\mathcal{X}}_{1,0}(s,t) = \left(\beta s, (\lambda+\beta)t + \alpha s^2 + \gamma s\right),$$

satisfies $\beta(\lambda+\beta) = 0$.

The precise description of the dynamics of the LCS $\Sigma_{1,0}$ is quite difficult. Because of that our study in this part will be focused on the singular systems.

Remark 5.5. As is said above, the preceding definition is indeed related to singularities of the induced vector field $\widehat{\mathcal{X}}_{1,0}$ since $\beta(\lambda+\beta) \neq 0$ if and only if $(0,0)$ is the only singularity of $\widehat{\mathcal{X}}_{1,0}$. Note that the techniques used to treat dynamics in singular and non-singular case are quite different. In fact, for the singular case, the fact that the solutions only depend on one of the components and on the control allows a direct algebraic approach for the understanding of the dynamics. However, for the non-singular case the solutions depend on both components which forces exponential behavior in each one of them. Therefore, the precise understanding of the solutions in the non-singular case requires a more geometrical approach to analyze controllability and control sets.

It follows from the singularity imposed on the associated vector field that we should treat the possibilities $\beta(\lambda+\beta) = 0$. In what follows, we do a detailed analysis of the possible control sets of the one-input LCS on $\mathbb{R}\times\mathbb{R}$. We divide such an analysis in two main cases depending on α together with their subcases (whenever they exist) depending on the eigenvalues of A. See the Figure 5.2 below.

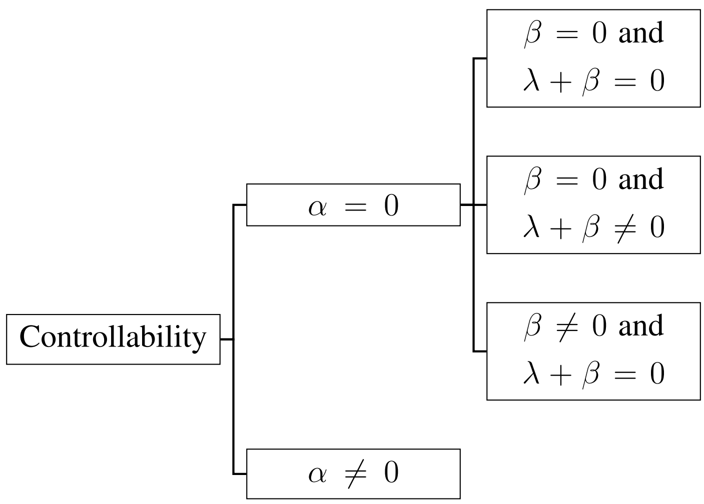


Figure 5.2 *Description of all possible cases to be analyzed on*

Controllability in the case $\alpha = 0$

In this subsection, we will be assuming that $\alpha = 0$, and hence our system does not have a quadratic term. The analysis for this case consists of four parts and is done by analyzing the possibilities for the values of β and $\lambda + \beta$ as follows:

- **The subcase $\beta = \lambda + \beta = 0$:**

Now consider the following result with these conditions.

Proposition 5.6. Assume $\alpha = \beta = \lambda + \beta = 0$ in the equation (5.1). Then $\Sigma_{1,0}$ reduces to the following system

$$(\Sigma_{1,0}) : \begin{cases} \dot{s} = \omega b \\ \dot{t} = (\gamma + a\omega)s + c\omega \end{cases}$$

where $\omega \in \Omega$ satisfies the LARC if and only if $b\gamma \neq 0$ and hence is controllable.

Proof. The first assertion follows immediately from the Proposition 5.4, i.e., it satisfies the LARC if and only if $b\gamma \neq 0$. For $\omega \neq 0$ and $\mathbf{v}_0 = (s_0, t_0)$, the solutions $\varphi(\tau, \mathbf{v}_0, \omega)$ of $\Sigma_{1,0}$ are given, component-wise, as

$$\varphi_1(\tau, \mathbf{v}_0, \omega) = s_0 + \omega b \tau$$
$$\varphi_2(\tau, \mathbf{v}_0, \omega) = \frac{b\omega(a\omega + \gamma)}{2}\tau^2 + (s_0(a\omega + \gamma) + c\omega)\tau + t_0.$$

A simple calculation shows that such a solution coincides with the parabola

$$\Gamma_{\mathbf{v}_0,\omega} := \left\{ \left(s, \frac{(a\omega + \gamma)}{2b\omega}(s - s_0)^2 + \frac{s_0(a\omega + \gamma) + c\omega}{b\omega}(s - s_0) + t_0 \right) \ : \ s \in \mathbb{R} \right\}$$

with concavity determined by the sign of $b\omega$. Let us analyze the case where $\gamma, b \in \mathbb{R}^+$, since other choices are treated similarly. For this choice, there exists $\varepsilon > 0$ such that $a\omega + \gamma > 0$ for $\omega \in (-\varepsilon, \varepsilon)$.

Let

$$\mathbf{v}_0 \neq \mathbf{v}_1 \in \mathbb{R}^2 \quad \text{and} \quad -\varepsilon < \omega_0 < 0 < \omega_1 < \varepsilon.$$

A trajectory connecting $\mathbf{v}_0$ to $\mathbf{v}_1$ can be constructed in the two steps as follows:

Step 1) v_0 belongs to the interior of the region determined by the parabola $\Gamma_{\mathbf{v}_1,\omega_1}$.

Since $b, \gamma \in \mathbb{R}^+$ we have that $\tau \to +\infty$ implies that

$$\varphi_1(\tau, \mathbf{v}_0, \omega_0) \to -\infty \quad \text{and} \quad \varphi_2(\tau, \mathbf{v}_0, \omega_0) \to -\infty.$$

As a consequence, there exists $\tau_0 > 0$ such that $\tilde{\mathbf{v}}_1 := \varphi(\tau_0, \mathbf{v}_0, \omega_0)$ belongs to $\Gamma_{\mathbf{v}_1,\omega_1}$. Write $\tilde{\mathbf{v}}_1 = (\tilde{s}_1, \tilde{t}_1)$ and consider the following cases:

(i) $\tilde{s}_1 \leq s_1$:

Since the solution starting at $\tilde{\mathbf{v}}_1$ associated to the control ω_1 lies on the parabola $\Gamma_{\mathbf{v}_1,\omega_1}$ and

$$\tau \to +\infty \qquad \Longrightarrow \qquad \varphi_1(\tau, \tilde{\mathbf{v}}_1, \omega_1) \to +\infty,$$

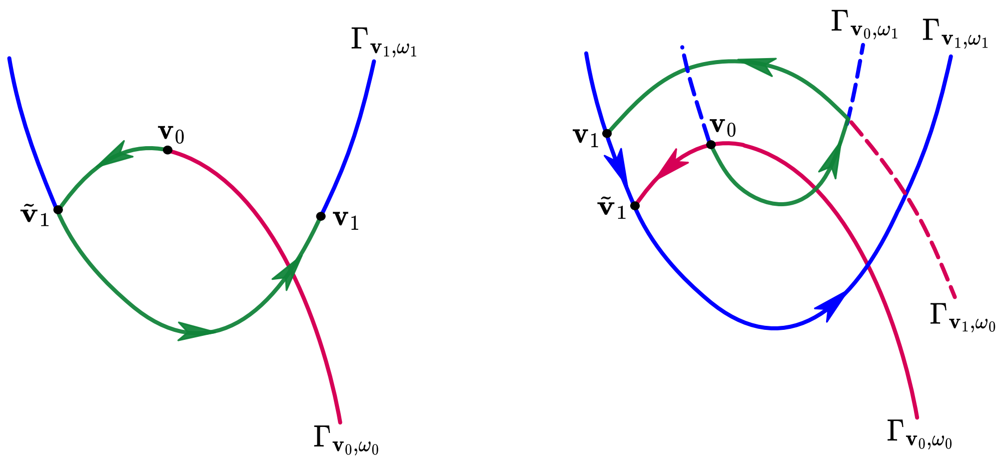
there exists $\tau_1 > 0$ such that $\varphi(\tau_1, \tilde{\mathbf{v}}_1, \omega_1) = \mathbf{v}_1$. By concatenation, we get a trajectory from $\mathbf{v}_0$ to $\mathbf{v}_1$ (Figure 5.3a).

(ii) $\tilde{s}_1 > s_1$:

Since the parabolas $\Gamma_{\mathbf{v}_1,\omega_0}$ and $\Gamma_{\mathbf{v}_0,\omega_0}$ coincides with solutions of $\Sigma_{1,0}$ for the constant control ω_0, they are parallels. Therefore, the assumption $\tilde{s}_1 > s_1$ implies that $\mathbf{v}_0$ lies in the interior of the region determined $\Gamma_{\mathbf{v}_1,\omega_0}$. On the other hand, $\tau \to +\infty$ implies

$$\varphi_1(\tau, \mathbf{v}_0, \omega_1) \to +\infty \qquad \text{and} \qquad \varphi_2(\tau, \mathbf{v}_0, \omega_1) \to +\infty,$$

and consequently, there exists $\tau_0 > 0$ such that $\tilde{\mathbf{v}}_0 := \varphi(\tau_0, \mathbf{v}_0, \omega_1)$ belongs to the parabola determined by $\mathbf{v}_1$ and ω_0. By writing $\tilde{\mathbf{v}}_0 = (\tilde{s}_0, \tilde{t}_0)$ it holds that $\tilde{s}_0 > s_1$ and hence, $\varphi(\tau_1, \tilde{\mathbf{v}}_0, \omega_0) = \mathbf{v}_1$ for some $\tau_1 > 0$. By concatenation we obtain a trajectory from $\mathbf{v}_0$ to $\mathbf{v}_1$ (Figure 5.3b).



(a) Trajectory starting inside $\Gamma_{\mathbf{v}_1,\omega_1}$ with $\tilde{s}_1 \leq s_1$

(b) Trajectories starting inside $\Gamma_{\mathbf{v}_1,\omega_1}$ with $\tilde{s}_1 > s_1$

Figure 5.3 *Trajectories starting inside the parabola* $\Gamma_{\mathbf{v}_1,\omega_1}$

Step 2) $\mathbf{v}_0$ belongs to the exterior of the region determined $\Gamma_{\mathbf{v}_1,\omega_1}$.

Let us show, in this case, the existence of a trajectory connecting $\mathbf{v}_0$ to a point in the interior $\Gamma_{\mathbf{v}_1,\omega_1}$, which by item (i) implies the result.

Since the interior of $\Gamma_{\mathbf{v}_1,\omega_1}$ is the set

$$\left\{(s,t)\in\mathbb{R}^2 \;:\; t > \frac{(a\omega_1+\gamma)}{2b\omega_1}(s-s_1)^2 + \frac{s_1(a\omega_1+\gamma)+c\omega_1}{b\omega_1}(s-s_1)+t_1\right\}$$

we construct a trajectory from $\mathbf{v}_0$ to a point in this region as follows:

(i) Since

$$\tau\to+\infty \qquad \Longrightarrow \qquad \varphi_1(\tau,\mathbf{v}_0,\omega_1)\to+\infty,$$

there exists $\tau_0>0$ such that the point $\varphi(\tau_0,\mathbf{v}_0,\omega_1):=\tilde{\mathbf{v}}_0=(\tilde{s}_0,\tilde{t}_0)$ satisfies $\tilde{s}_0>0$.

(ii) Now, with control $\omega=0$ we get that

$$\varphi(\tau,\tilde{\mathbf{v}}_0,0)=(\tilde{s}_0,\tilde{s}_0\gamma\tau+\tilde{t}_0),$$

which for $\tau>0$ large enough satisfies

$$\tilde{s}_0\gamma\tau+\tilde{t}_0 > \frac{(a\omega_1+\gamma)}{2b\omega_1}(\tilde{s}_0-s_1)^2+\frac{s_1(a\omega_1+\gamma)+c\omega_1}{b\omega_1}(\tilde{s}_0-s_1)+t_1,$$

assuring the existence of $\tau_1>0$ such that $\varphi(\tau_1,\tilde{\mathbf{v}}_0,0)$ belongs to the interior of $\Gamma_{\mathbf{v}_1,\omega_1}$ as stated.

■

- **The subcase $\beta=0$ and $\lambda+\beta\neq 0$:**

In this case, $b\lambda\neq 0$ and the diffeomorphism

$$f:\mathbb{R}^2\to\mathbb{R}^2, \qquad f(s,t)=\left(\frac{s}{b},t+\frac{\gamma}{\lambda}s\right),$$

conjugates our system $\Sigma_{1,0}$ in (5.1) with $\alpha=\beta=0$ such that $\lambda+\beta\neq 0$ to the following singular system

$$\Sigma_{1,0}:\begin{cases}\dot{s}=\omega\\ \dot{t}=\lambda t+\omega(c+as)\end{cases} \tag{5.2}$$

where the LARC is equivalent to $a^2 + c^2 \neq 0$. Let us assume that $\lambda < 0$ and $a \geq 0$, since the other cases are analogous.

For each $\omega \in \Omega$, let us define the function

$$F_\omega : \mathbb{R}^2 \to \mathbb{R}, \qquad F_\omega(s,t) = \lambda^2 t + \omega(\lambda(c + as) + a\omega).$$

By a straightforward calculation, one shows that, for $\omega \in \Omega$ and $\mathbf{v} = (s,t)$, the first coordinate of the solution of $\Sigma_{1,0}$ is given by $\varphi_1(\tau, \mathbf{v}, \omega) = s + \tau\omega$, and the second one is determined by the relation

$$F_\omega\left(\varphi(\tau, \mathbf{v}, \omega)\right) = \mathrm{e}^{\lambda\tau} F_\omega(\mathbf{v}). \tag{5.3}$$

Moreover, it holds that

$$F_{\omega_1}\left(\varphi(\tau, \mathbf{v}, \omega_0)\right) - F_{\omega_2}\left(\varphi(\tau, \mathbf{v}, \omega_0)\right) = a\omega_0(\omega_1 - \omega_2)\lambda\tau + F_{\omega_1}(\mathbf{v}) - F_{\omega_2}(\mathbf{v}) \tag{5.4}$$

for any $\tau \in \mathbb{R}, \mathbf{v} \in \mathbb{R}^2$ and $\omega_0, \omega_1, \omega_2 \in \Omega$. Note that $F_\omega^{-1}(0)$ is a line on the $\mathbb{R}^2$ whose inclination, with relation to the s-axis, and intersection with the t-axis are given, respectively, by

$$-\frac{a\omega}{\lambda} \quad \text{and} \quad -\frac{\omega}{\lambda^2}(c\lambda + a\omega).$$

By relation (5.3) and the assumption that $\lambda < 0$, the line $F_\omega^{-1}(0)$ is an asymptote of the curve $\tau \mapsto \varphi(\tau, \mathbf{v}, u)$ as $\tau \to +\infty$ (see Figure 5.4).

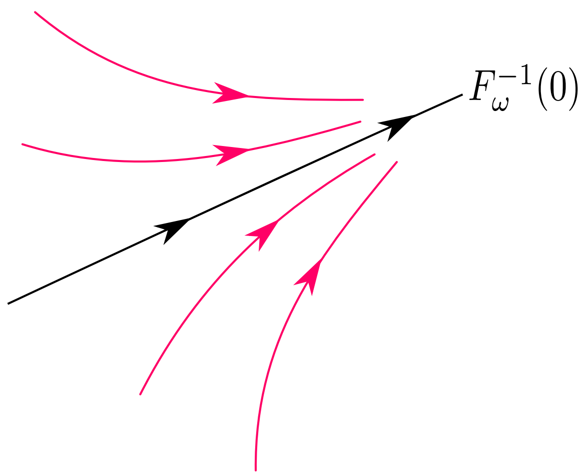


Figure 5.4 *The lines $F_\omega^{-1}(0)$ are asymptotes of the curve $\tau \mapsto \varphi(\tau, \mathbf{v}, u)$ as $\tau \to +\infty$*

Assume that $a \neq 0$, and let us define some regions with invariant properties. The first one is the region $\mathcal{C}^-$ given by

$$\mathcal{C}^- := \left\{ \mathbf{v} \in \mathbb{R}^2 \;:\; F_{\omega^-}(\mathbf{v}) < 0 \;\text{ and }\; F_{\omega^+}(\mathbf{v}) < 0 \right\}.$$

For the second one, let us consider the point $\mathbf{v}_a = (-c/a, 0)$ and define

$$\mathcal{C}^+ = \Big\{ (s,t) \in \mathbb{R}^2 \;:\; \exists \tau \geq 0 \;\text{ and }\; \omega \in \{\omega^-, \omega^+\} \;\text{ with } \\ s = \varphi_1(\tau, \mathbf{v}_a, \omega) \;\text{ and }\; t > \varphi_2(\tau, \mathbf{v}_a, \omega) \Big\}.$$

Geometrically, $\mathcal{C}^-$ is the set of points under the lines $F_{\omega^-}^{-1}(0)$ and $F_{\omega^+}^{-1}(0)$, and $\mathcal{C}^+$ the set of points over the curves

$$\tau \in (0, +\infty) \mapsto \varphi(\tau, \mathbf{v}_a, \omega^-) \quad \text{and} \quad \tau \in (0, +\infty) \mapsto \varphi(\tau, \mathbf{v}_a, \omega^+),$$

as depicted in Figure 5.5 below.

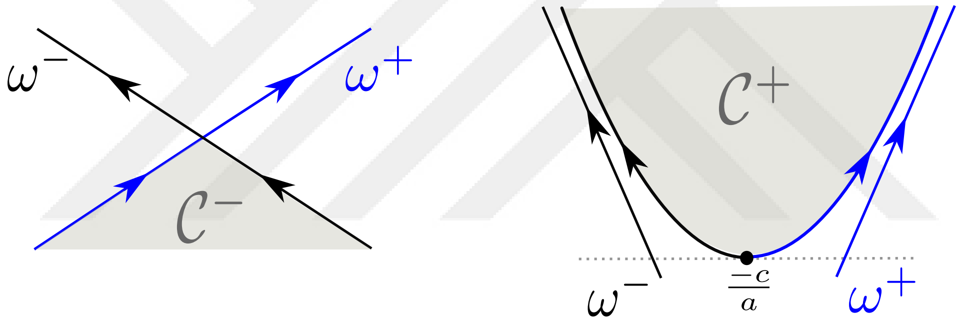


Figure 5.5 *A geometrical description of the regions $\mathcal{C}^-$ and $\mathcal{C}^+$*

The solutions starting at the point $\mathbf{v}_a$ for constant control can be explicitly calculated as follows: Since $F_\omega(\mathbf{v}_a) = a\omega^2$, relation (5.3) gives us that

$$\lambda^2 \varphi_2(\tau, \mathbf{v}_a, \omega) = -\omega(\lambda(a \underbrace{(-c/a + \tau\omega)}_{\varphi_1(\tau, \mathbf{v}_a, \omega)} + c) + a\omega) + a\omega^2 \mathrm{e}^{\lambda\tau} = a\omega^2(\mathrm{e}^{\lambda\tau} - \lambda\tau - 1),$$

implying that

$$\varphi(\tau, \mathbf{v}_a, \omega) = \left(-\frac{c}{a} + \tau\omega \,,\, \frac{a\omega^2}{\lambda^2}(\mathrm{e}^{\lambda\tau} - \lambda\tau - 1) \right). \tag{5.5}$$

The next lemma assures that $\mathcal{C}^-$ and $\mathcal{C}^+$ are invariant in negative time and also proves controllability in some regions determined by the lines $F_\omega^{-1}(0)$.

Lemma 5.1. *Assume $a > 0$. It holds:*

(1) The regions $\mathcal{C}^-$ and $\mathcal{C}^+$ are invariant in negative time.

(2) For any $\omega_1 < 0 < \omega_2$, the following region is controllable

$$\mathcal{C}(\omega_1, \omega_2) = \{\mathbf{v} \in \mathbb{R}^2 \ : \ F_{\omega_1}(\mathbf{v}) \cdot F_{\omega_2}(\mathbf{v}) < 0\}.$$

Proof. (1) Note that, for fixed $\mathbf{v} \in \mathbb{R}^2$, the map $\omega \mapsto F_\omega(\mathbf{v})$ is a polynomial with a maximum degree equal to two. Since we are assuming $a \geq 0$ we have that

$$F_{\omega^-}(\mathbf{v}) < 0 \quad \text{and} \quad F_{\omega^+}(\mathbf{v}) < 0 \quad \Longleftrightarrow \quad F_\omega(\mathbf{v}) < 0, \ \ \forall \omega \in \Omega.$$

Let $\omega, \widehat{\omega} \in \Omega$ and define the function

$$\psi : \mathbb{R} \to \mathbb{R}, \qquad \psi(\tau) = F_\omega(\varphi(\tau, \mathbf{v}, \widehat{\omega})).$$

In order to show the invariance of $\mathcal{C}^-$ in negative time, it is enough to show that $\psi(\tau) < 0$ if $\tau < 0$. However, relations (5.3) and (5.4), allow us to rewrite ψ as

$$\begin{aligned}\psi(\tau) =& F_\omega(\varphi(\tau, \mathbf{v}, \widehat{\omega})) - F_{\omega_2}(\varphi(\tau, \mathbf{v}, \widehat{\omega})) + F_{\omega_2}(\varphi(\tau, \mathbf{v}, \widehat{\omega})) \\ =& a\widehat{\omega}(\omega - \widehat{\omega})\lambda\tau + F_\omega(\mathbf{v}) + (\mathrm{e}^{\lambda\tau} - 1)F_{\widehat{\omega}}(\mathbf{v}).\end{aligned}$$

Since,

$$\widehat{\omega}(\omega - \widehat{\omega}) \leq 0 \qquad \Longrightarrow \qquad \psi(\tau) < 0 \qquad \text{for} \qquad \tau < 0,$$

we can assume, w.l.o.g., that $\widehat{\omega}(\omega - \widehat{\omega}) > 0$. On the other hand, if $a\widehat{\omega}(\omega - \widehat{\omega}) + F_{\widehat{\omega}}(\mathbf{v}) > 0$, we have that

$$a\widehat{\omega}(\omega - \widehat{\omega}) + F_{\widehat{\omega}}(\mathbf{v}) = \underbrace{\lambda^2 t}_{F_0(\mathbf{v})<0} + \widehat{\omega}(\lambda(as + c) + a\omega) \implies \widehat{\omega}(\lambda(as + c) + a\omega) > 0,$$

and hence,

$$\begin{aligned}0 > F_\omega(\mathbf{v}) &= a\widehat{\omega}(\omega - \widehat{\omega}) + F_{\widehat{\omega}}(\mathbf{v}) + (\omega - \widehat{\omega})(\lambda(as + c) + a\omega) \\ &= a\widehat{\omega}(\omega - \widehat{\omega}) + F_{\widehat{\omega}}(\mathbf{v}) + \frac{1}{\widehat{\omega}^2}\Big[\widehat{\omega}(\omega - \widehat{\omega})\Big]\Big[\widehat{\omega}(\lambda(as + c) + a\omega)\Big] > 0,\end{aligned}$$

which is absurd. Therefore, we get $a\widehat{\omega}(\omega - \widehat{\omega}) + F_{\widehat{\omega}}(\mathbf{v}) \leq 0$. In addition to that, if $\tau < 0$, we have

$$\begin{aligned}\psi(\tau) &= a\widehat{\omega}(\omega - \widehat{\omega})\lambda\tau + F_\omega(\mathbf{v}) + (\mathrm{e}^{\lambda\tau} - 1)F_{\widehat{\omega}}(\mathbf{v}) \\ &= \left(a\widehat{\omega}(\omega - \widehat{\omega}) + F_{\widehat{\omega}}(\mathbf{v})\right)\lambda\tau + F_\omega(\mathbf{v}) + (\mathrm{e}^{\lambda\tau} - \lambda\tau - 1)F_{\widehat{\omega}}(\mathbf{v}) < 0\end{aligned}$$

showing the invariance in negative time of $\mathcal{C}^-$.

Let us now show the invariance of $\mathcal{C}^+$ in negative time. Note first that, by the very definition,

$$(s,t) \in \mathcal{C}^+ \quad \Longrightarrow \quad \{(s, t+\rho) \,:\, \rho \geq 0\} \subset \mathcal{C}^+,$$

and hence,

$$\varphi(\tau, (s,t), 0) = (s, \mathrm{e}^{\lambda\tau} t) \in \{(s, t+\rho) \,:\, \rho \geq 0\} \quad \text{when} \quad \tau < 0,$$

showing that $\varphi(\tau, \mathcal{C}^+, 0) \subset \mathcal{C}^+$. On the other hand, the expression of the first coordinate of the solutions of the system together with equation (5.5) imply that, for any $\mathbf{v} \in \mathcal{C}^+$, the curve $\tau \in \mathbb{R} \mapsto \varphi(\tau, \mathbf{v}, \omega) \in \mathbb{R}^2$ intersects $-c/a \times (0, +\infty)$ at exactly one point. Since solutions of ODEs are either parallel or coincident, in order to show the invariance of $\mathcal{C}^+$, it is enough to show that the curves $\tau \in (-\infty, 0) \mapsto \varphi(\tau, \mathbf{v}_a, \omega)$ for $\omega \in \Omega$ with $\omega \neq 0$ do not leave $\mathcal{C}^+$.

Let then $\omega \in \Omega$ with $\omega \neq 0$ and $\tau_1 < 0$. There exists $\tau_2 > 0$ such that $\tau_1 \omega = \tau_2 \omega^i$, where

$$\omega^i = \begin{cases} \omega^+, & \omega < 0 \\ \omega^-, & \omega > 0 \end{cases},$$

Therefore,

$$\begin{aligned} \varphi_2(\tau_1, \mathbf{v}_a, \omega) &= \frac{a\omega^2}{\lambda^2}(\mathrm{e}^{\lambda\tau_1} - \lambda\tau_1 - 1) = \frac{a(\tau_1\omega)^2}{\lambda^2}\frac{1}{\tau_1^2}(\mathrm{e}^{\lambda\tau_1} - \lambda\tau_1 - 1) \\ &> \frac{a(\tau_2\omega^i)^2}{\lambda^2}\frac{1}{\tau_2^2}(\mathrm{e}^{\lambda\tau_2} - \lambda\tau_2 - 1) = \varphi_2(\tau_2, \mathbf{v}_a, \omega^i) \end{aligned}$$

where, for the inequality, we used that the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(\tau) = \begin{cases} \frac{1}{\tau^2}(\mathrm{e}^{\lambda\tau} - \lambda\tau - 1), & \tau \neq 0 \\ \frac{\lambda^2}{2}, & \tau = 0 \end{cases},$$

is strictly decreasing when $\lambda < 0$. As a consequence, $\varphi(\tau, \mathcal{C}^+, \omega) \subset \mathcal{C}^+$ for any $\tau < 0$, showing the result.

(2) In order to show this item, it is enough to show that for any $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ satisfying

$$F_{\omega_1}(\mathbf{v}_1) > 0, \quad F_{\omega_2}(\mathbf{v}_1) < 0 \quad \text{and} \quad F_{\omega_1}(\mathbf{v}_2) < 0, \quad F_{\omega_2}(\mathbf{v}_2) > 0,$$

there exists a closed orbit passing by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Define the curve

$$\gamma_1 : \mathbb{R} \to \mathbb{R}, \qquad \gamma_1(\tau) = F_{\omega_2}(\varphi(\tau, \mathbf{v}_1, \omega_1)).$$

As in the previous item, we have that

$$\gamma_1(\tau) = a\omega_1(\omega_2 - \omega_1)\lambda\tau + F_{\omega_2}(\mathbf{v}_1) + (\mathrm{e}^{\lambda\tau} - 1)F_{\omega_1}(\mathbf{v}_1).$$

Derivation gives us that

$$\gamma_1'(\tau) = \lambda[a\omega_1(\omega_2 - \omega_1) + \mathrm{e}^{\lambda\tau}F_{\omega_1}(\mathbf{v}_1)] \quad \text{and} \quad \gamma''(\tau) = \lambda^2\mathrm{e}^{\lambda\tau}F_{\omega_1}(\mathbf{v}_1),$$

showing that γ_1' is strictly increasing. Since, $F_{\omega_1}(\mathbf{v}_1) > 0$ and $\omega_1(\omega_2 - \omega_1) < 0$ we conclude that γ' has exactly one zero. As a consequence,

$$\lim_{\lambda\tau \to \pm\infty} \gamma(\tau) = +\infty \quad \text{and} \quad \gamma_1(0) = F_{\omega_2}(\mathbf{v}_1) < 0,$$

imply that the curve $\tau \in \mathbb{R} \mapsto \varphi(\tau, \mathbf{v}_1, \omega_1)$ crosses the line $F_{\omega_2}^{-1}(0)$ exactly two times, one in positive time and one in negative time (see Figure 5.6). A similar analysis of the curve

$$\gamma_2 : \mathbb{R} \to \mathbb{R}, \qquad \gamma_2(\tau) = F_{\omega_1}(\varphi(\tau, \mathbf{v}_2, \omega_2)),$$

allows us to conclude the same for the curve $\tau \in \mathbb{R} \mapsto \varphi(\tau, \mathbf{v}_2, \omega_2)$ and the line $F_{\omega_1}^{-1}(0)$.

Since for $i = 1, 2$ the line $F_{\omega_i}^{-1}(0)$ is an asymptote to the curve $\tau \mapsto \varphi(\tau, \mathbf{v}_i, \omega_i)$ as $\tau \to +\infty$, the previous analysis imply the existence of real numbers $\tau_1, \tau_2, \rho_1, \rho_2$ satisfying

$$\tau_2 < 0 < \tau_1 \quad \text{and} \quad \rho_1 < 0 < \rho_2$$

and

$$\varphi(\tau_1, \mathbf{v}_1, \omega_1) = \varphi(\tau_2, \mathbf{v}_2, \omega_2) \quad \text{and} \quad \varphi(\rho_1, \mathbf{v}_1, \omega_1) = \varphi(\rho_2, \mathbf{v}_2, \omega_2).$$

Therefore, the piecewise constant functions

$$\omega_{12}(\tau) = \begin{cases} \omega_1, & \tau \in [0, \tau_1] \\ \omega_2, & \tau \in (\tau_1, \tau_1 - \tau_2] \end{cases} \quad \text{and} \quad \omega_{21}(\tau) = \begin{cases} \omega_2, & \tau \in [0, \rho_2] \\ \omega_1, & \tau \in (\rho_2, \rho_2 - \rho_1] \end{cases}$$

satisfies,

$$\varphi(\tau_1 - \tau_2, \mathbf{v}_1, \omega_{12}) = \varphi(-\tau_2, \varphi(\tau_1, \mathbf{v}_1, \omega_1), \omega_2) = \mathbf{v}_2$$
$$\varphi(\rho_2 - \rho_1, \mathbf{v}_2, \omega_{21}) = \varphi(-\rho_1, \varphi(\rho_2, \mathbf{v}_2, \omega_2), \omega_1) = \mathbf{v}_1,$$

which assures the existence of the periodic orbit between $\mathbf{v}_1$ and $\mathbf{v}_2$ as stated (see Figure 5.6).

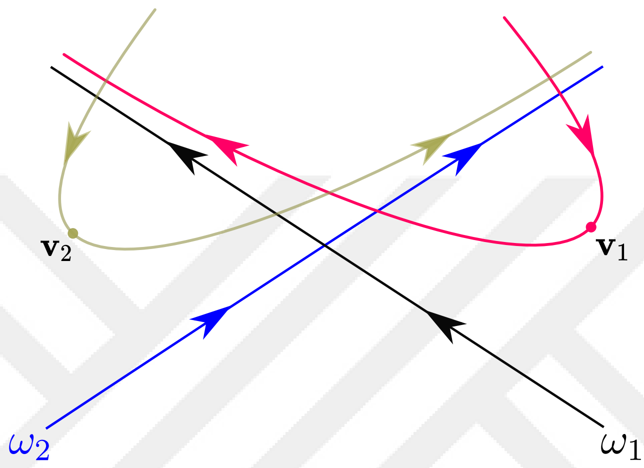


Figure 5.6 *A periodic orbit through* $\mathbf{v}_1$ *and* $\mathbf{v}_2$

■

We are now in a position to describe the only control set of $\Sigma_{1,0}$.

Theorem 5.7. *The singular control system* $\Sigma_{1,0}$ *in (5.2) admits a unique control set* $\mathcal{C}$ *whose closure satisfies:*

(1) If $a = 0$ *then* $\overline{\mathcal{C}} = \mathbb{R} \times -\frac{c}{\lambda}\Omega$.

(2) If $a > 0$ *then* $\overline{\mathcal{C}} = \mathbb{R}^2 \setminus (\mathcal{C}^+ \cup \mathcal{C}^-)$.

In both cases, $\mathcal{C}$ *is closed when* $\lambda < 0$ *and open if* $\lambda > 0$

Proof. (1) Since, by Proposition 5.1, the subset $-\frac{c}{\lambda}\Omega$ is the unique control set of the control system

$$\dot{t} = \lambda t + c\omega, \qquad \omega \in \Omega,$$

and hence it is enough to show by Proposition 3.5 that the fiber $\mathbb{R} \times \{0\}$ is controllable. For any $\mathbf{v}_0 = (s_0, 0), \mathbf{v}_1 = (s_1, 0) \in \mathbb{R} \times \{0\}$ there exists, by the continuous dependence of the initial conditions, $\omega, \omega_0, \omega_1 \in \operatorname{int}\Omega$, $\tau_0, \tau_1 > 0$ and $\tilde{s}_0, \tilde{s}_1 \in \mathbb{R}$ satisfying

$$\begin{cases} \varphi_2(\tau_0, \mathbf{v}_0, \omega_0) = \left(\tilde{s}_0, -\frac{c}{\lambda}\omega\right) \\ \varphi_2\left(\tau_1, \left(\tilde{s}_1, -\frac{c}{\lambda}\omega\right), \omega_1\right) = \mathbf{v}_1, \text{ and} \\ (\tilde{s}_1 - \tilde{s}_0)\omega > 0. \end{cases}$$

A trajectory of $\Sigma_{1,0}$ connecting $\mathbf{v}_0$ to $\mathbf{v}_1$ is constructed by concatenation as:

(i) Start at $\mathbf{v}_0$ to steer $\left(\tilde{s}_0, -\dfrac{c}{\lambda}\omega\right)$ in time $\tau_0 > 0$ with control $\omega_0 \in \Omega$;

(ii) Use the time $\tau = \frac{\tilde{s}_1 - \tilde{s}_0}{\omega} > 0$ and the control ω to move from $\left(\tilde{s}_0, -\dfrac{c}{\lambda}\omega\right)$ to

$$\varphi\left(\tau, \left(\tilde{s}_0, -\frac{c}{\lambda}\omega\right), \omega\right) = \left(\tilde{s}_0 + \tau\omega, -\frac{c}{\lambda}\omega\right) = \left(\tilde{s}_1, -\frac{c}{\lambda}\omega\right);$$

(iii) With time $\tau_1 > 0$ and control $\omega_1 \in \Omega$ go from $\left(\tilde{s}_1, -\dfrac{c}{\lambda}\omega\right)$ to $\mathbf{v}_1$.

By the arbitrariness of $\mathbf{v}_0$ and $\mathbf{v}_1$ we get the controllability of $\mathbb{R} \times \{0\}$, it proves the item (1).

(2) By Lemma 5.1, if $\lambda < 0$, the set $\mathcal{C}^+ \cup \mathcal{C}^-$ is invariant in negative time and hence, $\mathbb{R}^2 \setminus (\mathcal{C}^+ \cup \mathcal{C}^-)$ is positively invariant. Therefore, if we show that controllability holds in $\operatorname{int}(\mathbb{R}^2 \setminus (\mathcal{C}^+ \cup \mathcal{C}^-))$ the result follows.

However,

$$\operatorname{int}(\mathbb{R}^2 \setminus (\mathcal{C}^+ \cup \mathcal{C}^-)) = \mathcal{C}_1 \cap \mathcal{C}_2 \cup \mathcal{C}(\omega^-, \omega^+),$$

where

$$\mathcal{C}_1 := \Big\{(s,t) \in \mathbb{R}^2 \ : \ \exists \tau \geq 0 \text{ and } \omega \in \{\omega^-, \omega^+\} \text{ with } s = \varphi_1(\tau, \mathbf{v}_a, \omega) \text{ and } t < \varphi_2(\tau, \mathbf{v}_a, \omega)\Big\}$$

and

$$\mathcal{C}_2 := \{(s,t) \in \mathbb{R}^2 \ : \ F_{\omega^+}(s,t) \geq 0 \text{ and } F_{\omega^-}(s,t) \geq 0\}.$$

Since by Lemma 5.1 controllability holds in $\mathcal{C}(\omega^-, \omega^+)$, it is enough to show that, for any point in $\mathcal{C}_1 \cap \mathcal{C}_2$, there exists a trajectory starting and finishing in $\mathcal{C}(\omega^-, \omega^+)$.

Let us first note that the lines $F_\omega^{-1}(0)$ can be parametrized by the curve

$$s \in \mathbb{R} \mapsto \mathbf{v}_\omega(s) = \Big(s, -\frac{\omega}{\lambda^2}(\lambda(as+c)+a\omega)\Big) \in \mathbb{R}^2,$$

and hence, for $\omega_1, \omega_2 \in \Omega$, we get that

$$F_{\omega_1}(\mathbf{v}_{\omega_2}(s)) = (\omega_1 - \omega_2)(\lambda(as+c) + a(\omega_1+\omega_2)).$$

As a consequence, for any $\omega \in \Omega$, there exists $s_\omega > 0$ such that

$$\mathbf{v}_\omega(s) \in \mathcal{C}(\omega^-, \omega^+), \quad \text{for} \quad |s| \geq s_\omega.$$

Since the lines $F_\omega^{-1}(0)$ are asymptotes to the curve $\tau \in \mathbb{R} \mapsto \varphi(\tau, \mathbf{v}, \omega)$, we get that

$$\varphi(\tau_0, \mathbf{v}, \omega) \in \mathcal{C}(\omega^-, \omega^+), \ \text{ for some } \tau_0 > 0,$$

showing that we can reach $\mathcal{C}(\omega^-, \omega^+)$ from any point in $\mathcal{C}_1 \cap \mathcal{C}_2$ (see Figure 5.7).

Next, let us construct, for a given $\mathbf{v} \in \mathcal{C}_1 \cap \mathcal{C}_2$, an orbit starting in $\mathcal{C}(\omega^-, \omega^+)$ and finishing on $\mathbf{v}$. Since solutions of ODEs are parallel or coincident, there exists $\tau_1 \geq 0$ and $t < 0$ such that

$$\varphi(\tau, (-c/a, t), \omega^i) = \mathbf{v}, \quad \text{where} \quad \omega^i = \begin{cases} \omega^-, & as+c \leq 0 \\ \omega^+, & as+c > 0 \end{cases}$$

On the other hand, the fact that

$$F_\omega^{-1}(0) \cap \{s = -c/a\} = \left\{\left(-c/a, -\frac{\omega^2}{\lambda^2}\right)\right\},$$

implies the existence of $\omega_* \in \Omega$ such that

$$\mathbf{v}_1 = F_{\omega_*}^{-1}(0) \cap \varphi\left([0,\tau_1], (-c/a, t), \omega^i\right), \quad \text{with} \quad \omega_*\omega^i < 0.$$

By the previous discussion, we can take $\mathbf{v}_2 \in F_{\omega_*}^{-1}(0) \cap \mathcal{C}(\omega^-, \omega^+)$ satisfying,

$$\mathbf{v}_1 = \varphi(\tau_2, \mathbf{v}_2, \omega_*), \qquad \text{for some } \tau_2 > 0.$$

Therefore,

$$\varphi(\tau_1 - \rho, \varphi(\tau_2, \mathbf{v}_2, \omega_*), \omega^i) = \mathbf{v}, \quad \text{where} \quad \mathbf{v}_1 = \varphi(\rho, (-c/a, t), \omega^i), \quad \rho \in [0, \tau_1],$$

showing that we can reach any $\mathbf{v} \in \mathcal{C}_1 \cap \mathcal{C}_2$ from a point in $\mathcal{C}(\omega^-, \omega^+)$ (see Figure

5.7) and concluding the proof.

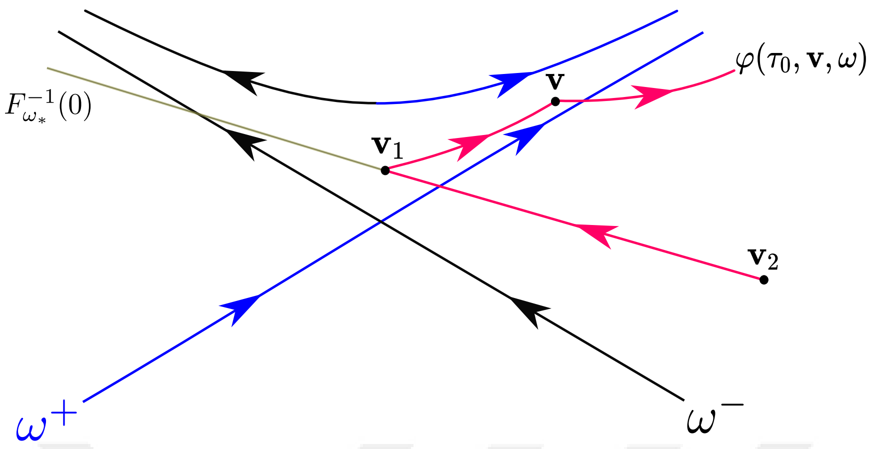


Figure 5.7 *Orbit through a point* $\mathbf{v} \in \mathcal{C}_1 \cap \mathcal{C}_2$ *starting and ending in* $\mathcal{C}(\omega^-, \omega^+)$

■

Finally, we explain the last subcase of the case $\alpha = 0$.

- **The subcase** $\beta \neq 0$ **and** $\lambda + \beta = 0$**:**

In this case, under the LARC, the diffeomorphism

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(s,t) = \left(-\frac{\beta s}{b}, -\frac{\beta}{b}s + \frac{\gamma}{b}t\right),$$

conjugates our system $\Sigma_{1,0}$ in (5.1) with $\alpha = \lambda + \beta = 0$ such that $\beta \neq 0$ to the following system

$$\Sigma_{1,0} : \begin{cases} \dot{s} = \beta(s - \omega) \\ \dot{t} = (a\omega + \gamma)s \end{cases} \tag{5.6}$$

where $\omega \in \Omega$. It satisfies LARC if and only if $a^2 + c^2 \neq 0$.
As before, let us assume that $\beta < 0$. In this case, the fact that Ω is the unique control set of the control system (see, Proposition 5.1)

$$\dot{s} = \beta(s - \omega), \qquad \omega \in \Omega,$$

implies that any control set of $\Sigma_{1,0}$ has to be inside the subset $\Omega \times \mathbb{R}$. Moreover, since the first component of the solutions of $\Sigma_{1,0}$ for constant control is given by

$$\varphi_1(\tau, \mathbf{v}_0, \omega) = \mathrm{e}^{\beta\tau}(s_0 - \omega) + \omega, \quad \mathbf{v}_0 := (s_0, t_0)$$

it holds that

$$\varphi_{\tau,\omega}(\Omega \times \mathbb{R}) \subset \Omega \times \mathbb{R}, \quad \forall \tau \geq 0.$$

Theorem 5.8. *Let $\Sigma_{1,0}$ denote the singular linear control system in (5.6). Then it holds that:*

(1) If $\gamma \neq 0$ then $\mathcal{C} = \Omega \times \mathbb{R}$ is the only control set of $\Sigma_{1,0}$.

(2) If $\gamma = 0$ then $\{(0, t)\}$ are distinct control sets of $\Sigma_{1,0}$.

Proof. (1) By Proposition 3.5, we only have to show that the fiber $\{0\} \times \mathbb{R}$ is controllable. Moreover, let us assume that $\gamma > 0$ since the other possibility is analogous. Let then $\mathbf{v}_0 = (0, t_0)$ and $\mathbf{v}_1 = (0, t_1)$ satisfy $t_0 < t_1$.

Then,

$$\tau = \frac{t_1 - t_0}{\gamma} \quad \Longrightarrow \quad \varphi(\tau, \mathbf{v}_0, 0) = (0, t_0 + \gamma\tau) = (0, t_1) = \mathbf{v}_1.$$

On the other hand, by the continuous dependence of the initial conditions, there exists $\omega \in \operatorname{int}\Omega$ such that $(a\omega + \gamma)\omega < 0$ and

$$\varphi_1(\tau_1, \mathbf{v}_1, \omega_1) = (\omega, \tilde{t}_1) \quad \text{and} \quad \varphi_1(\tau_2, (\omega, \tilde{t}_0), \omega_2) = \mathbf{v}_0, \quad \text{with} \quad \tilde{t}_0 < \tilde{t}_1,$$

for some $\omega_1, \omega_2 \in \Omega$ and $\tau_1, \tau_2 > 0$. A trajectory of $\Sigma_{1,0}$ starting in $\mathbf{v}_1$ and finishing in $\mathbf{v}_0$ is constructed by concatenation as:

(i) Start from $\mathbf{v}_1$ to reach $(0, \tilde{t}_1)$ in time $\tau_1 > 0$ with the control $\omega_1 \in \Omega$;

(ii) With time $\tau = \frac{\tilde{t}_0 - \tilde{t}_1}{(a\omega+\gamma)\omega} > 0$ and control ω go from $(0, \tilde{t}_1)$ to

$$\varphi(\tau, (0, \tilde{t}_1), \omega) = (\omega, \tilde{t}_1 + (a\omega + \gamma)\omega\tau) = (\omega, \tilde{t}_0);$$

(iii) Go from $(\omega, \tilde{t}_0)$ to $\mathbf{v}_0$ in time $\tau_2 > 0$ with control $\omega_2 \in \Omega$.

(2) If $\gamma = 0$, then $a \neq 0$, and we will assume that $a > 0$, since the other possibility

is analogous. Define the function

$$F : \Omega \times \mathbb{R} \to \mathbb{R}, \qquad F(s,t) := t + \frac{a\sigma}{\beta}s - \frac{a\sigma^2}{\beta}\ln(s+\sigma),$$

where $\sigma > 0$ satisfies $\sigma > |\omega|$ for all $\omega \in \Omega$.

Let $\omega \in \Omega$ and $\mathbf{v}_0 := (s_0, t_0) \in \operatorname{int}\Omega \times \mathbb{R}$ and write $\varphi(\tau, \mathbf{v}_0, \omega) = (s,t)$. Then,

$$\frac{d}{d\tau}F(\varphi(\tau,\mathbf{v}_0,\omega)) = \dot{t} + \frac{a\sigma}{\beta}\dot{s} - \frac{a\sigma^2}{\beta}\frac{\dot{s}}{s+\sigma} = a\omega s + \frac{a\sigma}{\beta}\beta(s-\omega) - \frac{a\sigma^2}{\beta}\frac{\beta(s-\omega)}{s+\sigma}$$

$$\frac{a\omega s(s+\sigma) + a\sigma(s-\omega)(s+\sigma) - a\sigma^2(s-\omega)}{s+\sigma} = \frac{a(\omega+\sigma)}{s+\sigma}s^2.$$

By concatenation, we easily conclude that,

$$F(\varphi(\tau,\mathbf{v}_0,\omega)) > F(\mathbf{v}_0), \quad \text{if} \quad \omega \neq 0 \quad \text{or} \quad s_0 \neq 0,$$

implying that any control set of $\Sigma_{1,0}$ has to be contained in one of the curves $F^{-1}(\cdot)$ and have the first component equal to zero. Since both properties happen only at exactly one point, we conclude that $\{(0,t)\}$ are the only control sets of $\Sigma_{1,0}$ inside $\Omega \times \mathbb{R}$, concluding the result.

■

Controllability in the case $\alpha \neq 0$

Let us now analyze the case where $\alpha \neq 0$. By Proposition 5.4, such a condition implies that $\gamma = 0$, too. Moreover, if $\lambda \neq \beta$, the map

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(s,t) = \left(s, t + \frac{\alpha}{2(\lambda-\beta)}s^2\right),$$

is a diffeomorphism that conjugates our system $\Sigma_{1,0}$ in (5.1) with $\alpha \neq 0$ such that $\lambda \neq \beta$ to the following system

$$\Sigma_{1,0} : \begin{cases} \dot{s} = \beta s + \omega b \\ \dot{t} = (\lambda+\beta)t + \omega(c + as) \end{cases}$$

that has no quadratic term. As a consequence, we only have to analyze singular LCSs where $\alpha \neq 0$ and $\lambda = \beta$, since the other possibilities were studied in the preceding sections. However, the only singular LCS where $\lambda = \beta$ and $\alpha \neq 0$ is given by

$$\Sigma_{1,0}: \begin{cases} \dot{s} = \omega b \\ \dot{t} = \alpha s^2 + \omega(c + as) \end{cases} \tag{5.7}$$

for which we have the following

Proposition 5.9. A singular linear control system $\Sigma_{1,0}$ in the equation (5.7) admits only one-point control sets given by the singularities of the drift.

Proof. The diffeomorphism

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(s,t) = \left(s, t - \frac{2c}{b}s\right),$$

conjugates $\Sigma_{1,0}$ in (5.7) to the following system

$$\Sigma_{1,0}: \begin{cases} \dot{s} = \omega b \\ \dot{t} = \alpha s^2 + 2a\omega s \end{cases} \tag{5.8}$$

Define the below function for the conjugated system in (5.8)

$$G : \mathbb{R}^2 \to \mathbb{R}, \qquad G(s,t) = 6\sigma t - 6\frac{a}{b}\sigma s^2 + 2s^3,$$

where $\sigma \in \mathbb{R}$ satisfies $\sigma\alpha + \omega b > 0$ for all $\omega \in \Omega$. Then, for any $\omega \in \Omega$ the function

$$\tau \mapsto G(\varphi(\tau, \mathbf{v}_0, \omega)), \qquad \mathbf{v}_0 = (s_0, t_0),$$

satisfies

$$\begin{aligned} \frac{d}{d\tau}G(\varphi(\tau, \mathbf{v}_0, \omega)) &= 6\sigma\dot{t} - 12\frac{a}{b}\sigma s\dot{s} + 6s^2\dot{s} \\ &= 6\left(\sigma(\alpha s^2 + 2a\omega s) - 2\frac{a}{b}\sigma\omega b s + \omega b s^2\right) = 6(\sigma\alpha + \omega b)s^2, \end{aligned}$$

which, as in Theorem 5.8 allows us to conclude that the only control sets are the singularities of the drift, that is, $\{(0,t)\}, t \in \mathbb{R}$ are the control sets of $\Sigma_{1,0}$. ■

5.2 Controllability of LCS's on the 3D Homogeneous Spaces

In this section, we show that under the LARC, a one-input LCS on $(\mathbb{T} \times \mathbb{R}) \times \mathbb{T}$ is always controllable. By Proposition 4.2, we have that a one-input linear control system $\Sigma_{0,1}$ on $L\backslash\mathbb{H} \simeq (\mathbb{T}\times\mathbb{R})\times\mathbb{T}$ has the form

$$\Sigma_{0,1} : \begin{cases} \dot{[u]} = \alpha s + \omega a \\ \dot{s} = \omega b \\ \dot{[t]} = \frac{1}{2}\alpha s^2 + \gamma s + \omega(c + as) \end{cases}$$

where $\omega \in \Omega$ with $a, b, c, \alpha, \gamma \in \mathbb{R}$ and $\alpha = 0$ if $\gamma \neq 0$.
The upcoming proposition provides a characterization of the LARC for the system $\Sigma_{0,1}$, a key factor in the handling of controllability issues.

Proposition 5.10. The one-input LCS $\Sigma_{0,1}$ on $(\mathbb{T}\times\mathbb{R})\times\mathbb{T}$ satisfies the LARC if and only if $b\alpha \neq 0$.

Proof. By calculating the brackets of

$$\widehat{\mathcal{X}}_{0,1}\Big(([u], s), [t]\Big) = \Big(\alpha s, 0, \frac{1}{2}\alpha s^2 + \gamma s\Big) \quad \text{and} \quad \widehat{B}_{0,1}\Big(([u], s), [t]\Big) = \Big(a, b, c + as\Big),$$

we obtain that

$$\Big[\widehat{B}_{0,1}, \widehat{\mathcal{X}}_{0,1}\Big] = \Big(b\alpha, 0, b(\alpha s + \gamma)\Big) \quad \text{with} \quad \Big[\widehat{B}_{0,1}, [\widehat{B}_{0,1}, \widehat{\mathcal{X}}_{0,1}]\Big] = \Big(0, 0, b\alpha\Big),$$

and any other bracket (of any length) is always zero. Therefore, taking any $\mathbf{v} \in (\mathbb{T}\times\mathbb{R})\times\mathbb{T}$, we have that

$$\begin{aligned} \operatorname{span}_{\mathcal{LA}}\{\widehat{\mathcal{X}}_{0,1}, \widehat{B}_{0,1}\}(\mathbf{v}) &= \operatorname{span}\Big\{\widehat{\mathcal{X}}_{0,1}, \widehat{B}_{0,1}, \Big[\widehat{B}_{0,1}, \widehat{\mathcal{X}}_{0,1}\Big], \Big[\widehat{B}_{0,1}, [\widehat{B}_{0,1}, \widehat{\mathcal{X}}_{0,1}]\Big]\Big\}(\mathbf{v}) \\ &= \mathbb{R}^3, \end{aligned}$$

if and only if $b\alpha \neq 0$. ■

Theorem 5.11. *Under the LARC, the one-input LCS $\Sigma_{0,1}$ is controllable.*

Proof. Let us take two points $\mathbf{v}_0 = \Big(([u_0], s_0), [t_0]\Big)$ and $\mathbf{v}_1 = \Big(([u_1], s_1), [t_1]\Big)$ in $(\mathbb{T}\times\mathbb{R})\times\mathbb{T}$ and assume that $s_1 \in \mathbb{R}$ is irrational. Since,

$$\phi_2(\tau, \mathbf{v}_0, \omega) = b\omega\tau + s_0,$$

there always exists $\omega_0 \in \Omega$ and $\tau_0 > 0$ such that

$$\phi(\tau_0, \mathbf{v}_0, \omega_0) = \Big(([\hat{u}_1], s_1), [\hat{t}_1]\Big) =: \hat{\mathbf{v}}_1.$$

On the other hand, by the LARC $\alpha \neq 0$ and $\gamma = 0$, which gives us that,

$$\phi(\tau, \hat{\mathbf{v}}_1, 0) = \Big(\big([\hat{u}_1 + \tau \cdot \alpha s_1], s_1\big), [\hat{t}_1 + \tau \cdot \alpha s_1^2]\Big).$$

Since the line

$$\tau \in (0, +\infty) \mapsto \big(\tau \cdot \alpha s_1, \tau \cdot \alpha s_1^2\big) \in \mathbb{R}^2,$$

has slope equal to s_1, which by assumption is irrational, the curve

$$\tau \in (0, +\infty) \mapsto \Big([\tau \cdot \alpha s_1], [\tau \cdot \alpha s_1^2]\Big) \in \mathbb{T} \times \mathbb{T} := \mathbb{T}^2,$$

is dense in the two dimensional torus $\mathbb{T}^2$. As a consequence, the curve

$$\tau \in (0, +\infty) \mapsto \Big([\hat{u}_1 + \tau \cdot \alpha s_1], s_1, [\hat{t}_1 + \tau \cdot \alpha s_1^2]\Big) \in (\mathbb{T} \times \{s_1\}) \times \mathbb{T},$$

is dense in $(\mathbb{T} \times \{s_1\}) \times \mathbb{T}$, and hence, there exists $\tau_k > 0$ such that

$$\phi(\tau_k, \hat{\mathbf{v}}_1, 0) \to \mathbf{v}_1 \ \text{ as } \ k \to +\infty \qquad \Longrightarrow \qquad \mathbf{v}_1 \in \overline{\mathcal{O}^+(\hat{\mathbf{v}}_1)} \subset \overline{\mathcal{O}^+(\mathbf{v}_0)},$$

where for the inclusion we used that $\hat{\mathbf{v}}_1 = \phi(\tau_1, \mathbf{v}_0, \omega_1)$.

Since the subset

$$\Big\{\big(([u], s), [t]\big) \in (\mathbb{T} \times \mathbb{R}) \times \mathbb{T} \ : \ s \in \mathbb{R} - \mathbb{Q}\Big\},$$

is dense in $(\mathbb{T} \times \mathbb{R}) \times \mathbb{T}$, we conclude that $(\mathbb{T} \times \mathbb{R}) \times \mathbb{T} \subset \overline{\mathcal{O}^+(\mathbf{v}_0)}$.

In order to finalize, let $\mathbf{v}_1 \in (\mathbb{T} \times \mathbb{R}) \times \mathbb{T}$. By the LARC, the negative orbit $\mathcal{O}^-(\mathbf{v}_1)$ has nonempty interior (see Remark 2.8). Therefore,

$$(\mathbb{T} \times \mathbb{R}) \times \mathbb{T} = \overline{\mathcal{O}^+(\mathbf{v}_0)} \qquad \Longrightarrow \qquad \mathcal{O}^-(\mathbf{v}_1) \cap \mathcal{O}^+(\mathbf{v}_0) \neq \emptyset,$$

and hence, there exists $\mathbf{v}_2 \in (\mathbb{T} \times \mathbb{R}) \times \mathbb{T}$, $\tau_1, \tau_2 > 0$, $\omega_1, \omega_2 \in \mathcal{U}$ such that

$$\mathbf{v}_1 = \phi(\tau_1, \mathbf{v}_2, \omega_1) \qquad \text{and} \qquad \mathbf{v}_2 = \phi(\tau_2, \mathbf{v}_0, \omega_2),$$

implying that $\mathbf{v}_1 = \phi\Big(\tau_1, \phi(\tau_2, \mathbf{v}_0, \omega_2), \omega_1\Big) \in \mathcal{O}^+(\mathbf{v}_0)$ and consequently we obtain $(\mathbb{T} \times \mathbb{R}) \times \mathbb{T} = \mathcal{O}^+(\mathbf{v}_0)$. Since $\mathbf{v}_0$ is arbitrary, we achieve that $\Sigma_{0,1}$ is controllable.

■

6
CONCLUSION

The thesis is a detailed study of the topic of linear control systems on homogeneous spaces of the 3D Heisenberg Lie group. The first sections establish the basic framework by providing the definition of a linear vector field and introducing essential properties and facts related to LCSs on Lie groups and their homogeneous manifolds. Importantly, the study adopts a special format, interpreting the Heisenberg group as the Cartesian product $\mathbb{R}^2 \times \mathbb{R}$ rather than the conventional group of upper triangular matrices. This choice of form allows a structured presentation of key aspects such as group multiplication, invariant and linear vector fields, and Lie brackets etc. The thesis proceeds to outline the conditions essential for projecting an LCS onto a homogeneous space, emphasizing the classification of closed subgroups, both discrete and non-normal, of $\mathbb{H}$ based on invariance criteria under the flow of a linear vector field.

Building on this foundation, the study defines and characterizes, up to equivalence, various LCSs on the homogeneous spaces of $\mathbb{H}$. Subsequently, a detailed analysis is conducted to investigate the controllability issue and the control sets for each determined dynamical system. The investigation is divided into two main parts, with the first part focusing on the classification of both normal and non-normal closed subgroups of $\mathbb{H}$. The emphasis is on studying the structures of homogeneous spaces by non-normal subgroups, and classifying all possible linear control systems on the homogeneous spaces of $\mathbb{H}$ by their closed subgroups, complementing the existing literature. The second part of the study deals with the controllability of LCSs on corresponding homogeneous spaces, such as $L\backslash\mathbb{H} \simeq \mathbb{R} \times \mathbb{T}$ and $L\backslash\mathbb{H} \simeq \mathbb{R} \times \mathbb{R}$. The complicated dynamics of these systems are carefully studied, providing a challenging yet compelling analysis. After these examinations, we focus on the controllability of a zero-dimensional (discrete) closed subgroup $L = \mathbb{Z}\mathbf{e}_1 \times \mathbb{Z}$ and its associated homogeneous space $L\backslash\mathbb{H} \simeq \mathbb{T} \times \mathbb{R} \times \mathbb{T}$.

The thesis concludes the journey through the intricacies of LCSs on homogeneous spaces of the 3D Heisenberg Lie group, revealing new insights and leading the way for future explorations in this type of dynamics. We think that this study on a special nilpotent Lie group will lay the foundation for future studies on general nilpotent Lie groups and higher dimensional Lie groups, shed light on the challenging dynamics of LCSs on homogeneous spaces and pave the way for future research in related fields.

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