Free Vibration Analysis of a Functionally Graded Beam with Finite Elements Method

Assist. Prof. C. Demir[1] and F. E. Oz[2]

[1] Yildiz Technical University, Istanbul, Turkey, e-mail: cdemir@yildiz.edu.tr

[2] Yildiz Technical University, Istanbul, Turkey, e-mail: fatihoz@yildiz.edu.tr

Abstract In this study, free vibration analysis of a functionally graded (FG) beam under various boundary conditions is carried out. Natural frequencies of the FG beam are analysed by using Finite Elements method. The system of equations of motion is derived by using Lagrange's equations with the assumption of Euler-Bernoulli beam theory. The material properties of the beam are assumed to vary through thickness according to power-law form. Different boundary conditions are attained by applying different stiffness values to the springs connected at the ends. The model is validated by comparing the results with previous studies. The effects of various material distributions and spring support values on the natural frequency parameters of the FG beam are discussed in detail.

Key words: Functionally Graded Materials, Euler-Bernoulli beam, Natural Frequency, Finite Element Method.

1 Introduction

Functionally Graded Materials (FGM) are basically inhomogeneously composite materials, in which the material and mechanical properties vary smoothly and continuously from one surface to another.

Understanding the static and dynamic behaviour of functionally graded structures such as beams, plates and shells have become very important in the last decade due to their wide range of application areas. In the literature, the number of studies on the behaviour of functionally graded (FG) beams is less than FG plates and shells. Besides, finite element modeling of FG beams is very limited. Sankar [1] investigated the elasticity solution based on the Euler-Bernoulli beam theory of FG beams subjected to sinusoidal transverse loading. Chakraborty [2] developed a new beam finite element, based on the first-order shear deformation theory (FSDBT), to study the thermoelastic behaviour of FG beams. Both the exponential and power-law material properties are examined with the variation through thickness. Aydogdu and Taskin [3] investigated the free vibration of simply supported functionally graded beams by using the Euler-Bernoulli and higher order beam theories. Simsek and Kocaturk [4] investigated the free and forced vibration of a

FG simply-supported beam subjected to a concentrated moving harmonic load. The material properties are assumed to vary through thickness according to power-law method and the equations of motion is derived by using Lagrange's method. In a recent study by Alshorbagy et al. [5] the free vibration analysis of a FG beam is investigated by using numerical finite elements method. Euler-Bernoulli beam theory is used and the equations of motion are derived by variational approach. The material properties are assumed to vary through thickness and longitudinal direction according to the power-law. The effects of different boundary conditions are investigated.

In this study, free vibration analysis of a FG beam with a slenderness ratio of 100 under various boundary conditions is carried out. Non-dimensional frequency parameters of the beam are analysed by using Finite Elements method. The system of equations of motion is derived by using Lagrange's equations under the assumption of Euler-Bernoulli beam theory. The effects of various material distribution and boundary conditions are discussed in details. The material properties of the beam are assumed to vary through thickness according to power-law form and the ratio of Young's Modulus of the constituents. Different boundary conditions are attained by applying different stiffness to the springs at the ends of the model from Free-Free (F-F) to Simply Supported (S-S). The effects of each value are presented. The model is validated by comparing the results with previous studies.

2 Theory and Formulations

2.1 Euler-Bernoulli Beam Theory

A spring supported FG beam with a length L, modulus of elasticity E(z), moment of inertia I(z) and mass density ρ(z) is considered as shown in Figure 1.

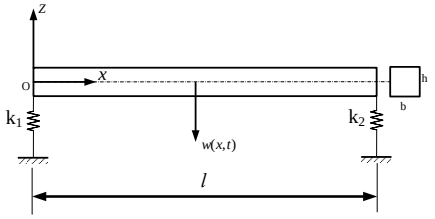


Fig. 1. Elastically Supported FG Beam

The material properties E(z), I(z), ρ(z) and ν(z) (which are represented by P(z)) of the functionally graded beam varies in the thickness direction according to the power law distribution:

$$\mathbf{P}(z) = (P_U - P_L)\left(\frac{z}{h} + \frac{1}{2}\right)^n + P_L \tag{1}$$

The spring stiffness are shown with $k_1 = k_2 = k_s$. n is the power-law exponent. P_U and P_L represent the material properties in the upper and lower surfaces respectively. The lower surface is considered as steel and the upper surface is defined according to $\mathbf{P}_{ratio} = P_U / P_L$. Axial strain of an Euler-Bernoulli beam is written as follows:

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u_0(x,t)}{\partial x} - z \cdot \frac{\partial w_0(x,t)}{\partial x^2} \tag{2}$$

By using Hooke's law, the stress is obtained as follows:

$$\sigma_x = E(z)\varepsilon_x \tag{3}$$

u and w are axial and transverse displacements respectively.

Potential and Kinetic Energies of the beam and Potential Energy of the springs are given in (4) and (5) respectively:

$$\mathbf{U} = \frac{1}{2}\int_V (\sigma_x \cdot \varepsilon_x) dV \tag{4}$$

$$\mathbf{T} = \frac{1}{2}\int_V \rho(z) \cdot \left(\dot{u}^2 + \dot{w}^2\right) dV \tag{5}$$

$\dot{u}$ and $\dot{w}$ denotes the velocities in axial and transverse directions respectively.

2.2 Finite element modeling (FEM)

In the theory of finite element modeling, generic displacements are written by using nodal displacements and shape functions according to the loading type. The displacements induced as a result of axial loading and bending can be written as follows:

$$\mathbf{u(x)} = \left[N(x)\right]_u \cdot \left\{q(t)\right\}_u \quad , \quad \mathbf{w(x)} = \left[N(x)\right]_w \cdot \left\{q(t)\right\}_w \tag{6}$$

[N(x)] and {q(t)} represents the shape functions and nodal displacements respectively. By using (6), the energy functions are written as follows:

$$\mathbf{U} = \frac{1}{2}\int_0^L \left[A_{xx}\left(\left[N'\right]_u \cdot \left\{q\right\}_u\right)^2 - 2B_{xx}\left(\left[N'\right]_u \cdot \left\{q\right\}_u\right)\left(\left[N''\right]_w \cdot \left\{q\right\}_w\right) + D_{xx}\left(\left[N''\right]_w \cdot \left\{q\right\}_w\right)^2\right] dx \tag{7}$$

$$\mathbf{T} = \frac{1}{2}\int_0^L \left[I_A\left(\left[N\right]_u \cdot \left\{\dot{q}\right\}_u\right)^2 - 2I_B\left(\left[N\right]_u \cdot \left\{\dot{q}\right\}_u\right)\left(\left[N'\right]_w \cdot \left\{\dot{q}\right\}_w\right) + I_A\left(\left[N\right]_w \cdot \left\{\dot{q}\right\}_w\right)^2 + I_D\left(\left[N'\right]_w \cdot \left\{\dot{q}\right\}_w\right)^2\right] dx \tag{8}$$

Where:

$$\left(\mathbf{A}_{xx}, \mathbf{B}_{xx}, \mathbf{D}_{xx}\right) = \int_A E(z) \cdot (1, z, z^2) dA, \quad \left(\mathbf{I}_A, \mathbf{I}_B, \mathbf{I}_D\right) = \int_A \rho(z) \cdot (1, z, z^2) dA \tag{9}$$

Superscripts of [N] in (7) and (8) mention the 1th and 2nd derivatives with respect to x. (7) and (8) yields to (10) in terms of Finite Element Modelling

$$\mathbf{U} = \frac{1}{2}\{q\}^T [K]\{q\}, \qquad \mathbf{T} = \frac{1}{2}\{\dot{q}\}^T [M]\{\dot{q}\}, \tag{10}$$

The potential energy of the springs is given in (11) as follows:

$$\mathbf{V}_S = \frac{1}{2}k_s\left[q_1(t)\right]^2 + \frac{1}{2}k_s\left[q_{p-1}(t)\right]^2 \tag{11}$$

[K] is the stiffness matrix which includes the spring effects and [M] is the mass matrix of the model. By applying Lagrange's method (12) for each degree of freedom, the problem is reduced to an eigenvalue problem and *p* represents the total number of degrees of freedom of the model.

$$\mathbf{I} = T - (U + V_S), \qquad \frac{\partial I}{\partial q_i} - \frac{d}{dt}\frac{\partial I}{\partial \dot{q}_i} = 0, \qquad i = 1, p \tag{12}$$

Nodal displacements are time-dependent functions:

$$\{q(t)\} = \{\bar{q}\}e^{i\omega t} \tag{13}$$

Thus, the equation of motion can be written as follows:

$$\left[[K] + [K_s] - \omega^2[M]\right]\{\bar{q}\}e^{i\omega t} = 0 \tag{14}$$

To validate results with previous studies the problem is reduced to non-dimensional eigenvalue problem as shown in (16) by applying non-dimensional coefficients shown in (15).

$$\kappa = \frac{k_s L^3}{EI}, \qquad \lambda^2 = \omega L^2\sqrt{\frac{\rho_L A}{E_L I}} \tag{15}$$

κ is the non-dimensional coefficient for spring stiffness which is plugged into appropriate cell in [K], $I = bh^3/12$ and λ^2 is the non-dimensional frequency of the model which is calculated with [M]. Thus the equation of motion:

$$\left[\bar{K}\right]\{\bar{q}\} - \lambda^2\left[\bar{M}\right]\{\bar{q}\} = 0 \tag{16}$$

The eigenvalues λ are found from the condition that the determinant of linear homogeneous equations given by (16) must vanish.

3 Results and Discussion

In this study, the mass ratio is considered as constant, $\rho_{ratio}=1$. Table 1 shows the variation of first non-dimensional frequencies with different E ratio and power-law exponent. It is seen that the model gives convenient results when compared with previous studies [4]. By taking the advantage of waterfall plotting in MATLAB, the variation of first three non-dimensional frequencies with respect to E_r, n and κ are presented in Figures 2 and 3. They show that the behaviour of each mode is similar. Free-Free boundary condition is represented with $\kappa=0$ and a Simply-Supported boundary condition is represented with $\kappa=\infty$ in Figures 2 and 3. As in classical solutions, it is found that at $\kappa=0$, the non-dimensional frequencies are higher than $\kappa > 0$. Also it should be noted that the homogenous beam condition is obtained when $E_r=1$. The first three non-dimensional frequencies of a free-free Euler-Bernoulli homogenous beam are shown in Table 2.

Table 1. 1st natural frequency parameters for material distribution of a simply-supported beam

	E ratio	*n=0*	*n=0.1*	*n=0.2*	*n=1*	*n=2*	*n=3*	*n=10*
Present	**0.25**	2.2214	2.3758	2.4626	2.7054	2.8071	2.8640	3.0100
[4]		2.2213	2.3752	2.4621	2.7053	2.8071	2.8639	3.0100
Present	**0.50**	2.6417	2.7121	2.7590	2.8961	2.9476	2.9764	3.0578
[4]		5.2831	5.4238	5.5176	5.7918	5.8948	5.9524	6.1152
Present	**1.00**	3.1415	3.1415	3.1415	3.1415	3.1415	3.1415	3.1415
[4]		3.1415	3.1415	3.1415	3.1415	3.1415	3.1415	3.1415
Present	**2.00**	3.7359	3.6791	3.6317	3.4440	3.3784	3.3519	3.2743
[4]		3.7359	3.6793	3.6320	3.4440	3.3784	3.3519	3.2742
Present	**3.00**	4.1345	4.0492	3.9757	3.6555	3.5332	3.4882	3.3758
[4]		4.1344	4.0495	3.9761	3.6555	3.5331	3.4881	3.3757
Present	**4.00**	4.4428	4.3388	4.2476	3.8260	3.6514	3.5887	3.4566
[4]		4.4427	4.3392	4.2481	3.8259	3.6513	3.5886	3.4565

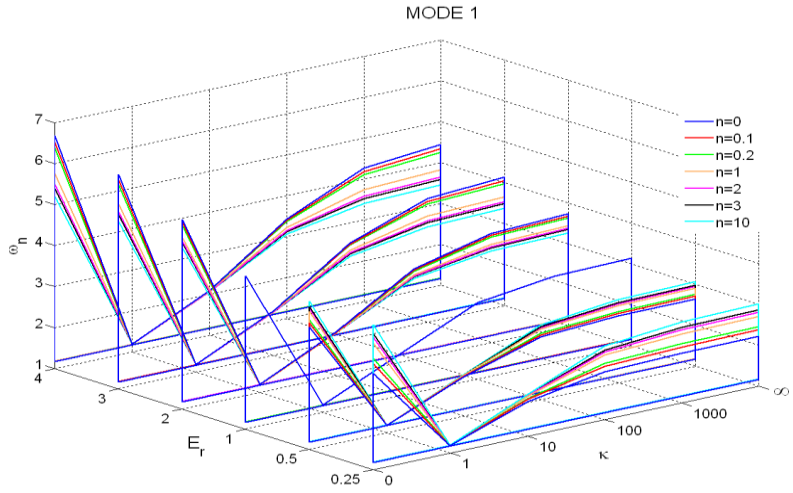


Figure 2. 1st natural frequency parameters for different material distribution and boundary conditions

Table 2. Non-dimensional frequency parameters of a free-free homogenous beam.

Classical	λ_1	λ_2	λ_3	*Present*	λ_1	λ_2	λ_3
	4.73004	7.8532	10.9956		4.7295	7.85142	10.99133

According to Table 1 and Figure 2, the first dimensionless natural frequency parameters increase while n increases from 0 to 10, in other words, while the material changes from pure alumina to steel, when E_r <1 and decrease while n increases when E_r>1. It can be observed in Table 1 and Figure 2 that the variation of E_r and κ is more effective on the frequency parameters than the variation of the n.

Figures 2 and 3 show that, the natural frequencies and the difference between frequencies of each n, increase with increasing κ when $\kappa > 1$ and decrease when $\kappa < 1$. Since rigid modes are dominant for $\kappa \leq 10$, n doesn't cause a significant

change in frequencies for these boundary conditions. This can be understood by paying attention to the differences between frequencies of each n.

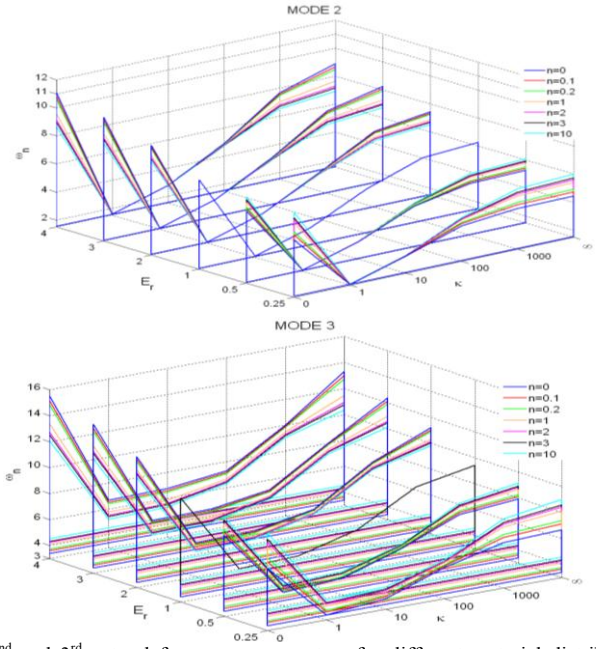


Figure 3. 2nd and 3rd natural frequency parameters for different material distribution and boundary conditions

4 Conclusions

It is observed from this study that parameters, κ , E_r and n affect the dynamic behavior of FG beam. The variation of E_r and κ is more effective on the frequency parameters than the variation of the n.

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