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Differential geometry of $\mathbf{GL}_h(1|1)$

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Abstract

We construct a right-invariant differential calculus on the quantum supergroup $\mathrm{GL}_h(1|1)$ and obtain the h-deformed superalgebra of $\mathrm{GL}_h(1|1)$.

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I. INTRODUCTION

In the last few years, the theory of quantum (super) groups like GL(2), GL(1|1), etc., were generalized in two ways. Both of the generalizations are based on the deformation of the algebra of functions on the matrix (super) groups generated by coordinate functions T^i_j which normally commute. These deformations of Lie (super) groups are algebraic structures depending on one (or more) continuous parameter. We have a standard Lie (super) group for particular values of the deformation parameters. Quantum (super) groups[1–3] present the examples of (graded) Hopf algebras. They have found application in diverse areas of physics and mathematics[4].

The q-deformation of Lie (super) groups can be realized on a quantum (super) space in which coordinates are noncommuting[2]. Recently the differential calculus on noncommutative (super) space has been intensively studied both by mathematicians and mathematical physicists. There is much activity in differential geometry on quantum groups. Throughout the recent development of differential calculus on the quantum groups two principal concepts are readily seen. First of them, formulated by Woronowicz[5], is known as bicovariant differential calculus on the quantum groups. Another concept, introduced by Woronowicz[6] and Schirmacher *et al*[7] proceeds from the requirement of a calculus only. There are many papers in this field[8]. We shall consider the second concept.

Another type of deformation, the so called h-deformation, which is a new class of quantum deformations of Lie groups and Lie algebras has recently been intensively studied[9]. This deformation may be obtained as a contraction of the q-deformation[10]. There is much interest in studies relating to various aspects of the h-deformed algebra. The differential geometry of $\mathrm{SL}_h(2)$ was given in[11]. In this work, we introduce a right-invariant differential calculus on the quantum supergroup $\mathrm{GL}_h(1|1)$. This quantum supergroup was obtained in ref. 12 using a contraction procedure given in Ref. 10.

Let us briefly discuss the content of the paper. In the second section, the basic notations of the Hopf algebra structure on the quantum supergroup $\mathrm{GL}_h(1|1)$ are introduced. In the third section we shall obtain the commutation relations for the group parameters (the matrix elements) and their differentials so we have a differential algebra. This differential algebra (extended algebra) has a Hopf algebra structure. Later, we shall construct the Cartan-Maurer one-forms and obtain the needed commutation relations. Using these commutation relations, we shall describe the quantum superalgebra for the vector fields (superalgebra generators) for $\mathrm{GL}_h(1|1)$ and derive the commutation relations between the group parameters and the algebra generators. We shall

also show that the obtained quantum superalgebra can be rederived using the partial derivatives and their relations.

II. THE ALGEBRA OF FUNCTIONS ON $\mathrm{GL}_h(1|1)$

Elementary properties of quantum supergroup $\mathrm{GL}_h(1|1)$ are described in Ref. 12. We state briefly the properties we are going to need in this work. Here we denote q-deformed objects by primed quantities. Unprimed quantities will be represent transformed coordinates. As usual, it known that even (bosonic) objects commute with everything and odd (Grassmann) objects anticommute among themselves. In this work, to obtain the quantum supergroup $\mathrm{GL}_h(1|1)$ [12], we shall only assume that odd elements β and γ anticommute with the 'new' deformation parameter h.

Let us begin with the q-deformed counterparts of GL(1|1). The quantum supergroup $\mathrm{GL}_q(1|1)$ is defined by the matrices of the form

$$T' = \begin{pmatrix} a' & \beta' \\ \gamma' & d' \end{pmatrix},$$

where the matrix entries satisfy the following commutation relations[2,3]

$$\begin{gathered} a'\beta' = q\beta' a', \qquad d'\beta' = q\beta' d', \\ a'\gamma' = q\gamma' a', \qquad d'\gamma' = q\gamma' d', \\ \beta'\gamma' + \gamma'\beta' = 0, \qquad \beta'^2 = 0 = \gamma'^2, \\ a'd' = d'a' + (q - q^{-1})\gamma'\beta'. \end{gathered} \tag{1}$$

We now consider the following similarity transformation[10]:

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = g^{-1} T' g, \tag{2}$$

where

$$g = \begin{pmatrix} 1 & 0 \\ h/(q-1) & 1 \end{pmatrix}, \qquad h^2 = 0. \tag{3}$$

Assuming β and γ that anticommute with the Grassmann number h and substituting (2) into (1), we arrive at the following relations[12]

$$\begin{gathered} a\beta = \beta a, \qquad a\gamma = \gamma a + ha^2(1 - \mathcal{D}_h^{-1}), \\ d\beta = \beta d, \qquad d\gamma = \gamma d + hd^2(\mathcal{D}_h - 1), \\ \beta^2 = 0, \qquad \gamma^2 = h\gamma d(1 - \mathcal{D}_h), \\ \beta\gamma = -\gamma\beta + h\beta d(1 - \mathcal{D}_h), \\ ad = da + h\beta d(\mathcal{D}_h - 1), \end{gathered} \tag{4}$$

where

$$\mathcal{D}_h = ad^{-1} - \beta d^{-1}\gamma d^{-1} \tag{5}$$

is the quantum superdeterminant of T. It can be checked that $\mathcal{D}_h$ commutes with all matrix elements of T. Note that, by imposing the relation

$$\mathcal{D}_h = 1$$

as an interesting case, we obtain the classical special supergroup SL(1|1), instead of $\mathrm{SL}_h(1|1)$. In other words, the restriction of the superdeterminant to unity does not give the quantum supergroup $\mathrm{SL}_h(1|1)$. It known that, in the q-deformed case, the restriction to unity ($\mathcal{D}_q = ad^{-1} - \beta d^{-1}\gamma d^{-1} = 1$) gives the quantum supergroup $\mathrm{SL}_q(1|1)$.

Let us denote the algebra generated by the elements a, β, γ, d with the relations (4) by $\mathcal{A}$. We know that the algebra $\mathcal{A}$ is a graded Hopf algebra with the following co-structures: the usual coproduct

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}, \qquad \Delta(T^i_j) = T^i_k \otimes T^k_j, \tag{6}$$

the counit

$$\varepsilon : \mathcal{A} \longrightarrow \mathcal{C}, \qquad \varepsilon(T^i_j) = \delta^i_j, \tag{7}$$

and the coinverse $S : \mathcal{A} \longrightarrow \mathcal{A}$,

$$S(T) = T^{-1} = \begin{pmatrix} a^{-1} + a^{-1}\beta d^{-1}\gamma a^{-1} & -a^{-1}\beta d^{-1} \\ -d^{-1}\gamma a^{-1} & d^{-1} + d^{-1}\gamma a^{-1}\beta d^{-1} \end{pmatrix}. \tag{8}$$

It is not difficult to verify the following properties of the co-structures:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \tag{9a}$$

$$\mu \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mu' \circ (\mathrm{id} \otimes \varepsilon) \circ \Delta, \tag{9b}$$

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \varepsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta, \tag{9c}$$

where id denotes the identity mapping, $\mu : \mathcal{C} \otimes \mathcal{A} \longrightarrow \mathcal{A}$, $\mu' : \mathcal{A} \otimes \mathcal{C} \longrightarrow \mathcal{A}$ are the canonical isomorphisms, defined by $\mu(k \otimes a) = ka = \mu'(a \otimes k)$, $\forall a \in \mathcal{A}$, $\forall k \in \mathcal{C}$ and m is the multiplication map $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$, $m(a \otimes b) = ab$.

The multiplication in $\mathcal{A} \otimes \mathcal{A}$ follows the rule

$$(A \otimes B)(C \otimes D) = (-1)^{p(B)p(C)} AC \otimes BD, \tag{10}$$

where $p(X)$ is the z_2-grade of X.

III. DIFFERENTIAL CALCULUS ON $\mathrm{GL}_h(1|1)$

In this section, we shall build up the right-invariant differential calsulus on the quantum supergroup $\mathrm{GL}_h(1|1)$. The differential calculus on the quantum

supergroups involves functions on the supergroup, differentials and differential forms.

A. Differential algebra

We first note that the properties of the exterior differential. We can introduce the exterior differential d to be an operator that is nilpotent and obeys the graded Leibniz rule:

$$\mathsf{d}^2 = 0, \tag{11a}$$

and

$$\mathsf{d}(fg) = (\mathsf{d}f)g + (-1)^{p(f)} f(\mathsf{d}g), \tag{11b}$$

where f and g are functions of the group parameters. Note that, since the deformation parameter h is an odd (Grassmann) number it must be anticommute with the exterior differential d. In fact

$$\mathsf{d}(hf) = (-1)^{p(h)} h(\mathsf{d}f) = -h(\mathsf{d}f) \implies \mathsf{d}h = -h\mathsf{d}. \tag{11c}$$

We have seen, in the previous section, that $\mathcal{A}$ is an associative algebra generated by the matrix elements of T with the relations (4). A differential algebra on $\mathcal{A}$ is a z_2-graded associative algebra Γ equipped with an operator d given in (11). Also the algebra Γ has to be generated by $\mathcal{A} \cup \mathsf{d}\mathcal{A}$.

Firstly, we wish to obtain the relations between the matrix elements of T and their differentials. To do this, we shall use the method of ref. 13. For this reason, we decompose the algebra $\mathcal{A}$ into subalgebras. We denote by $\mathcal{A}_{a'\beta'}$ the algebra generated by the elements a' and β' with the relations

$$a'\beta' = q\beta' a', \qquad \beta'^2 = 0. \tag{12}$$

Then, a possible set of commutation relations between generators of $\mathcal{A}_{a'\beta'}$ and $\mathsf{d}\mathcal{A}_{\mathsf{d}a'\mathsf{d}\beta'}$ is of the form

$$\begin{aligned} a'\ \mathsf{d}a' &= A_1 \mathsf{d}a'\ a', \\ a'\ \mathsf{d}\beta' &= F_{11}\mathsf{d}\beta'\ a' + F_{12}\mathsf{d}a'\ \beta', \\ \beta'\ \mathsf{d}a' &= F_{21}\mathsf{d}a'\ \beta' + F_{22}\mathsf{d}\beta'\ a', \\ \beta'\ \mathsf{d}\beta' &= A_2 \mathsf{d}\beta'\ \beta', \end{aligned} \tag{13}$$

where the coefficients A_i and F_{ij} are related to the complex deformation parameter q. To determine them we use the consistency of calculus (see, for details, ref. 13). Continuing in this way, we can obtain the other relations.

Let us now substitute the matrix elements of $\mathsf{d}T'$,

$$\mathsf{d}T' = \begin{pmatrix} \alpha' & b' \\ c' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha - \frac{h}{q-1} b & b \\ c + \frac{h}{q-1}(\delta - \alpha) & \delta - \frac{h}{q-1} b \end{pmatrix} \tag{14}$$

and T' into (13). After rather complicated and tedious calculations by using the consistency of calculus, as the final result one has the following commutation relations

$$
\begin{gathered}
a\alpha = \alpha a + h(\alpha\beta - ba), \qquad ab = ba - hb\beta, \\
ac = ca + h(\alpha a - c\beta + \delta a), \\
a\delta = \delta a + h(ba + \delta\beta), \\
\beta\alpha = -\alpha\beta + hb\beta, \qquad \beta b = b\beta, \\
\beta c = c\beta + h(\alpha + \delta)\beta, \qquad \beta\delta = -\delta\beta - hb\beta, \\
\gamma\alpha = -\alpha\gamma + h(\alpha a + \alpha d + b\gamma), \\
\gamma b = b\gamma + hb(a + d), \\
\gamma c = c\gamma + h(\alpha\gamma + ca + cd + \delta\gamma), \\
\gamma\delta = -\delta\gamma + h(\delta a + \delta d - b\gamma), \\
d\alpha = \alpha d - h(\alpha\beta + bd), \qquad db = bd + hb\beta, \\
dc = cd + h(\alpha d + c\beta + \delta d), \\
d\delta = \delta d + h(bd - \delta\beta).
\end{gathered}
\tag{15}
$$

It is easy to verify that the deformation parameter h anticommutes with α and δ. In fact, since $ah = ha$ we have

$$0 = \mathsf{d}(ah - ha) = \alpha h + h\alpha.$$

To find the commutation relations between differentials, we apply the exterior differential d on the relations (15) and use the nilpotency of d with (11c). Then it is easy to see that

$$
\begin{gathered}
\alpha b = b\alpha + hb^2, \qquad \alpha c = c\alpha + h(cb + \delta\alpha), \\
\delta b = b\delta - hb^2, \qquad \delta c = c\delta - h(cb - \alpha\delta), \\
\alpha^2 = h\alpha b, \qquad \alpha\delta = -\delta\alpha + h(\delta - \alpha)b, \\
\delta^2 = -h\delta b, \qquad bc = cb + h(\delta + \alpha)b.
\end{gathered}
\tag{16}
$$

These relations are the relations of $\mathrm{Gr}_h(1|1)$ in ref. 14. Note that, the central element of $\mathsf{d}\mathcal{A}$, which is generated by the elements α, b, c, δ with the relations (16), is

$$\hat{\mathcal{D}} = bc^{-1} - \alpha c^{-1}\delta c^{-1}. \tag{17}$$

However, the element $\hat{\mathcal{D}}$ also commutes with the generators of $\mathcal{A}$. So the element $\hat{\mathcal{D}}$ is the central element of the algebra $\mathcal{A}$, too. Thus $\hat{\mathcal{D}}$ is the central element of the differential algebra Γ.

An interesting note is also the following. The central element of the q-deformed differential algebra[13] is only the element $\hat{\mathcal{D}}$. However the superdeterminant of $T \in \mathrm{GL}_h(1|1)$ commutes with the generators of $\mathsf{d}\mathcal{A}$, too. So the superdeterminant $\mathcal{D}_h$ is also a central element for the h-deformed differential algebra Γ.

B. Hopf algebra structure on Γ

We first note that consistency of a differential calculus with commutation relations (4) means that the algebra Γ is a graded associative algebra generated by the elements of the set $\{a, \beta, \gamma, d, \alpha, b, c, \delta\}$. So, it is sufficent to only describe the actions of co-maps on the subset $\{\alpha, b, c, \delta\}$.

We consider a map $\phi_R : \Gamma \longrightarrow \Gamma \otimes \mathcal{A}$ such that

$$\phi_R \circ \mathsf{d} = (\mathsf{d} \otimes \mathrm{id}) \circ \Delta. \tag{18}$$

and define a map Δ_R as follows:

$$\Delta_R(u_1 \mathsf{d}v_1 + \mathsf{d}v_2 u_2) = \Delta(u_1)\phi_R(\mathsf{d}v_1) + \phi_R(\mathsf{d}v_2)\Delta(u_2). \tag{19}$$

Then it can be checked that the map Δ_R leaves invariant the relations (15) and (16). One can also check that the following identities are satisfied:

$$(\Delta_R \otimes \mathrm{id}) \circ \Delta_R = (\mathrm{id} \otimes \Delta) \circ \Delta_R \qquad (\mathrm{id} \otimes \epsilon) \circ \Delta_R = \mathrm{id}. \tag{20}$$

However, we do not have a coproduct for the differential algebra because the map ϕ_R does not gives an analog for the derivation property (11), yet. So we consider another map $\phi_L : \Gamma \longrightarrow \mathcal{A} \otimes \Gamma$ such that

$$\phi_L \circ \mathsf{d} = (\tau \otimes \mathsf{d}) \circ \Delta \tag{21}$$

and a map Δ_L with again (19) by replacing L with R. Here $\tau : \Gamma \longrightarrow \Gamma$ is the linear map of degree zero which gives $\tau(u) = (-1)^{p(u)}u$. The map Δ_L also leaves invariant the relations (15) and (16), and the following identities are satisfied:

$$(\mathrm{id} \otimes \Delta_L) \circ \Delta_L = (\Delta \otimes \mathrm{id}) \circ \Delta_L \qquad (\epsilon \otimes \mathrm{id}) \circ \Delta_L = \mathrm{id}. \tag{22}$$

To denote the coproduct, counit and coinverse which will be defined on the algebra Γ with those of $\mathcal{A}$ may be inadvisable. For this reason, we shall denote them with a different notation. Let us define the map $\hat{\Delta}$ as

$$\hat{\Delta} = \Delta_R + \Delta_L \tag{23}$$

which will allow us to define the coproduct of the differential algebra. We denote the restriction of $\hat{\Delta}$ to the algebra $\mathcal{A}$ by Δ and the extension of Δ to the differential algebra Γ by $\hat{\Delta}$. It is possible to interpret the relation

$$\hat{\Delta}|_{\mathcal{A}} = \Delta \tag{24}$$

as the definition of $\hat{\Delta}$ on the generators of $\mathcal{A}$ and (23) as the definition of $\hat{\Delta}$ on differentials. One can see that $\hat{\Delta}$ is a coproduct for the differential algebra Γ where

$$\hat{\Delta}(\mathsf{d}T^i_j) = \mathsf{d}T^i_k \otimes T^k_j + (-1)^{p(T^i_k)} T^i_k \otimes \mathsf{d}T^k_j. \tag{25}$$

It isnot difficult to verify the following conditions:

a) Γ is an $\mathcal{A}$-bimodule,

b) Γ is an $\mathcal{A}$-bicomodule with left and right coactions Δ_L and Δ_R, respectively, making Γ a left and right $\mathcal{A}$-comodule with (20) and (22), and

$$(\Delta_L \otimes \mathrm{id}) \circ \Delta_R = (\mathrm{id} \otimes \Delta_R) \circ \Delta_L \tag{26}$$

which is the $\mathcal{A}$-bimodule property. So, the triple $(\Gamma, \Delta_L, \Delta_R)$ is a bicovariant bimodule over Hopf algebra $\mathcal{A}$. In additional, since

c) (Γ, d) is a first order differential calculus over $\mathcal{A}$, and

d) d is both a left and a right comodule map, i.e. for all $u \in \mathcal{A}$

$$(\tau \otimes \mathsf{d})\Delta(u) = \Delta_L(\mathsf{d}u), \qquad (\mathsf{d} \otimes \mathrm{id})\Delta(u) = \Delta_R(\mathsf{d}u), \tag{27}$$

the quadruple $(\Gamma, d, \Delta_L, \Delta_R)$ is a first order bicovariant differential calculus over Hopf algebra $\mathcal{A}$.

Now let us return Hopf algebra structure of Γ. If we define a counit $\hat{\epsilon}$ for the differential algebra as

$$\hat{\epsilon} \circ \mathsf{d} = \mathsf{d} \circ \epsilon = 0 \tag{28}$$

and

$$\hat{\epsilon}|_{\mathcal{A}} = \epsilon, \qquad \epsilon|_{\Gamma} = \hat{\epsilon}. \tag{29}$$

we have

$$\hat{\epsilon}(\mathsf{d}T^i_j) = 0, \tag{30}$$

where

$$\hat{\epsilon}(u_1 \mathsf{d}v_1 + \mathsf{d}v_2 u_2) = \epsilon(u_1)\hat{\epsilon}(\mathsf{d}v_1) + \hat{\epsilon}(\mathsf{d}v_2)\epsilon(u_2). \tag{31}$$

Here we used the fact that $\mathsf{d}(1) = 0$.

As the next step we obtain a coinverse $\hat{S}$. For this, it suffices to define $\hat{S}$ such that

$$\hat{S} \circ \mathsf{d} = \mathsf{d} \circ S \tag{32}$$

and

$$\hat{S}|_{\mathcal{A}} = S, \qquad S|_{\Gamma} = \hat{S} \tag{33}$$

where

$$\hat{S}(u_1 \mathsf{d}v_1 + \mathsf{d}v_2 u_2) = \hat{S}(\mathsf{d}v_1)S(u_1) + S(u_2)\hat{S}(\mathsf{d}v_2). \tag{34}$$

Thus the action of $\hat{S}$ on the generators α, b, c and δ is as follows:

$$\hat{S}(\mathsf{d}T^i_j) = -(-1)^{p[(T^{-1})^i_k]}(T^{-1})^i_k \mathsf{d}T^k_l (T^{-1})^l_j. \tag{35}$$

Note that it is easy to check that $\hat{\epsilon}$ and $\hat{S}$ leave invariant the relations (15) and (16). Consequently, we can say that the structure $(\Gamma, \hat{\Delta}, \hat{\epsilon}, \hat{S})$ is a graded Hopf algebra.

C. Cartan-Maurer one-forms and their relations

To complete the differential geometric scheme we need the Cartan-Maurer one-forms. As in analogy with the right-invariant one-forms on a Lie group in classical differential geometry, one can construct the matrix valued one-form Ω where

$$\Omega = \mathsf{d}T \; T^{-1}. \tag{36}$$

So we can write the matrix elements (right one-forms) of Ω as follows

$$\begin{aligned} w_1 = \alpha A + bC, \qquad u = \alpha B + bD, \\ w_2 = \delta D + cB, \qquad v = cA + \delta C, \end{aligned} \tag{37}$$

where $T^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. We now wish to find the commutation relations of the matrix entries of T with those of Ω. So we need the commutation relations between the matrix elements of T and T^{-1}, which may be computed directly, as follows:

$$\begin{aligned} aA = Aa + h(A - D)\beta, \qquad aB = Ba, \\ aC = Ca + h(1 - \mathcal{D}_h), \qquad aD = Da, \\ \beta A = A\beta, \qquad \beta B = -B\beta, \\ \beta C = -C\beta + h(D - A)\beta, \qquad \beta D = D\beta, \\ \gamma A = A\gamma + h(1 - \mathcal{D}_h^{-1}), \\ \gamma B = -B\gamma + h(A - D)\beta, \\ \gamma C = -C\gamma, \qquad \gamma D = D\gamma + h(\mathcal{D}_h - 1), \\ dA = Ad, \qquad dC = Cd + h(\mathcal{D}_h^{-1} - 1), \\ dB = Bd, \qquad dD = Dd + h(D - A)\beta. \end{aligned} \tag{38}$$

Using these relations, we now find the commutation relations of the matrix entries of T with those of Ω :

$$
\begin{gathered}
aw_1 = w_1 a - hua, \qquad au = ua, \\
av = va + h(w_1 + w_2)a, \qquad aw_2 = w_2 a + hua, \\
\beta w_1 = -w_1 \beta + hu\beta, \qquad \beta u = u\beta, \\
\beta v = v\beta + h(w_1 + w_2)\beta, \qquad \beta w_2 = -w_2 \beta - hu\beta, \\
\gamma w_1 = -w_1 \gamma + h(2w_1 a + u\gamma), \qquad \gamma u = u\gamma + 2hua, \\
\gamma v = v\gamma + h(w_1 \gamma + 2va + w_2 \gamma), \qquad \gamma w_2 = -w_2 \gamma + h(2w_2 a - u\gamma), \\
dw_1 = w_1 d - h(2w_1 \beta + ud), \qquad du = ud + 2hu\beta, \\
dv = vd + h(w_1 d + 2v\beta + w_2 d), \qquad dw_2 = w_2 d + h(ud - 2w_2 \beta).
\end{gathered}
\tag{39}
$$

To obtain the commutation relations among the right Cartan-Maurer one-forms, we use the commutation relations of the matrix elements of T^{-1} with the differentials of the group parameters which are given in the following:

$$
\begin{gathered}
A\alpha = \alpha A + h(bA - \alpha B), \qquad Ab = bA + hbB, \\
Ac = cA - h(\alpha A + \delta A - cB), \qquad A\delta = \delta A - h(bA + \delta B), \\
B\alpha = -\alpha B - hbB, \qquad Bb = bB, \\
Bc = cB - h(\alpha + \delta)B, \qquad B\delta = -\delta B + hbB, \\
C\alpha = -\alpha C - h(\alpha A + \alpha D + bC), \qquad Cb = bC - hb(A + D), \\
Cc = cC - h(\alpha C + cA + cD + \delta C), \\
C\delta = -\delta C + h(bC - \delta A - \delta D), \\
D\alpha = \alpha D + h(\alpha B + bD), \qquad Db = bD - hbB, \\
Dc = cD - h(\alpha D + cB + \delta D), \qquad D\delta = \delta D + h(\delta B - bD).
\end{gathered}
\tag{40}
$$

Using these relations, we obtain the commutation relations of the right Cartan-Maurer forms with the differentials of the matrix elements of T as follows:

$$
\begin{gathered}
w_1 \alpha = -\alpha w_1 - h\alpha u, \qquad w_1 b = bw_1 - hbu, \\
w_1 c = cw_1 - hcu, \qquad w_1 \delta = -\delta w_1 - h\delta u, \\
u\alpha = \alpha u, \qquad ub = bu, \\
uc = cu, \qquad u\delta = \delta u, \\
v\alpha = \alpha v + h\alpha(w_1 - w_2), \qquad vb = bv - hb(w_1 - w_2),
\end{gathered}
\tag{41}
$$

$$vc = cv - hc(w_1 - w_2), \qquad v\delta = \delta v + h\delta(w_1 - w_2),$$
$$w_2\alpha = -\alpha w_2 - h\alpha u, \qquad w_2 b = bw_2 - hbu,$$
$$w_2 c = cw_2 - hcu, \qquad w_2\delta = -\delta w_2 - h\delta u.$$

We now obtain the commutation relations of the right Cartan-Maurer forms

$$w_1 u = uw_1 - 2hu^2, \qquad w_2 u = uw_2,$$
$$w_1 v = vw_1 + 2h(w_1 w_2 - uv), \qquad w_2 v = vw_2,$$
$$w_1 w_2 = -w_2 w_1 - 2huw_2,$$
$$w_1^2 = -2huw_1, \qquad w_2^2 = 0, \tag{42}$$
$$uv = vu - 2huw_2.$$

It can be checked that the elements $\mathcal{D}_h$ and $\hat{\mathcal{D}}$ commute with the Cartan-Maurer one-forms, i.e, both of the $\mathcal{D}_h$ and $\hat{\mathcal{D}}$ are still central elements. In the q-deformation, $\mathcal{D}_q$ does not commute with the Cartan Maurer forms.

Of course, the relations (4), (15), (16) and (38)-(42) can be obtained with the help of a matrix R that acts on the square tensor space of the supergroup. The matrix R is a solution of the quantum supergroup equation. The quantum supergroup relations (4) follows from the equation

$$RT_1T_2 = T_2T_1R,$$

where, in usual gradin tensor notation, $T_1 = T \otimes I$ and $T_2 = I \otimes T$ and

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -h & 1 & 0 & 0 \\ h & 0 & 1 & 0 \\ 0 & h & h & 1 \end{pmatrix}.$$

The relations (15) are equivalent to the equation

$$T_1'\hat{T}_2 = R^{-1}\hat{T}_2T_1R,$$

where

$$T_1' = (-1)^{p(T_1)}T_1, \qquad \hat{T}_2 = \mathsf{d}T_2.$$

Applying the exterior differential d on both sides of the above equation, one has

$$(\hat{T}_1)'\hat{T}_2 = R\hat{T}_2'\hat{T}_1R, \qquad (\hat{T}_1)' = \mathsf{d}(T_1'),$$

which is equivalent to the relations (16). Similarly, the relations (39), (41) and (42 can be written, in a compact form, as follows, respectively:

$$T_1'\Omega_2 = R^{-1}\Omega_2RT_1,$$

$$\hat{T}_1'\Omega_2 = R\Omega_2' R\hat{T}_1,$$
$$\Omega_1' R^{-1}\Omega_2 R = -R\Omega_2' R\Omega_1.$$

Note that one can check that the action of d on (39), (41) and also (42) is consistent. These relations allow us to evaluate the superalgebra of $\mathrm{GL}_h(1|1)$ by relating the generators of the superalgebra to the right one-forms.

IV. QUANTUM SUPERALGEBRA

The commutation relations of Cartan-Maurer forms allow us to construct the algebra of the generators. To obtain the quantum superalgebra of the algebra generators we first write the Cartan-Maurer forms as

$$\alpha = w_1 a + u\gamma, \qquad b = w_1\beta + ud,$$
$$c = w_2 d + v\beta, \qquad \delta = w_2\gamma + va. \tag{43}$$

The differential d can then the expressed in the form

$$\mathsf{d} = w_1 T_1 + w_2 T_2 + u\nabla_+ + v\nabla_-. \tag{44}$$

Here T_1, T_2 and $\nabla_\pm$ are the quantum algebra generators. We now shall obtain the commutation relations of these generators. Considering an arbitrary function f of the matrix elements of T and using the nilpotency of the exterior differential d one has

$$(\mathsf{d}w_i)T_i f + (\mathsf{d}u_i)\nabla_i f = w_i \mathsf{d}T_i f - u_i \mathsf{d}\nabla_i f, \tag{45}$$

where

$$w_i \in \{w_1, w_2\}, \qquad u_i \in \{u, v\}, \qquad \nabla_i \in \{\nabla_+, \nabla_-\}.$$

So we need the four two-forms. To obtain these, using the nilpotency of the differential d, we can write $\mathsf{d}\Omega$ of the form

$$\mathsf{d}\Omega = \sigma_3\Omega\sigma_3\Omega, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{46}$$

In terms of the two-forms, these become

$$\mathsf{d}w_1 = w_1^2 - uv, \qquad \mathsf{d}u = w_1 u - uw_2,$$
$$\mathsf{d}w_2 = w_2^2 - vu, \qquad \mathsf{d}v = w_2 v - vw_1. \tag{47}$$

Using the Cartan-Maurer equations we find the following commutation relations for the quantum superalgebra:

$$T_1 T_2 - T_2 T_1 = 2h\nabla_- T_1,$$

$$
\begin{gathered}
T_1\nabla_+ - \nabla_+ T_1 = -\nabla_+ + 2h(T_1^2 - T_1), \\
T_2\nabla_+ - \nabla_+ T_2 = \nabla_+ - 2h(T_2T_1 + T_2 - \nabla_+\nabla_-), \\
T_1\nabla_- - \nabla_- T_1 = \nabla_-, \\
T_2\nabla_- - \nabla_- T_2 = -\nabla_-, \\
\nabla_+^2 = -2hT_1\nabla_+, \qquad \nabla_-^2 = 0, \\
\nabla_-\nabla_+ + \nabla_+\nabla_- = T_1 + T_2 - 2h\nabla_- T_1.
\end{gathered}
\tag{48}
$$

The commutation relations (48) of the algebra generators should be consistent with monomials of the matrix elements of T. To do this, we evaluate the commutation relations between the generators of algebra and the matrix elements of T. The commuation relations of the generators with the matrix elements can be extracted from the Leibniz rule:

$$
\mathsf{d}(af) = (\mathsf{d}a)f + a(\mathsf{d}f) \Longrightarrow (w_iT_i + u_i\nabla_i)a = \mathsf{d}a + a(w_iT_i + u_i\nabla_i), \tag{49}
$$

etc. This yields

$$
\begin{gathered}
T_1 a = a + aT_1 - ha\nabla_-, \\
T_1\beta = \beta + \beta T_1 + h\beta\nabla_-, \\
T_1\gamma = \gamma T_1 + h(2aT_1 + \gamma\nabla_-), \\
T_1 d = dT_1 + h(2\beta T_1 - d\nabla_-), \\
T_2 a = aT_2 - ha\nabla_-, \\
T_2\beta = \beta T_2 + h\beta\nabla_-, \\
T_2\gamma = \gamma + \gamma T_2 + h(2aT_2 + \gamma\nabla_-), \\
T_2 d = d + dT_2 + h(2\beta T_2 - d\nabla_-), \\
\nabla_+ a = \gamma + a\nabla_+ - ha(T_1 - T_2), \\
\nabla_+\beta = d - \beta\nabla_+ - h\beta(T_1 - T_2), \\
\nabla_+\gamma = -\gamma\nabla_+ - h(2a\nabla_- + \gamma T_1 - \gamma T_2), \\
\nabla_+ d = d\nabla_+ + h(2\beta\nabla_+ - dT_1 + dT_2), \\
\nabla_- a = a\nabla_-, \qquad \nabla_-\beta = -\beta\nabla_-, \\
\nabla_-\gamma = a - \gamma\nabla_- - 2ha\nabla_-, \\
\nabla_- d = \beta + d\nabla_- + 2h\beta\nabla_-.
\end{gathered}
\tag{50}
$$

V. CONCLUSION

To conclude, we introduce here commutation relations between the group parameters and their partial derivatives and thus illustrate the connection between the relations in sec. IV, and the relations which will be now obtained.

To proceed, let us first obtain the relations of the group parameters with their partial derivatives. We know that the right exterior differential d can be expressed in the form

$$\mathsf{d}f = (\alpha\partial_a + b\partial_\beta + c\partial_\gamma + \delta\partial_d)f. \tag{51}$$

Then, replacing f with af, etc. we obtain the following commutation relations

$$\begin{gathered}
\partial_a a = 1 + a\partial_a - h(\beta\partial_a + a\partial_\gamma), \\
\partial_a \beta = \beta\partial_a + h\beta\partial_\gamma, \\
\partial_a \gamma = \gamma\partial_a + h(a\partial_a + d\partial_a + \gamma\partial_\gamma), \\
\partial_a d = d\partial_a + h(\beta\partial_a - d\partial_\gamma), \\
\partial_\beta a = a\partial_\beta - h(a\partial_a - a\partial_d + \beta\partial_\beta), \\
\partial_\beta \beta = 1 - \beta\partial_\beta - h\beta(\partial_a - \partial_d), \\
\partial_\beta \gamma = -\gamma\partial_\beta - h(a\partial_\beta + \gamma\partial_a - \gamma\partial_d + d\partial_\beta), \\
\partial_\beta d = d\partial_\beta + h(\beta\partial_\beta - d\partial_a + d\partial_d), \\
\partial_\gamma a = a\partial_\gamma - h\beta\partial_\gamma, \qquad \partial_\gamma \beta = -\beta\partial_\gamma, \\
\partial_\gamma \gamma = 1 - \gamma\partial_\gamma - h(a\partial_\gamma + d\partial_\gamma), \\
\partial_\gamma d = d\partial_\gamma + h\beta\partial_\gamma, \\
\partial_d a = a\partial_d - h(a\partial_\gamma + \beta\partial_d), \\
\partial_d \beta = \beta\partial_d + h\beta\partial_\gamma, \\
\partial_d \gamma = \gamma\partial_d + h(a\partial_d + \gamma\partial_\gamma + d\partial_d), \\
\partial_d d = 1 + d\partial_d + h(\beta\partial_d - d\partial_\gamma).
\end{gathered} \tag{52}$$

We thus find the commutation relations between the derivatives. These relations can be obtained by using the nilpotency of the right exterior differential d and they have the form

$$\begin{gathered}
\partial_a\partial_\beta = \partial_\beta\partial_a + h(\partial_d\partial_a - \partial_\beta\partial_\gamma - \partial_a^2), \\
\partial_d\partial_\beta = \partial_\beta\partial_d - h(\partial_a\partial_d + \partial_\beta\partial_\gamma - \partial_d^2), \\
\partial_a\partial_\gamma = \partial_\gamma\partial_a, \qquad \partial_d\partial_\gamma = \partial_\gamma\partial_d, \\
\partial_\beta\partial_\gamma = -\partial_\gamma\partial_\beta + h\partial_\gamma(\partial_a - \partial_d), \\
\partial_\beta^2 = h\partial_\beta(\partial_a - \partial_d), \qquad \partial_\gamma^2 = 0,
\end{gathered} \tag{53}$$

$$\partial_a \partial_d = \partial_d \partial_a + h\partial_\gamma(\partial_d - \partial_a).$$

The (graded) Hopf algebra structure for ∂ is given by

$$\begin{aligned}
\Delta(\partial_a) &= \partial_a \otimes \partial_a + \partial_\beta \otimes \partial_\gamma, \qquad \Delta(\partial_\beta) = \partial_a \otimes \partial_\beta + \partial_\beta \otimes \partial_d, \\
\Delta(\partial_d) &= \partial_d \otimes \partial_d + \partial_\gamma \otimes \partial_\beta, \qquad \Delta(\partial_\gamma) = \partial_\gamma \otimes \partial_a + \partial_d \otimes \partial_\gamma, \\
\varepsilon(\partial_a) &= 1 = \varepsilon(\partial_d), \qquad \varepsilon(\partial_\beta) = 0 = \varepsilon(\partial_\gamma), \\
S(\partial_a) &= \partial_a^{-1} + \partial_a^{-1}\partial_\beta\partial_d^{-1}\partial_\gamma\partial_a^{-1}, \qquad S(\partial_\beta) = -\partial_a^{-1}\partial_\beta\partial_d^{-1}, \\
S(\partial_d) &= \partial_d^{-1} + \partial_d^{-1}\partial_\gamma\partial_a^{-1}\partial_\beta\partial_d^{-1}, \qquad S(\partial_\gamma) = -\partial_d^{-1}\partial_\gamma\partial_a^{-1},
\end{aligned} \tag{54}$$

provided that the formal inverses ∂_a^{-1} and ∂_d^{-1} exist. However these co-maps do not leave invariant the relations (52).

We know, from Sec. IV, that the right exterior differential d can be expressed in the form (31), which we repeat here,

$$\mathsf{d}f = (w_1 T_1 + u\nabla_+ + v\nabla_- + w_2 T_2)f. \tag{55}$$

Considering (51) together (55) and using (43) one has

$$\begin{aligned}
T_1 &= a\partial_a + \beta\partial_\beta, \qquad \nabla_+ = \gamma\partial_a + d\partial_\beta, \\
T_2 &= d\partial_d + \gamma\partial_\gamma, \qquad \nabla_- = a\partial_\gamma + \beta\partial_d.
\end{aligned} \tag{56}$$

Using the relations (52) and (53) one can check that the relations of the generators in (56) coincide with (48). It can also be verified that, the action of the generators in (56) on the group parameters coincide with (50).

The classical limit $h \longrightarrow 0$ of the right-invariant differential calculus is the undeformed (ordinary) differential calculus.

Note that if we make the identification

$$u \longrightarrow \frac{x}{2}, \qquad w_1 \longrightarrow \theta,$$

where x and θ are the coordinates of superplane, we have

$$x\theta = \theta x + hx^2, \theta^2 = -hx\theta.$$

One of the interesting problems may be to construct linear connections[15] on the h-superplane.

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