Research Article

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On weakly classical 1-absorbing prime submodules

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Abstract: In this paper, we study weakly classical 1-absorbing prime submodules of a nonzero unital module M over a commutative ring R having a nonzero identity. A proper submodule N of M is said to be a weakly classical 1-absorbing prime submodule, if for each $m \in M$ and nonunits $a, b, c \in R$, $0 \neq abcm \in N$ implies that $abm \in N$ or $cm \in N$. We give various examples and properties of weakly classical 1-absorbing prime submodules. Also, we investiage the weakly classical 1-absorbing prime submodules of tensor product $F \otimes M$ of a (faithfully) flat R-module F and any R-module M. Also, we prove that if every proper submodule of an R-module M is weakly classical 1-absorbing prime, then $\mathrm{Jac}(R)^3 M = 0$. In terms of this result, we characterize modules over local rings in which every proper submodule is weakly classical 1-absorbing prime.

Keywords: weakly classical prime submodules; weakly classical 1-absorbing prime submodules; weakly classical 2-absorbing submodules

MSC 2020: 13A15; 13C05; 13C13

1 Introduction

In this article, all rings under consideration are assumed to be commutative with nonzero identity and all modules are assumed to be nonzero unital. Let R always denote such a ring and M denote such an R-module. A proper submodule N of M is called a *prime submodule* if whenever $xm \in N$ for some $x \in R$ and $m \in M$, then $x \in (N:_R M)$ or $m \in N$, where $(N:_R M)$ is the annihilator $\mathrm{Ann}(M/N)$ of R-module M/N, that is, $(N:_R M) = \{x \in R : xM \subseteq N\}$. In this case, $(N:_R M)$ is a prime ideal of R. However, the converse is not true in general. For instance, consider $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Q}$ and note that $N = 2\mathbb{Z} \times \mathbb{Z}$ is not a prime submodule since $2\left(0, \frac{1}{2}\right) \in N$, $2 \notin \left(N:_{\mathbb{Z}} \mathbb{Z} \oplus \mathbb{Q}\right)$ and $\left(0, \frac{1}{2}\right) \notin N$. But $\left(N:_{\mathbb{Z}} \mathbb{Z} \oplus \mathbb{Q}\right) = 0$ is a prime ideal of $\mathbb{Z}$. Several authors have extended the notion of prime ideals to modules (see, [1–4]). In 2004, Behboodi and Koohy introduced the concept of weakly prime submodules (see, [5, 6]). According to [6], a proper submodule N of M is called a *weakly prime submodule* if whenever $xym \in N$ for some $x, y \in R$ and $m \in M$, then $xm \in N$ or $ym \in N$. By definition, it is clear that every prime submodule is also weakly prime but the converse is not true in general. For instance, the submodule $N = 2\mathbb{Z} \oplus 0$ of $\mathbb{Z}$-module

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$\mathbb{Z} \oplus \mathbb{Q}$ is a weakly prime submodule which is not prime. It can be easily seen that a proper submodule N of M is weakly prime if and only if $(N:_R m) = \{x \in R: xm \in N\}$ is a prime ideal of R for every $m \notin N$. So far, the concept of weakly prime submodules has drawn attention of many authors and has been studied in many papers. For example, Azizi in his paper [7, Lemma 3.2 and Theorem 3.3] studied the weakly prime submodules of tensor product $F \otimes M$ for a flat (faithfully flat) R-module F and any R-module M. Here, we should mention that after some initial works, many authors studied the same notion under the name "classical prime submodule", and in this paper we prefer to use classical prime submodule instead of weakly prime submodule. In 2016, Mostafanasab et al. gave a new generalization of classical prime submodules which is called weakly classical prime submodule as follows: a proper submodule N of M is called a weakly classical prime submodule if whenever $0 \neq abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ [8]. It is clear that zero submodule in any module is trivially weakly classical prime. However, zero submodule need not be classical prime in general. For instance, take the zero submodule $N = (\bar{0})$ of $\mathbb{Z}$-module $\mathbb{Z}_9$ (where $\mathbb{Z}_9$ denotes the factor ring $\mathbb{Z}/9\mathbb{Z}$). Then N is weakly classical prime which is not classical prime. In a recent study, Yassine et al. gave a new generalization of prime ideal which is called 1-absorbing prime ideal and they characterized local rings in terms of 1-absorbing prime ideals. A proper ideal Q of R is said to be a 1-absorbing prime ideal if whenever $xyz \in Q$ for some nonunits $x, y, z \in R$, then $xy \in Q$ or $z \in Q$ [9]. For more on 1-absorbing prime ideals, see [10–12]. Afterwards, Koç et al. defined the concept of weakly 1-absorbing prime ideals as follows: a proper ideal Q of R is said to be a weakly 1-absorbing prime if whenever $0 \neq xyz \in Q$ for some nonunits $x, y, z \in R$, then $xy \in Q$ or $z \in Q$ [13]. Among many results in the paper [13], the authors studied weakly 1-absorbing prime ideals in the ring of real valued continuous functions $C(X)$ on a topological space X, and also characterized rings in which every proper ideal is weakly 1-absorbing prime. According to [13, Theorem 14], every proper ideal of a ring R is a weakly 1-absorbing prime if and only if (R, M) is a local ring with $M^3 = 0$ or R is a direct product of two fields. Motivated by the studies of [8, 13], in this paper we introduce and study weakly classical 1-absorbing prime submodules of modules over commutative rings. A proper submodule N of M is said to be a *weakly classical 1-absorbing prime submodule* if whenever $0 \neq xyzm \in N$ for some nonunits $x, y, z \in R$ and $m \in M$, then $xym \in N$ or $zm \in N$.

In the following table, we provide a compilation of the definitions of various families of submodules (and related ideals) that are directly related to the discussions and results developed in this work. These definitions constitute the fundamental building blocks employed throughout the paper, and their proper understanding is crucial for a clear comprehension of the subsequent analyses and proofs. Here, N is considered as a proper submodule of the R-module M, and Q be a proper ideal of the ring R.

Classical prime submodule	$abm \in N \Rightarrow am \in N$ or $bm \in N$.
Weakly classical prime submodule	$0 \neq abm \in N \Rightarrow am \in N$ or $bm \in N$.
1-absorbing prime ideal	$xyz \in Q$ with x, y, z nonunits $\Rightarrow xy \in Q$ or $z \in Q$.
Weakly 1-absorbing prime ideal	$0 \neq xyz \in Q$ with x, y, z nonunits $\Rightarrow xy \in Q$ or $z \in Q$.
Weakly classical 1-absorbing prime submodule	$0 \neq xyzm \in N$ with x, y, z nonunits $\Rightarrow xym \in N$ or $zm \in N$.

For the sake of completeness, now we will give some notions and notations which will be followed in the sequel. Let R be a ring and M be an R-module. For every element $x \in R$ ($m \in M$), principal ideal (cyclic submodule) generated by $x \in R$ ($m \in M$) is denoted by Rx (Rm). Also, for any subset K of M and any submodule N of M, the residual of N by K will be designated by $(N:_R K) = \{x \in R: xK \subseteq N\}$. In particular, if $N = (0)$ is the zero submodule and $K = \{m\}$ is a singleton for some $m \in M$, we prefer to use $\mathrm{Ann}_R(m)$ instead of $((0):_R K)$. Also, we prefer to use $\mathrm{Ann}_R(M)$ to denote the annihilator of M instead of $((0):_R M)$ [14]. Recall from [14] that M is said to be a *faithful module* if $\mathrm{Ann}_R(M)$ is the zero ideal of R. Recall from [15] that an R-module M is said to be a *multiplication module* if every submodule N of M has the form IM for some ideal I of R. In this case, $N = (N:_R M)M$. Ameri in his paper [16] defined product of two submodules of multiplication modules as follows: Let M be a multiplication module and $N = IM, K = JM$ be two submodules of M. Then the product of N and K is defined by $NK = IJM$. This product is independent of the represetantions of N and K (see, [16, Theorem 3.4]). In particular, we use Nm to denote the product of N and Rm for every submodule N of M and $m \in M$. Also for any submodule N of M and any subset J of R, the residual of N by J is defined by $(N:_M J) = \{m \in M: Jm \subseteq N\}$.

Among other results in this paper, we investigate the relations between weakly classical 1-absorbing prime submodules and weakly 1-absorbing prime ideals (see, Theorems 1, 2 and Corollary 1). We investigate the stability of weakly classical 1-absorbing prime submodules under homomorphism, in localization and in factor modules (see, Theorems 3, 4 and Corollaries 2, 3). Recall from that [17] that a proper submodule N of M is said to be a weakly 1-absorbing prime submodule if whenever $0 \neq abm \in N$ for some nonunits $a, b \in R$ and $m \in M$, then $ab \in (N:_R M)$ or $m \in N$. Recall from [18] that a proper submodule N of M is said to be a weakly semiprime if for each $a \in R$ and $m \in M$, $0 \neq a^2m \in N$ implies that $am \in N$. An R-module M is said to be a reduced module if whenever $a^2m = 0$ for some $a \in R$ and $m \in M$, then $am = 0$ [19]. In Proposition 1, we investigate the relations between weakly classical 1-absorbing prime submodules and other classical submodules such as weakly 1-absorbing prime submodules, weakly classical prime submodules and weakly semiprime submodules. Also, we prove that in any cyclic module, weakly 1-absorbing prime submodules and weakly classical 1-absorbing prime submodules are equal (see, Proposition 2). We give two very long lists whose all items are characterizing weakly classical 1-absorbing prime submodules (see, Theorems 6 and 9). Also, we determine weakly classical 1-absorbing prime submodules of multiplication modules (Theorem 10). In algebra, Prime avoidance lemma states that if an ideal is contained in a finite union of prime ideals, then it must be contained in one of those primes. If we replace prime ideals with any ideals, then the claim is not true in general. For instance, consider $R = \mathbb{Z}_2[x, y, z]/(x, y, z)^2$ and $I = \{\overline{0}, \overline{x}, \overline{y}, \overline{x+y}\}$. Let $P = \{\overline{0}, \overline{x}\}$, $Q = \{\overline{0}, \overline{y}\}$ and $J = \{\overline{0}, \overline{x+y}\}$. Then $I \subseteq P \cup Q \cup J$ and I is not contained in one of the ideals P, Q, J of R. One can naturally ask for which rings, a similar version of prime avoidance lemma holds for any ideals: "When does $I \subseteq \bigcup_{k=1}^{n} J_k$ imply $I \subseteq J_k$ for some $1 \leq k \leq n$ ". In 1975, Quartararo and Butts investigated the algebraic properties of such rings. According to [20] a ring R is said to be a u-ring if an ideal is contained in a finite union of ideals, then it must be contained in one of those ideals. Also, R is called a um-ring if M is an R-module and $M = \bigcup_{i=1}^{n} N_i$ for some submodules N_i of M, then $M = N_i$ for some $1 \leq i \leq n$. The authors in [20] showed that Prüfer domains and Bėzout rings are examples of u-rings. We give a characterization of weakly classical 1-absorbing prime submodules of modules over um-rings (see, Theorem 9). Furthermore, we investigate the weakly classical 1-absorbing prime submodules of direct product of modules (see, Theorems 11 and 12). Recall from [21] that an R-module F is said to be a flat R-module if for each exact sequence $K \to L \to M$ of R-modules, the sequence $F \otimes K \to F \otimes L \to F \otimes M$ is also exact. Also, F is said to be a faithfully flat, the sequence $K \to L \to M$ is exact if and only if $F \otimes K \to F \otimes L \to F \otimes M$ is exact. Azizi in [7, Lemma 3.2] showed that if M is an R-module, N is a submodule of M and F is a flat R-module, then $(F \otimes N:_{F\otimes M} a) = F \otimes (N:_M a)$ for every $a \in R$. In Theorem 13, we characterize weakly classical 1-absorbing prime submodules of tensor product $F \otimes M$ of a (faithfully) flat R-module F and any R-module M. Recall from [14] that the intersection of all maximal ideals of a ring R is denoted by $\mathrm{Jac}(R)$ which is called the Jacobson radical of R. Finally, we show that for any R-module M, if every proper submodule of M is weakly classical 1-absorbing prime submodule, then $\mathrm{Jac}(R)^3M = 0$. In terms of this result, for any module M over a local ring $(R, \mathfrak{m})$, every proper submodule of M is weakly classical 1-absorbing prime if and only if $\mathfrak{m}^3M = 0$ (see, Theorem 14).

2 Characterization of weakly classical 1-absorbing prime submodules

Throghout this section, M will denote a nonzero unital module over a commutative ring R having a nonzero identity.

Definition 1. A proper submodule N of an R-module M is called a weakly classical 1-absorbing prime if whenever $0 \neq abcm \in N$ for some nonunits $a, b, c \in R$ and $m \in M$, then $abm \in N$ or $cm \in N$.

Recall from [22] that a proper submodule N of M is said to be a classical 1-absorbing prime submodule if $abcm \in N$ for some nonunits $a, b, c \in R$ and $m \in M$ implies that $abm \in N$ or $cm \in N$. By definition, it is clear that every classical 1-absorbing submodule is also weakly classical 1-absorbing submodule. However, the converse is not true in general. See the following example.

Example 1. (Weakly classical 1-absorbing prime $\not\Rightarrow$ Classical 1-absorbing prime). Let p, q, r be three distinct prime numbers and consider $\mathbb{Z}$-module $M = \mathbb{Z}_{pqr}$. Suppose that $N = (\bar{0})$. Then N is clearly a weakly classical 1-absorbing. Since $p \cdot q \cdot r \cdot \bar{1} \in N$, $p \cdot q \cdot \bar{1} \notin N$ and $r \cdot \bar{1} \notin N$, it follows that N is not classical 1-absorbing prime submodule.

The following example shows that there exist nonzero weakly classical 1-absorbing prime submodule which is not classical 1-absorbing prime.

Example 2. Consider $\mathbb{Z}_6$-module $M = \prod_{i=1}^{\infty}\mathbb{Z}_6$ and the submodule $N = \bigoplus_{i=1}^{\infty}\mathbb{Z}_6$. Clearly N consists of all sequence $x = (x_n)_{n\in\mathbb{N}}$ where $x_n \neq \bar{0}$ for finitely many $n \in \mathbb{N}$. Choose $x = (x_n)_{n\in\mathbb{N}} \in M - N$. Since all ideals of $\mathbb{Z}_6$ are $(\bar{0})$, $(\bar{2})$, $(\bar{3})$ and $\mathbb{Z}_6$, we get $(N:_R x) = (\bar{0})$, $(\bar{2})$ or $(\bar{3})$ which are weakly 1-absorbing prime ideals. Then by Theorem 1, N is clearly a weakly classical 1-absorbing prime submodule of M. Now, take $x = (x_n)_{n\in\mathbb{N}}$ as constant sequence $\bar{1}$, that is, $x_n = \bar{1}$ for all $n \in \mathbb{N}$. Then clearly $(N:_R x) = (\bar{0})$ which is not a 1-absorbing prime ideal. By [22, Theorem 2 (ix)], N is not a classical 1-absorbing prime submodule of M.

Theorem 1. *Let M be an R-module and N a proper submodule of M. If $(N:_R m)$ is a weakly 1-absorbing prime ideal of R for each $m \in M\backslash N$, then N is a weakly classical 1-absorbing prime submodule of M. The converse is true provided that* $\mathrm{Ann}_R(m) = 0$.

Proof. Let $(N:_R m)$ be a weakly 1-absorbing prime ideal of R for each $m \in M\backslash N$, and choose $0 \neq abcm \in N$ for some nonunits $a, b, c \in R$ and $m \in M$. If $m \in N$, then we are done. So, suppose that $m \in M\backslash N$. Hence $0 \neq abc \in (N:_R m)$ and by assumption, we have $ab \in (N:_R m)$ or $c \in (N:_R m)$, that is, $abm \in N$ or $cm \in N$. For the converse, suppose that N is a weakly classical 1-absorbing prime submodule of M, and choose $m \in M\backslash N$ such that $\mathrm{Ann}_R(m) = 0$. Let $0 \neq abc \in (N:_R m)$ for some nonunits $a, b, c \in R$. Then we get $0 \neq abcm \in N$. Since N is a weakly classical 1-absorbing prime submodule of M, we have either $abm \in N$ or $cm \in N$. Hence $ab \in (N:_R m)$ or $c \in (N:_R m)$. Consequently, $(N: m)$ is a weakly 1-absorbing prime ideal of R. □

In the converse of previous theorem, the condition "$\mathrm{Ann}_R(m) = 0$" is necessary. On the other words, even if N is a weakly classical 1-absorbing prime submodule of M and $m \in M\backslash N$ with $\mathrm{Ann}_R(m) \neq 0$, then $(N:_R m)$ may not be weakly 1-absorbing prime ideal of R. See, the following example.

Example 3. Let p, q be two distinct prime numbers and consider $\mathbb{Z}$-module $M = \mathbb{Z}_{p^2q^2}$. Let $N = (\bar{0})$ and $m = \bar{p}$. Then note that N is a weakly classical 1-absorbing prime submodule and $m \in M\backslash N$ with $\mathrm{Ann}_{\mathbb{Z}}(m) \neq 0$. However, $(N:_{\mathbb{Z}} M) = pq^2\mathbb{Z}$ is not a weakly 1-absorbing prime ideal of $\mathbb{Z}$.

Let M be an R-module. Recall from [14] that the set of all torsion elements is denoted by $T_R(M) = \{m \in M: \mathrm{Ann}_R(m) \neq 0\}$. It is worthwhile to note that $T_R(M)$ is not always a submodule of M. For instance, if we consider $R = \mathbb{Z} \times \mathbb{Z}$-module $M = \mathbb{Q} \times \mathbb{Q}$, then clearly $T_R(M) = (0) \times \mathbb{Q} \cup \mathbb{Q} \times (0)$ is not a submodule of M. If $T_R(M) = M$, M is said to be a torsion module. If there exists $m \in M - T_R(M)$, then we say that M is a non-torsion module.

Theorem 2. *Let M be a non-torsion module and N a proper submodule of M such that $T_R(M) \subseteq N$. Then N is a weakly classical 1-absorbing submodule of M if and only if $(N:_R m)$ if a weakly 1-absorbing prime ideal of R for each $m \in M\backslash N$.*

Proof. Assume that $T_R(M) \subseteq N$. Then note that $M - N \subseteq M - T_R(M)$ which means that $\mathrm{Ann}_R(m) = 0$ for each $m \in M - N$. The rest follows from Theorem 1. □

Corollary 1. *Let R be a ring and I a proper ideal of R. Then I is a weakly classical 1-absorbing prime submodule of R-module R if and only if I is a weakly 1-absorbing prime ideal of R.*

Proof. Let I be a weakly classical 1-absorbing prime submodule of R-module R. Then by Theorem 1, we have $(I:_R 1) = I$ is a weakly 1-absorbing prime ideal of R. Now, let I be a weakly 1-absorbing prime ideal and choose nonunits $a, b, c \in R$ and $m \in M$ such that $0 \neq abcm = ab(cm) \in I$. Since I is a weakly 1-absorbing prime ideal, we have $ab \in I$ or $cm \in I$. Which implies that $abm \in I$ or $cm \in I$. □

Theorem 3. *Let M, M' be two R-modules and $f: M \to M'$ be an R-homomorphism.*

(1) Suppose that f is a monomorphism and N' is a weakly classical 1-absorbing prime submodule of M' with $f^{-1}(N') \neq M$, then $f^{-1}(N')$ is a weakly classical 1-absorbing submodule of M.

(2) Suppose that f is an epimorphism and N is a weakly classical 1-absorbing prime submodule of M containing $\mathrm{Ker}(f)$*, then $f(N)$ is a weakly classical 1-absorbing prime submodule of M'.*

Proof. (1): Suppose that N' is a weakly classical 1-absorbing prime submodule of M' with $f^{-1}(N') \neq M$. Let $0 \neq abcm \in f^{-1}(N')$ for some nonunits $a, b, c \in R$ and $m \in M$. Since f is a monomorphism, $0 \neq f(abcm) = abcf(m) \in N'$. Since N' is a weakly classical 1-absorbing prime submodule, we have $abf(m) \in N'$ or $cf(m) \in N'$. Which implies that $f(abm) \in N'$ or $f(cm) \in N'$. Therefore, $abm \in f^{-1}(N')$ or $cm \in f^{-1}(N')$. Consequently, $f^{-1}(N')$ is a weakly classical 1-absorbing prime submodule of M.

(2): Assume that N is a weakly classical 1-absorbing prime submodule of M. Let $a, b, c \in R$ be nonunits and $m' \in M'$ be such that $0 \neq abcm' \in f(N)$. By assumption, there exists $m \in M$ such that $m' = f(m)$ and so $0 \neq abcm' = f(abcm) \in f(N)$. Since $\mathrm{Ker}(f) \subseteq N$, we have $0 \neq abcm \in N$. As N is a weakly classical 1-absorbing prime submodule of M, we have either $abm \in N$ or $cm \in N$. Hence, $f(abm) = abm' \in f(N)$ or $f(cm) = cm' \in f(N)$. Consequently, $f(N)$ is a weakly classical 1-absorbing prime submodule of M'. □

As an immediate consequence of Theorem 3 (2), we have the following explicit result.

Corollary 2. *Let M be an R-module and $L \subseteq N$ be two submodules of M. If N is a weakly classical 1-absorbing prime submodule of M, then N/L is a weakly classical 1-absorbing prime submodule of M/L.*

The converse of previous corollary need not be true. See, the following example.

Example 4. Let p be a prime number and consider $\mathbb{Z}$-module $M = \mathbb{Z}^3_{p^3}$. Let $L = \mathbb{Z}_{p^3} \times (\bar{0}) \times (\bar{0}) = N$. Then N/L is a zero submodule which is clearly weakly classical 1-absorbing prime. Since $p^3(\bar{1}, \bar{1}, \bar{1}) \in N$ and $p^2(\bar{1}, \bar{1}, \bar{1}) \notin N$, it follows that N is not a weakly classical 1-absorbing prime submodule of M.

Corollary 3. *Let K and N be two submodules of M with $K \subseteq N \subseteq M$. If K is a weakly classical 1-absorbing prime submodule of M and N/K is a weakly classical 1-absorbing prime submodule of M/K, then N is a weakly classical 1-absorbing prime submodule of M.*

Proof. Assume that K is a weakly classical 1-absorbing prime submodule of M and N/K is a weakly classical 1-absorbing prime submodule of M/K. Let $a, b, c \in R$ be nonunits and $m \in M$ with $0 \neq abcm \in N$. Assume that $abcm \in K$. Since K is a weakly classical 1-absorbing prime submodule, we get either $abm \in K \subseteq N$ or $cm \in K \subseteq N$. Now, assume that $abcm \notin K$. Then we obtain $0 \neq abc(m + K) \in N/K$. Since N/K is a weakly classical 1-absorbing prime submodule, we have $ab(m + K) \in N/K$ or $c(m + K) \in N/K$ which implies that $abm \in N$ or $cm \in N$, as needed. □

Theorem 4. *Let M be an R-module, N be a submodule of M and S be a multiplicative subset of R. If N is a weakly classical 1-absorbing prime submodule of M such that $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a weakly classical 1-absorbing prime submodule of $S^{-1}M$.*

Proof. Let N be a weakly classical 1-absorbing prime submodule of M and $S^{-1}N \neq S^{-1}M$. Suppose that $0 \neq \frac{a}{s}\frac{b}{t}\frac{c}{u}\frac{m}{v} \in S^{-1}N$ for some nonunits $\frac{a}{s}, \frac{b}{t}, \frac{c}{u} \in S^{-1}R$ and $\frac{m}{v} \in S^{-1}M$. Then there exists $w \in S$ such that $0 \neq wabcm \in N$. Since N is a weakly classical 1-absorbing prime, then we have $ab(wm) \in N$ or $c(wm) \in N$. Thus $\frac{a}{s}\frac{b}{t}\frac{m}{v} = \frac{wabm}{wstv} \in S^{-1}N$ or $\frac{c}{u}\frac{m}{v} = \frac{wcm}{wuv} \in S^{-1}N$. Therefore, $S^{-1}N$ is a weakly classical 1-absorbing prime submodule of $S^{-1}M$. □

Proposition 1. *Let N be a proper submodule of M.*

(1) If N is a weakly 1-absorbing prime submodule of M, then N is a weakly classical 1-absorbing prime submodule of M.

(2) If N is a weakly classical prime submodule of M, then N is a weakly classical 1-absorbing prime submodule and weakly semiprime submodule of M.

(3) Suppose that M is a reduced module. Then N is a weakly classical prime submodule of M if and only if N is a weakly semiprime submodule and weakly classical 1-absorbing prime submodule of M.

Proof. (1), (2): Clear.

(3) ($\Rightarrow$): follows from (2). ($\Leftarrow$): Suppose that N is a weakly classical 1-absorbing prime submodule and weakly semiprime submodule of a reduced module M. Choose $a, b \in R$ and $m \in M$ such that $0 \neq abm \in N$. Without loss of generality, we may assume that a, b are nonunits. Since M is reduced, we have $0 \neq a^2bm \in N$. As N is a weakly classical 1-absorbing prime submodule, we get either $a^2m \in N$ or $bm \in N$. If $a^2m \in N$, then $am \in N$ because N is weakly semiprime. Thus, N is weakly classical prime submodule of M. □

Proposition 2. *Let M be a cyclic R-module. Then a proper submodule N of M is a weakly 1-absorbing prime submodule if and only if it is weakly classical 1-absorbing prime submodule of M.*

Proof. ($\Rightarrow$) Follows from Proposition 1 (1). ($\Leftarrow$) Let $M = Rm$ for some $m \in M$ and N be a weakly classical 1-absorbing prime submodule of M. Suppose that $0 \neq abx \in N$ for some nonunits $a, b \in R$ and $x \in M$. If $Rx = Rm = M$, then we conclude that $ab \in (N :_R M)$. So assume that $Rx \neq M$. Then there exists nonunit $c \in R$ such that $x = cm$. Therefore $0 \neq abx = abcm \in N$. Since N is a weakly classical 1-absorbing prime submodule of M, we have either $abm \in N$ or $cm \in N$. Hence $ab \in (N :_R M)$ or $x = cm \in N$. Consequently, N is a weakly 1-absorbing prime submodule of M. □

The following definition shows that whenever N is a weakly classical 1-absorbing prime submodule of M which is not a classical 1-absorbing prime submodule, there exist some nonunits $a, b, c \in R$ and $m \in M$ such that N has an (a, b, c, m) classical 1-quadruple-zero.

Definition 2. Let N be a proper submodule of M and a, b, c be nonunits in R, $m \in M$. If N is a weakly classical 1-absorbing prime submodule and $abcm = 0$, $abm \notin N$, $cm \notin N$, then (a, b, c, m) is called a classical 1-quadruple-zero of N.

Theorem 5. *Let N be a weakly classical 1-absorbing prime submodule of M, and suppose that $abcK \subseteq N$ for some nonunits $a, b, c \in R$ and some submodule K of M. If (a, b, c, k) is not a classical 1-quadruple-zero of N for every $k \in K$, then $abK \subseteq N$ or $cK \subseteq N$.*

Proof. Suppose that (a, b, c, k) is not a classical 1-quadruple-zero of N for every $k \in K$ and $abcK \subseteq N$. Assume on the contrary that $abK \nsubseteq N$ and $cK \nsubseteq N$. Then there are $k_1, k_2 \in K$ such that $abk_1 \notin N$ and $ck_2 \notin N$. If $abck_1 \neq 0$, then we have $ck_1 \in N$, because $abk_1 \notin N$ and N is a weakly classical 1-absorbing prime. So assume that $abck_1 = 0$. Since (a, b, c, k_1) is not a classical 1-quadruple-zero of N, we conclude that $ck_1 \in N$. In both cases, we have $ck_1 \in N$. Likewise, $abk_2 \in N$. On the other hand, note that $abc(k_1 + k_2) \in N$. A similar argument above shows that $ab(k_1 + k_2) \in N$ or $c(k_1 + k_2) \in N$. As $abk_2 \in N$ and $ck_1 \in N$, we have $abk_1 \in N$ or $ck_2 \in N$ which both of them are contradictions. Hence, we obtain $abK \subseteq N$ or $cK \subseteq N$. □

Definition 3. Let N be a weakly classical 1-absorbing prime submodule of M, and suppose that $IJLK \subseteq N$ for some proper ideals I, J, L of R and some submodule K of M. We say that N is a free classical 1-quadruple-zero with respect to $IJLK$ if (a, b, c, k) is not a classical 1-quadruple-zero of N for every elements $a \in I$, $b \in J$, $c \in L$ and $k \in K$.

Remark 1. Let N be a weakly classical 1-absorbing prime submodule of M and suppose that $IJLK \subseteq N$ for some proper ideals I, J, L of R and some submodule K of M such that N is a free classical 1-quadruple-zero with respect to $IJLK$. Hence, if $a \in I$, $b \in J$, $c \in L$ and $k \in K$, then $abk \in N$ or $ck \in N$.

Corollary 4. *Let N be a weakly classical 1-absorbing prime submodule of M, and suppose that $IJLK \subseteq N$ for some proper ideals I, J, L of R and some submodule K of M. If N is a free classical 1-quadruple-zero with respect to $IJLK$, then $IJK \subseteq N$ or $LK \subseteq N$.*

Proof. Suppose that N is a free classical 1-quadruple-zero with respect to $IJLK$. Assume that $IJK \nsubseteq N$ and $LK \nsubseteq N$. Then there are $a \in I$, $b \in J$, $c \in L$ with $abK \nsubseteq N$ and $cK \nsubseteq N$. Since $abcK \subseteq N$ and N is a free classical 1-quadruple-zero with respect to $IJLK$, then Theorem 5 implies that $abK \subseteq N$ or $cK \subseteq N$, which is a contradiction. Consequently, we have $IJK \subseteq N$ or $LK \subseteq N$. □

Let M be an R-module and N be a submodule M. For every $a \in R$, $\{m \in M : am \in N\}$ is denoted by $(N:_M a)$. It is easy to see that $(N:_M a)$ is a submodule of M containing N. In the next theorem, we give a long list whose each item characterizes weakly classical 1-absorbing prime submodules.

Theorem 6. *Let M be an R-module and N be a proper submodule M. The following conditions are equivalent.*

(1) N is a weakly classical 1-absorbing prime submodule;

(2) For every nonunits $a, b, c \in R$, $(N:_M abc) = (0:_M abc) \cup (N:_M ab) \cup (N:_M c)$;

(3) For every nonunits $a, b \in R$ and $m \in M$ with $abm \notin N$; $(N:_R abm) = (0:_R abm) \cup (N:_R m)$;

(4) For every nonunits $a, b \in R$ and $m \in M$ with $abm \notin N$; $(N:_R abm) = (0:_R abm)$ or $(N:_R abm) = (N:_R m)$;

(5) For every nonunits $a, b \in R$, $m \in M$ and every proper ideal I of R with $0 \neq abIm \subseteq N$, either $abm \in N$ or $\mathrm{Im} \subseteq N$;

(6) For every nonunits $a, b \in R$, $m \in M$ and every proper ideal I of R with $0 \neq abIm \subseteq N$, either $aIm \subseteq N$ or $bm \in N$;

(7) For every proper ideal I of R, nonunit $a \in R$ and $m \in M$ with $aIm \nsubseteq N$, $(N:_R aIm) = (0:_R aIm)$ or $(N:_R aIm) = (N:_R m)$;

(8) For every proper ideals I, J, K of R and $m \in M$ with $0 \neq IJKm \subseteq N$, either $IJm \subseteq N$ or $Km \subseteq N$.

Proof. (1) ⇒ (2): Suppose that N is a weakly classical 1-absorbing prime submodule of M. Let $m \in (N:_M abc)$. Then $abcm \in N$. If $abcm = 0$, then $m \in (0:_M abc)$. Assume that $abcm \neq 0$. Since N is a weakly classical 1-absorbing prime submodule, we have $abm \in N$ or $cm \in N$. Hence $m \in (N:_M ab)$ or $m \in (N:_M c)$ and so $m \in (0:_M abc) \cup (N:_M ab) \cup (N:_M c)$. Consequently, $(N:_M abc) = (0:_M abc) \cup (N:_M ab) \cup (N:_M c)$.

(2) ⇒ (3): Let $abm \notin N$ for some nonunits $a, b \in R$ and $m \in M$. Assume that $c \in (N:_R abm)$. Then $abcm \in N$, and so $m \in (N:_M abc)$. Thus by part (2), $m \in (0:_M abc) \cup (N:_M ab) \cup (N:_M c)$. Since $abm \notin N$, then $m \notin (N:_M ab)$ which implies that $m \in (0:_M abc)$ or $m \in (N:_M c)$. Therefore $c \in (0:_R abm)$ or $c \in (N:_R m)$. Consequently, $c \in (0:_R abm) \cup (N:_R m)$. Thus $(N:_R abm) = (0:_R abm) \cup (N:_R m)$.

(3) ⇒ (4): By the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.

(4) ⇒ (5): Assume that $0 \neq abIm \subseteq N$ for some nonunits $a, b \in R$, $m \in M$ and a proper ideal I of R. Hence $I \subseteq (N:_R abm)$ and $I \nsubseteq (0:_R abm)$. If $abm \in N$, then we are done. So, assume that $abm \notin N$. Therefore by part (4), we have that $I \subseteq (N:_R m)$ which implies that $\mathrm{Im} \subseteq N$.

(5) ⇒ (6): Let $0 \neq abIm \subseteq N$ for some proper ideal I of R, nonunits $a, b \in R$ and $m \in M$. Assume that $aIm \nsubseteq N$. Then there exists $x \in I$ such that $axm \notin N$. Then note that $ax(Rb)m \subseteq N$. If $0 \neq ax(Rb)m$, then by part (5) we have $(Rb)m \subseteq N$ which completes the proof. So assume that $ax(Rb)m = 0$. Since $0 \neq abIm$, there exists $y \in I$ such that $ay(Rb)m \neq 0$. This implies that $0 \neq a(x+y)(Rb)m \subseteq N$. As $0 \neq ay(Rb)m \subseteq N$, again by part (5), we have $aym \in N$ or $(Rb)m \subseteq N$. Let $aym \in N$. Then we have $a(x+y)m \notin N$. As $0 \neq a(x+y)(Rb)m \subseteq N$, by part (5), we get $(Rb)m \subseteq N$ which implies that $bm \in N$. In the other case, we have $bm \in N$ which completes the proof.

(6) ⇒ (7): Let $aIm \nsubseteq N$ for some nonunit $a \in R$, some proper ideal I of R and $m \in M$. Choose $b \in (N:_R aIm)$. Then we have $abIm \subseteq N$. If $abIm = 0$, then we get $b \in (0:_R aIm)$. Now, assume that $abIm \neq 0$. Then by part (6), we get $bm \in N$ which implies that $b \in (N:_R m)$. Thus, we conclude that $(N:_R aIm) = (0:_R aIm) \cup (N:_R m)$. This implies that $(N:_R aIm) = (0:_R aIm)$ or $(N:_R aIm) = (N:_R m)$.

(7) ⇒ (8): Let $0 \neq IJKm \subseteq N$ for some proper ideals I, J, K of R and $m \in M$. Assume that $IJm \nsubseteq N$. Then there exists $a \in J$ such that $aIm \nsubseteq N$. Since $aIKm \subseteq N$, we have $K \subseteq (N:_R aIm) = (0:_R aIm)$ or $K \subseteq (N:_R aIm) = (N:_R m)$ by part (7). This gives $aIKm = 0$ or $Km \subseteq N$. In the second case, we are done. So we may assume that $aIKm = 0$. Since $0 \neq IJKm$, there exists $b \in J$ such that $IbKm \neq 0$. Since $K \subseteq (N:_R bIm)$, again by part (7), we have $bIm \subseteq N$ or $Km \subseteq N$. So assume that $bIm \subseteq N$. Then $(a+b)Im \nsubseteq N$ and $0 \neq (a+b)IKm \subseteq N$. Since $K \subseteq (N:_R (a+b)Im)$, by part (7), we conclude that $Km \subseteq N$.

(8) ⇒ (1): Clear. □

Theorem 7. *Let N be a weakly classical 1-absorbing prime submodule of M and suppose that (a, b, c, m) is a classical 1-quadruple-zero of N for some nonunits $a, b, c \in R$ and $m \in M$. Then*

(1) $abcN = 0 = ab(N:_R M)m$.

(2) If $a, b \notin (N:_R cm)$, then $ac(N:_R M)m = bc(N:_R M)m = a(N:_R M)^2m = b(N:_R M)^2m = c(N:_R M)^2m = 0$. In particular, $(N:_R M)^3m = 0$.

Proof. (1): Let $abcN \neq 0$. Then there exists $n \in N$ with $abcn \neq 0$. This gives $0 \neq abcn = abc(m+n) \in N$. Since N is weakly classical 1-absorbing prime, we have $ab(m+n) \in N$ or $c(m+n) \in N$. Which implies that $abm \in N$ or $cm \in N$, which is a contradiction. Thus, $abcN = 0$. Now, we will show that $ab(N:_R M)m = 0$. Choose $x \in (N:_R M)$ and assume that $abxm \neq 0$. Then we have $0 \neq abxm = ab(c+x)m \in N$. If $c+x$ is unit, we conclude that $abm \in N$ which is a contradiction. Thus $c+x$ is nonunit. As N is a weakly classical 1-absorbing prime, we have $abm \in N$ or $(c+x)m \in N$. Then we obtain $abm \in N$ or $cm \in N$, again a contradiction. Thus, $ab(N:_R M)m = 0$. (2): Let $a, b \notin (N:_R cm)$, that is, $acm \notin N$ and $bcm \notin N$. Choose $x \in (N:_R M)$ and $acxm \neq 0$. Then we have $0 \neq a(b+x)cm \in N$. Note that $b+x$ must be nonunit. As N is weakly classical 1-absorbing prime, $a(b+x)m \in N$ or $cm \in N$ which yields that $abm \in N$ or $cm \in N$. This is a contradiction. Hence, we have $ac(N:_R M)m = 0$. Likewise, we have $bc(N:_R M)m = 0$. Now, we will show that $a(N:_R M)^2m = 0$. Suppose that $a(N:_R M)^2m \neq 0$. Then there exist $x, y \in (N:_R M)$ such that $axym \neq 0$. Then by $ab(N:_R M)m = ac(N:_R M)m = 0$, we have $0 \neq axym = a(b+x)(c+y)m \in N$. Since $abm \notin N$ and $acm \notin N$, $(b+x)$ and $(c+y)$ are nonunits. As N is weakly classical 1-absorbing prime, we have $a(b+x)m \in N$ or $(c+y)m \in N$. Which implies that $abm \in N$ or $cm \in N$. This is a contradiction. Thus, we have $a(N:_R M)^2m = 0$. Similarly one can prove that $b(N:_R M)^2m = c(N:_R M)^2m = 0$. Now, we will show that $(N:_R M)^3m = 0$. Suppose to the contrary. Then there exist $x, y, z \in (N:_R M)$ such that $xyzm \neq 0$. Then we have $0 \neq (a+x)(b+y)(c+z)m = xyzm \in N$. Also, $(a+x), (b+y), (c+z)$ are nonunits. Then we conclude that $(a+x)(b+y)m \in N$ or $(c+z)m \in N$. This implies that $abm \in N$ or $cm \in N$ which is a contradiction. Hence, $(N:_R M)^3m = 0$. □

A submodule N of an R-module M is called a *nilpotent submodule* if $(N:_R M)^k N = 0$ for some positive integer k [23]. In particular, an element $m \in M$ is nilpotent in M if Rm is a nilpotent submodule of M.

Theorem 8. *If N is a weakly classical 1-absorbing prime submodule of M that is not classical 1-absorbing prime submodule, then there exists classical 1-quadruple-zero (a, b, c, m) of N. If $a, b \notin (N:_R cm)$, then $(N:_R M)^3N = 0$ and thus N is nilpotent. Furthermore, if M is a multiplication module, then $N^4 = 0$.*

Proof. Suppose that N is a weakly classical 1-absorbing prime submodule of M that is not a classical 1-absorbing prime submodule. Then there exists a classical 1-quadruple-zero (a, b, c, m) of N for some nonunits $a, b, c \in R$ and $m \in M$. Suppose that $a, b \notin (N:_R cm)$. Now, we will show that $(N:_R M)^3N = 0$. Suppose that $(N:_R M)^3N \neq 0$. Then there are $x, y, z \in (N:_R M)$ and $n \in N$ such that $xyzn \neq 0$. By Theorem 7, we have $0 \neq (a+x)(b+y)(c+z)(m+n) = xyzn \in N$. Since N is a weakly classical 1-absorbing prime submodule, we have $(a+x)(b+y)(m+n) \in N$ or $(c+z)(m+n) \in N$. Then we have $abm \in N$ or $cm \in N$ which is a contradiction.

Thus, we obtain $(N:_R M)^3N = 0$ and N is nilpotent. Now, suppose that M is a multiplication module. Then it is clear that $N^4 = (N:M)^3N = 0$. □

Proposition 3. *Let M be a multiplication module and N a proper submodule of M. The following conditions are equivalent.*

(1) N is a weakly classical 1-absorbing prime submodule of M.

(2) If $0 \neq KLPm \subseteq N$ for some proper submodules K, L, P of M and $m \in M$, then either $KLm \subseteq N$ or $Pm \subseteq N$.

Proof. (1) ⇒ (2): Let $0 \neq KLPm \subseteq N$ for some proper submodules K, L, P of M and $m \in M$. Since M is a multiplication module, there are ideals I, J, Q of R such that $K = IM, L = JM$ and $P = QM$. Since $0 \neq KLPm \subseteq N$ we have $0 \neq IJQm \subseteq N$, and thus we have $IJm \subseteq N$ or $Qm \subseteq N$. Hence $KLm \subseteq N$ or $Pm \subseteq N$.

(2) ⇒ (1): Suppose that $0 \neq IJQm \subseteq N$ for some proper ideals I, J, Q of R and some $m \in M$. It is sufficient to set $K := IM, L := JM$ and $P := QM$ in part (2). Hence $0 \neq KLPm \subseteq N$ implies that $KLm \subseteq N$ or $Pm \subseteq N$. This implies that $IJm \subseteq N$ or $Qm \subseteq N$. Consequently, N is a weakly classical 1-absorbing prime submodule of M by Theorem 6. □

Now, we are ready to give a new characterization of weakly classical 1-absorbing prime submodules of modules over um-rings.

Theorem 9. *Let R be a um-ring and M be an R-module. For any proper submodule N of M, the followings are equivalent.*

(1) N is a weakly classical 1-absorbing prime submodule.

(2) For every nonunits $a, b, c \in R$, $(N:_M abc) = (N:_M ab)$ or $(N:_M abc) = (0:_M abc)$ or $(N:_M abc) = (N:_M c)$.

(3) For every nonunits $a, b, c \in R$ and every submodule L of M, $0 \neq abcL \subseteq N$ implies that $abL \subseteq N$ or $cL \subseteq N$.

(4) For every nonunits $a, b \in R$ and every submodule L of M with $abL \nsubseteq N$, $(N:_R abL) = (0:_R abL)$ or $(N:_R abL) = (N:_R L)$.

(5) For every nonunits $a, b \in R$, every proper ideal J of R and every submodule L of M with $0 \neq abJL \subseteq N$ implies that $abL \subseteq N$ or $JL \subseteq N$.

(6) For every nonunits $a \in R$, every proper ideals I, J of R and every submodule L of M with $0 \neq aIJL \subseteq N$ implies that $aIL \subseteq N$ or $JL \subseteq N$.

(7) For every proper ideals I, J of R and every submodule L of M with $IJL \nsubseteq N$, $(N:_R IJL) = (0:_R IJL)$ or $(N:_R IJL) = (N:_R L)$.

(8) For every proper ideals I, J, K of R and every submodule L of M, $0 \neq IJKL \subseteq N$ implies that $IJL \subseteq N$ or $KL \subseteq N$.

Proof. (1) ⇒ (2): Let $m \in (N:_M abc)$. Then we have $abcm \in N$. If $abcm = 0$, then $m \in (0:_M abc)$. Now, assume that $abcm \neq 0$. As N is a weakly classical 1-absorbing prime submodule, we have $abm \in N$ or $cm \in N$. Thus, we have $m \in (N:_M ab) \cup (N:_M c)$. This gives $(N:_M abc) = (N:_M ab) \cup (N:_M c) \cup (0:_M abc)$. As R is a *um*-ring, we have $(N:_M abc) = (N:_M ab)$ or $(N:_M abc) = (0:_M abc)$ or $(N:_M abc) = (N:_M c)$.

(2) ⇒ (3): Let $0 \neq abcL \subseteq N$. Then we have $L \subseteq (N:_M abc)$. Since $0 \neq abcL$, $(N:_M abc) \neq (0:_M abc)$. This gives $L \subseteq (N:_M abc) = (N:_M ab)$ or $L \subseteq (N:_M abc) = (N:_M c)$. Then we have $abL \subseteq N$ or $cL \subseteq N$.

(3) ⇒ (4): Suppose that $abL \nsubseteq N$ and take $x \in (N:_R abL)$. Then we have $abxL \subseteq N$. If $abxL = 0$, then we get $x \in (0:_R abL)$. Now, assume that $abxL \neq 0$. Then by part (3), we conclude that $xL \subseteq N$, that is, $x \in (N:_R L)$. This gives that $(N:_R abL) = (0:_R abL) \cup (N:_R L)$. The follows from the fact that if a submodule is a union of two submodules, then it must be equal to one of them.

(4) ⇒ (5): Let $0 \neq abJL \subseteq N$ and assume that $abL \nsubseteq N$. Then we have $J \subseteq (N:_R abL)$ and $(N:_R abL) \neq (0:_R abL)$. Thus by part (4), we get $J \subseteq (N:_R abL) = (N:_R L)$. This implies that $JL \subseteq N$ which proves (5).

(5) ⇒ (6): Suppose that $0 \neq aIJL \subseteq N$ and $JL \nsubseteq N$. Now, take $x \in I$. Since $0 \neq aIJL$, there exists $y \in I$ such that $ayJL \neq 0$. As $0 \neq ayJL \subseteq N$, by part (5), $ayL \subseteq N$. If $0 \neq axJL \subseteq N$, then similarly we have $axL \subseteq N$. If $0 = axJL$,

then $0 \neq ayJL = a(x+y)JL \subseteq N$. Then similar argument shows that $a(x+y)L \subseteq N$ which implies that $axL \subseteq N$. Hence, we conclude that $aIL \subseteq N$.

(6) $\Rightarrow$ (7): Suppose that $IJL \nsubseteq N$ and $a \in (N:_R IJL)$. Then $aIJL \subseteq N$ and there exists $x \in J$ such that $xIL \nsubseteq N$. Assume that $aIJL = 0$. Then we have $a \in (0:_R IJL)$. Now, assume that $aIJL \neq 0$. Then there exists $y \in J$ such that $0 \neq ayIL = yI(Ra)L \subseteq N$. Then by part (6), we conclude that $yIL \subseteq N$ or $(Ra)L \subseteq N$. In the second case, we have $a \in (N:_R L)$. So assume that $yIL \subseteq N$. If $0 \neq axIL = xI(Ra)L \subseteq N$, then by part (6), we have $(Ra)L \subseteq N$ which implies that $a \in (N:_R L)$. So we may assume that $axIL = 0$. In this case, we have $0 \neq ayIL = a(x+y)IL = (x+y)I(Ra)L \subseteq N$. Again by part (6), we conclude that $(x+y)IL \subseteq N$ or $(Ra)L \subseteq N$. If $(x+y)IL \subseteq N$, then we get $xIL \subseteq N$ because $yIL \subseteq N$. This is a contradiction. Thus we conclude that $(Ra)L \subseteq N$, that is, $a \in (N:_R L)$. By above arguments, we conclude that $(N:_R IJL) \subseteq (0:_R IJL) \cup (N:_R L)$. Since the other inclusion is always true, we have the equality $(N:_R IJL) = (0:_R IJL) \cup (N:_R L)$. In this case, we have $(N:_R IJL) = (0:_R IJL)$ or $(N:_R IJL) = (N:_R L)$.

(7) $\Rightarrow$ (8): Suppose that $0 \neq IJKL \subseteq N$ and $IJL \nsubseteq N$. Then we have $K \subseteq (N:_R IJL) \neq (0:_R IJL)$. Then by part (7), we have $K \subseteq (N:_R IJL) = (N:_R L)$ which implies that $KL \subseteq N$.

(8) $\Rightarrow$ (1): It is clear. □

By [22, Theorem 4], we know that if N is a classical 1-absorbing prime submodule of M, then $(N:_R M)$ is a 1-absorbing prime ideal of R. One can naturally ask whether this property can be extended to weakly classical 1-absorbing prime or not. Now, we give a negative answer to this question with the following example.

Example 5. Let p be a prime number and consider $\mathbb{Z}$-module $\mathbb{Z}_{p^2}$. Assume that $N = (\overline{0})$. Then N is trivially weakly classical 1-absorbing prime submodule. However, $\left(N:_{\mathbb{Z}} \mathbb{Z}_{p^2}\right) = p^2\mathbb{Z}$ is not a weakly 1-absorbing prime ideal.

The following is an immediate consequences of Theorem 6.

Corollary 5. *Let N be a proper submodule of M. Then N is a weakly classical 1-absorbing prime submodule of M if and only if for every nonunits $a, b \in R$ and $m \in M$, $(N:_R abm) = (0:_R abm)$ or $(N:_R abm) = (N:_R m)$ or $(N:_R abm) = R$.*

Theorem 10. *Let M be a finitely generated multiplication module such that $\mathrm{Ann}_R(M)$ is a 1-absorbing prime ideal of R. For a proper submodule N of M, the followings are equivalent.*

(1) N is a weakly classical 1-absorbing prime submodule of M.

(2) If $0 \neq N_1N_2N_3N_4 \subseteq N$ for some proper submodules N_1, N_2, N_3 of M and some submodule N_4 of M, then $N_1N_2N_4 \subseteq N$ or $N_3N_4 \subseteq N$.

(3) $(N:_R M)$ is a weakly 1-absorbing prime ideal of R.

(4) $N = PM$ for some weakly 1-absorbing prime ideal P of R with $\mathrm{Ann}_R(M) \subseteq P$.

Proof. (1) $\Rightarrow$ (2): Suppose that N is a weakly classical 1-absorbing prime submodule of M. Let $0 \neq N_1N_2N_3N_4 \subseteq N$ for some proper submodules N_1, N_2, N_3 of M and some submodule N_4 of M. Since M is multiplication, there exist proper ideals I_1, I_2, I_3 of R such that $N_i = I_iM$ for every $i = 1, 2, 3$. Then we have $0 \neq I_1I_2I_3N_4 \subseteq N$. Then by Theorem 9, $I_1I_2N_4 = N_1N_2N_4 \subseteq N$ or $I_3N_4 = N_3N_4 \subseteq N$.

(2) $\Rightarrow$ (1): It is similar to (1) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3): Suppose that N is a weakly classical 1-absorbing prime submodule, $a, b, c \in R$ are nonunits and $0 \neq abc \in (N:_R M)$. Then we have $abcM \subseteq N$. If $abcM = 0$, then $ab \in \mathrm{Ann}_R(M)$ or $c \in \mathrm{Ann}_R(M)$. Which implies that $abM \subseteq N$ or $cM \subseteq N$. Now, assume that $0 \neq abcM \subseteq N$. Then by Theorem 9, $abM \subseteq N$ or $cM \subseteq N$ which implies that $ab \in (N:_R M)$ or $c \in (N:_R M)$. Hence, $(N:_R M)$ is a weakly 1-absorbing prime ideal.

(3) $\Rightarrow$ (4): It is clear.

(4) $\Rightarrow$ (1): Let $0 \neq IJKL \subseteq N = PM$ for some proper ideals I, J, K of R and some submodule L of M. Then we have $IJK(L:_R M) \subseteq (PM:_R M) = P + \mathrm{Ann}_R(M) = P$ by [24, Corollary to Theorem 9]. As P is a weakly 1-absorbing prime ideal, we have $IJ(L:_R M) \subseteq P$ or $K(L:_R M) \subseteq P$. Then we have $IJL \subseteq N$ or $KL \subseteq N$. Then by Theorem 9, N is a weakly classical 1-absorbing prime submodule of M. □

Since the proof of the following lemma is similar to the previous proposition (1) ⇒ (3), we omit the proof.

Lemma 1. *Let M be a faithful R-module and N a weakly classical 1-absorbing prime submodule of M. Then $(N:_R M)$ is a weakly 1-absorbing prime ideal of R.*

Theorem 11. *Let M_1, M_2 be two R-modules and N_1 be a proper submodule of M_1. The following conditions are equivalent.*

(1) $N = N_1 \times M_2$ is a weakly classical 1-absorbing prime submodule of $M = M_1 \times M_2$.

(2) N_1 is a weakly classical 1-absorbing prime submodule of M_1 and if whenever (a, b, c, m) is a classical 1-quadruple-zero of N_1 for some nonunits $a, b, c \in R$ and $m \in M_1$, then $abc \in \mathrm{Ann}_R(M_2)$.

Proof. (1) ⇒ (2): Suppose that $N = N_1 \times M_2$ is a weakly classical 1-absorbing prime submodule of $M = M_1 \times M_2$. Let $0 \neq abcm \in N_1$ for some nonunits $a, b, c \in R$ and $m \in M_1$. Then we have $(0, 0) \neq abc(m, 0) = (abcm, 0) \in N$. Since N is a weakly classical 1-absorbing prime submodule of M, we have $ab(m, 0) = (abm, 0) \in N$ or $c(m, 0) = (cm, 0) \in N$. This implies that $abm \in N_1$ or $cm \in N_1$. Thus, N_1 is a weakly classical 1-absorbing prime submodule of M_1. Now, assume that (a, b, c, m) is a classical 1-quadruple-zero of N_1. Then we have $abcm = 0$, $abm \notin N_1$ and $cm \notin N_1$. Now, we will show that $abc \in \mathrm{Ann}_R(M_2)$. Suppose that $abc \notin \mathrm{Ann}_R(M_2)$. Then there exists $n \in M_2$ such that $abcn \neq 0$. This gives $(0, 0) \neq abc(m, n) = (0, abcn) \in N$. As N is a weakly classical 1-absorbing prime, we have $ab(m, n) \in N$ or $c(m, n) \in N$. Which implies that $abm \in N_1$ or $cm \in N_1$. Both of them are contradictions. Thus, $abc \in \mathrm{Ann}_R(M_2)$.

(2) ⇒ (1): Let $a, b, c \in R$ be nonunits and $(m, n) \in M = M_1 \times M_2$ be such that $(0, 0) \neq abc(m, n) \in N = N_1 \times M_2$. First assume that $abcm \neq 0$. Then by part (2), $abm \in N_1$ or $cm \in N_1$. So, we conclude that $ab(m, n) \in N$ or $c(m, n) \in N$. This shows that N is a weakly classical 1-absorbing prime. So assume that $abcm = 0$. If $abm \notin N_1$ and $cm \notin N_1$, then (a, b, c, m) is a classical 1-quadruple-zero of N_1. Then by part (2), $abcn = 0$ and so $abc(m, n) = (0, 0)$ which is a contradiction. Thus, we have $abm \in N_1$ or $cm \in N_1$. This argument shows $ab(m, n) \in N$ or $c(m, n) \in N$ which completes the proof. □

Proposition 4. *Let M_1, M_2 be two R-modules and N_1, N_2 be two proper submodules of M_1 and M_2, respectively. If $N = N_1 \times N_2$ is a weakly classical 1-absorbing prime submodule of $M = M_1 \times M_2$, then N_1 is a weakly classical 1-absorbing prime submodule of M_1 and N_2 is a weakly classical 1-absorbing prime submodule of M_2.*

Proof. One can easily verify the claim by a similar argument in the previous theorem. □

The following example shows that the converse of previous proposition is not true in general.

Example 6. Let $R = \mathbb{Z}$, $M = \mathbb{Z}^2$ and $N = p\mathbb{Z} \times q\mathbb{Z}$ where p, q are two distinct prime integers. Since, $p\mathbb{Z}$, $q\mathbb{Z}$ are prime ideals of $\mathbb{Z}$, then $p\mathbb{Z}$, $q\mathbb{Z}$ are weakly classical 1-absorbing prime $\mathbb{Z}$-submodules of $\mathbb{Z}$. Notice that $(0, 0) \neq ppq(1, 1) = (ppq, ppq) \in N$, but neither $pp(1, 1) \in N$ nor $q(1, 1) \in N$. So, N is not a weakly classical 1-absorbing prime submodule of M.

Let R_i be a commutative ring with identity and M_i be an R_i-module for every $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module with componentwise addition and scalar multiplication. Also each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 12. *Let M_i be a faithful multiplication R_i-module such that $\mathrm{Ann}_R(M_i)$ is not unique maximal ideal of R_i for each $i = 1, 2$. Suppose that $R = R_1 \times R_2$ is a decomposable ring and $M = M_1 \times M_2$. Further, assume that $N = N_1 \times N_2 \neq 0$ for some submodule N_i of M_i. The following statements are equivalent.*

(1) N is a weakly classical 1-absorbing prime submodule of M.

(2) $N = N_1 \times M_2$ for some classical prime submodule N_1 of M_1 or $N = M_1 \times N_2$ for some classical prime submodule N_2 of M_2.

(3) N is a classical prime submodule of M.

(4) N is a weakly classical prime submodule of M.

Proof. (1) ⇒ (2): Suppose that $N = N_1 \times N_2$ is a weakly classical 1-absorbing prime submodule of M. Since M_1, M_2 are faithful modules, so is M. As N is a weakly classical 1-absorbing prime submodule, by Lemma 1, $(N:_R M)$ is a weakly 1-absorbing prime ideal of R. Since N is nonzero and M is faithful multiplication module, $(N:_R M) = (N_1:_{R_1} M_1) \times (N_2:_{R_2} M_2)$ is nonzero. Then by the proof of [13, Theorem 10], we have $(N_1:_{R_1} M_1) = R_1$ or $(N_2:_{R_2} M_2) = R_2$.Without loss of generality, we may assume that $N = N_1 \times M_2$ for some proper submodule N_1 of M_1. Now, we will show that N_1 is a classical prime submodule of M_1. Let $abm \in N_1$ for some $a, b \in R_1$ and $m \in M_1$. Since $\mathrm{Ann}_R(M_2)$ is not unique maximal ideal of R_2, there exists a nonunit $t \notin \mathrm{Ann}_R(M_2)$. Then there exists $n \in M_2$ such that $tn \neq 0$. Then we have $0_M \neq (abm, tn) = (a,1)(1,t)(b,1)(m,n) \in N$. As N is a weakly classical 1-absorbing prime submodule of M, we have either $(a,1)(1,t)(m,n) \in N$ or $(b,1)(m,n) \in N$ which implies that $am \in N_1$ or $bm \in N_1$. Thus, N_1 is a classical prime submodule of M_1.

(2) ⇒ (3): Follows from [8, Theorem 2.35].

(3) ⇒ (4) ⇒ (1): Clear. □

Theorem 13. *Let R be a um-ring and M be an R-module.*

(1) If F is a flat R-module and N is a weakly classical 1-absorbing prime submodule of M such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a weakly classical 1-absorbing prime submodule of $F \otimes M$.

(2) Suppose that F is a faithfully flat R-module. Then N is a weakly classical 1-absorbing prime submodule of M if and only if $F \otimes N$ is a weakly classical 1-absorbing prime submodule of $F \otimes M$.

Proof. (1): Let $a, b, c \in R$ be nonunits. Then we get by Theorem 9, either $(N:_M abc) = (0:_M abc)$ or $(N:_M abc) = (N:_M ab)$ or $(N:_M abc) = (N:_M c)$. Assume that $(N:_M abc) = (0:_M abc)$. Then by [7, Lemma 3.2], we have $(F \otimes N:_{F\otimes M} abc) = F \otimes (N:_M abc) = F \otimes (0:_M abc) = (F \otimes 0:_{F\otimes M} abc) = (0:_{F\otimes M} abc)$. Now, suppose that $(N:_M abc) = (N:_M ab)$. Again by [7, Lemma 3.2], we have $(F \otimes N:_{F\otimes M} abc) = F \otimes (N:_M abc) = F \otimes (N:_M ab) = (F \otimes N:_{F\otimes M} ab)$. Similarly, we can show that if $(N:_M abc) = (N:_M c)$, then $(F \otimes N:_{F\otimes M} abc) = (F \otimes N:_{F\otimes M} c)$. Consequently, by Theorem 9, we deduce that $F \otimes N$ is a weakly classical 1-absorbing prime submodule of $F \otimes M$.

(2): Let N be a weakly classical 1-absorbing prime submodule of M and assume $F \otimes N = F \otimes M$. Then $0 \to F \otimes N \xrightarrow{\subseteq} F \otimes M \to 0$ is an exact sequence. Since F is a faithfully flat R-module, $0 \to N \to M \to 0$ is an exact sequence. So $N = M$, which is a contradiction. Thus we have $F \otimes N \neq F \otimes M$. Then $F \otimes N$ is a weakly classical 1-absorbing prime submodule of $F \otimes M$ by (1). Now for the converse, let $F \otimes N$ be a weakly classical 1-absorbing prime submodule of $F \otimes M$. Then we have $F \otimes N \neq F \otimes M$ and so $N \neq M$. Let $a, b, c \in R$ be nonunits. Then $(F \otimes N:_{F\otimes M} abc) = (0:_{F\otimes M} abc)$ or $(F \otimes N:_{F\otimes M} abc) = (F \otimes N:_{F\otimes M} ab)$ or $(F \otimes N:_{F\otimes M} abc) = (F \otimes N:_{F\otimes M} c)$ by Theorem 9. Assume that $(F \otimes N:_{F\otimes M} abc) = (0:_{F\otimes M} abc)$. Hence $F \otimes (N:_M abc) = (F \otimes N:_{F\otimes M} abc) = (0:_{F\otimes M} abc) = F \otimes (0:_M abc)$. So $0 \to F \otimes (0:_M abc) \xrightarrow{\subseteq} F \otimes (N:_M abc) \to 0$ is exact sequence. Since F is a faithfully flat R-module $0 \to (0:_M abc) \xrightarrow{\subseteq} (N:_M abc) \to 0$ is an exact sequence which implies that $(N:_M abc) = (0:_M abc)$. By a similar argument, we can deduce that if $(F \otimes N:_{F\otimes M} abc) = (F \otimes N:_{F\otimes M} ab)$ or $(F \otimes N:_{F\otimes M} abc) = (F \otimes N:_{F\otimes M} c)$, then $(N:_M abc) = (N:_M ab)$ or $(N:_M abc) = (N:_M c)$. Consequently, N is a weakly classical 1-absorbing prime submodule of M by Theorem 9. □

Corollary 6. *Let R be an um-ring, M be an R-module, and x be an indeterminate. If N is a weakly classical 1-absorbing prime submodule of M, then $N[x]$ is a weakly classical 1-absorbing prime submodule of $M[x]$.*

Proof. Assume that N is a weakly classical 1-absorbing prime submodule of M. Notice that $R[x]$ is a flat R-module. So by Theorem 13, $N[x] \simeq R[x] \otimes N$ is a weakly classical 1-absorbing prime submodule of $M[x] \simeq R[x] \otimes M$. □

Now, we investigate the modules over which every proper ideal is weakly classical 1-absorbing prime.

Theorem 14. *Let M be an R-module. The following statements are satisfied.*

(1) If every proper submodule of M is a weakly classical 1-absorbing prime. Then $\mathrm{Jac}(R)^3 M = 0$.

(2) Suppose that $(R, \mathfrak{m})$ is a local ring. Then every proper submodule of M is a weakly classical 1-absorbing prime if and only if $\mathfrak{m}^3 M = 0$.

Proof. (1): Suppose that every proper submodule of M is a weakly classical 1-absorbing prime. Now, we will show that $\mathrm{Jac}(R)^3M = 0$. Suppose to the contrary. Then there exist $x, y, z \in \mathrm{Jac}(R)$ and $m \in M$ such that $xyzm \neq 0$. Now, put $N = Rxyzm$. Since $0 \neq xyzm \in N$ and N is a weakly classical 1-absorbing prime submodule, we have either $xym \in N$ or $zm \in N$. Then there exists $a \in R$ such that $xym = axyzm$ or $zm = axyzm$. In the first case, we have $(1 - az)xym = 0$. Since $1 - az$ is unit, we conclude that $xym = 0$. In the second case, one can similarly show that $zm = 0$. Thus, we conclude, $xyzm = 0$ which is a contradiction. Hence, $\mathrm{Jac}(R)^3M = 0$.

(2): Suppose that $(R, \mathfrak{m})$ is a local ring. If every proper submodule of M is weakly classical 1-absorbing prime, then by (1), $\mathfrak{m}^3M = 0$. For the converse, note that $xyz = 0$ for every nonunits $x, y, z \in R$. In this case, every proper submodule of M is trivially weakly classical 1-absorbing prime submodule. □

The following example shows that the converse of Theorem 14 is not true in general.

Example 7. Consider the $R = \mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$ and the submodule $N = 2\mathbb{Z} \times 3\mathbb{Z}$. Then $\mathrm{Jac}(R) = (0)$ and this gives $\mathrm{Jac}(R)^3M = (0)$. Since $(0, 0) \neq 2 \cdot 5 \cdot 3 \cdot (1, 1) = (30, 30) \in N$, but $2 \cdot 5 \cdot (1, 1) = (10, 10) \notin N$ and $3 \cdot (1, 1) = (3, 3) \notin N$, it follows that N is not a weakly classical 1-absorbing prime submodule of M.

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