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On the regularized trace of a second order differential operator

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Abstract

We calculate a regularized trace formula for a second order differential operator with bounded operator coefficient given in a finite interval.
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1. Introduction

Let H be a separable Hilbert space. We denote the inner product in H by $(\cdot,\cdot)$ and the norm in H by $\|\cdot,\cdot\|$. Let $H_1 = L_2(H;[0,\pi])$ denotes the set of all measurable functions f with its values in H such that

$$\int_0^\pi \|f(x)\|^2 \mathrm{d}x < \infty.$$

If the inner product of two arbitrary elements f and g of the space H_1 is defined by

$$(f,g)_{H_1} = \int_0^\pi (f(x),g(x))\mathrm{d}x$$

then H_1 becomes a separable Hilbert space, (see [1]). We denote by $\sigma_\infty(H)$ the set of compact operators from H to H. If $A \in \sigma_\infty(H)$ then A^*A is a non-negative self-adjoint operator and $(A^*A)^{\frac{1}{2}} \in \sigma_\infty(H)$ (see [2]), where A^* is the adjoint operator of A. Let the non-zero eigenvalues of the $(A^*A)^{\frac{1}{2}}$ be $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_k$ $(0 \leqslant k \leqslant \infty)$. Here each eigenvalue is represented as many times as its multiplicity. We call $s_1, s_2, \ldots, s_k$ s-numbers of the operator A. We assume $s_j = 0$ $(j = k+1, k+2, \ldots)$ when $k < \infty$. This s-numbers of A is also denoted by $s_k(A)$ $(k = 1,2,\ldots)$. If A is a normal operator, that is $A^*A = AA^*$ then $s_k(A) = |\lambda_k(A)|$ $(k = 1,2,\ldots)$ (see [2]), where $\lambda_1(A), \lambda_2(A), \ldots, \lambda_k(A)$ are the non-zero eigenvalues of A. We denote by $\sigma_p(H)$ or σ_p $(p \geqslant 1)$ the set of all compact $A \in \sigma_\infty(H)$ operators satisfying

$$\sum_{j=1}^{\infty} s_j^p(A) < \infty, \quad (p \geqslant 1).$$

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The space σ_p ($p \geqslant 1$) is a separable Banach space with the norm

$$\|A\|_{\sigma_p(H)} = \left[\sum_{j=1}^{\infty} s_j^p(A)\right]^{\frac{1}{p}} \quad (A \in \sigma_p(H))$$

(see [2]).

For $p = 1$ the space $\sigma_1(H)$ is called the space of kernel operators. Thus an operator in $\sigma_1(H)$ is called a kernel operator. If $A \in \sigma_1(H)$ then for any linear bounded operator $B : H \to H$ we have AB, $BA \in \sigma_1(H)$ and

$$\|BA\|_{\sigma_1(H)} \leqslant \|B\|\|A\|_{\sigma_1(H)},$$
$$\|AB\|_{\sigma_1(H)} \leqslant \|B\|\|A\|_{\sigma_1(H)}.$$

If $A \in \sigma_1(H)$ and $\{e_k\}_{k=1}^{\infty} \subset H$ is any orthonormal basis then the series

$$\sum_{k=1}^{\infty}(Ae_k, e_k)$$

is convergent and the sum of this series does not depend on the choice of the basis $\{e_k\}_{k=1}^{\infty}$. The sum of this series is said to be the trace of matrix of A and is denoted by $\mathrm{tr}A$. We have

$$\mathrm{tr}A = \sum_{j=1}^{v(A)} \lambda_j(A).$$

Here each eigenvalue is counted as many times as its algebraic multiplicity, and $v(A)$ denotes the sum of algebraic multiplicity of non-zero eigenvalues of A.

The regularized trace of the differential operators can be regarded as a generalization of the trace of matrices and kernel operators. The regularized trace formulas for scalar differential operators are studied in [3–7] and in many other works. The regularized trace formulas for differential operator with operator coefficients are calculated in for example [8–12]. In this work, we give a different result on the latter.

Let us consider the operators L_0 and L in H_1 defined by

$$L_0[y] = -y''(x),$$
$$L[y] = -y''(x) + Q(x)y(x)$$

with the boundary conditions $y'(0) = y(\pi) = 0$. Suppose that the operator function $Q(x)$ in $L[y]$ satisfies the following conditions:

(1) For each x in $[0, \pi]$, $Q(x)$ has a weak derivative of second order and for $i = 0, 1, 2$, $Q^i(x) : H \to H$ is a self-adjoint operator in $[0, \pi]$.
(2) $\|Q\|_{H_1} < 1$.
(3) There is an orthonormal basis$\{\varphi_n\}_{n=1}^{\infty}$ in H such that

$$\sum_{n=1}^{\infty} \|Q(x)\varphi_n\|_{H_1} < \infty.$$

(4) For $i = 0, 1, 2$ the functions $\|Q^{(i)}(x)\|_{\sigma_1(H)}$ are bounded and measurable in $[0, \pi]$.

The spectrum of L_0 is the set $\{\mu_m\}_{m=0}^{\infty} = \left\{\left(m + \frac{1}{2}\right)^2\right\}_{m=0}^{\infty}$ which means $\sigma(L_0) = \{\mu_m\}_{m=0}^{\infty}$. Every point of this set is an eigenvalue of L_0 whose multiplicity is infinite. The orthonormal eigenfunctions corresponding to μ_m are of the form

$$\psi_{mn}^0 = \sqrt{\frac{2}{\pi}}\cos\left(m + \frac{1}{2}\right)x \cdot \varphi_n; \quad (n = 1, 2, \ldots). \tag{1.1}$$

Let R_λ^0 and R_λ be the resolvents of L_0 and L respectively. If $\lambda \notin \{\mu_m\}$ and $Q(x)$ satisfies the condition (3) above then $QR_\lambda^0 : H_1 \to H_1$ is a kernel operator, i.e. $QR_\lambda^0 \in \sigma_1(H_1)$. From this and the fact that $Q(x)$ satisfies the conditions (2) and (3), it can be seen that the spectrum of L is a subset of the union of pairwise disjoint intervals $F_m = [\mu_m - \|Q\|_{H_1}, \mu_m + \|Q\|_{H_1}]$ $(m = 0, 1, 2, \ldots)$. Each point of the spectrum of L which is different from μ_m in F_m is an isolated eigenvalue of multiplicity finite. However, μ_m can be an eigenvalue of L which has multiplicity finite or infinite. Moreover,

$$\lim_{n\to\infty} \lambda_{mn} = \mu_m,$$

where $\{\lambda_{mn}\}_{n=1}^{\infty}$ are the eigenvalues of L in F_m.

In this work, we obtain a formula for the sum of the series

$$\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn} - \mu_m) - \frac{1}{\pi}\int_0^{\pi} \operatorname{tr} Q(x)\mathrm{d}x\right]$$

and we call this formula the regularized trace formula for L.

2. Calculation of the regularized trace for the operator L

By definition of $Q(x)$, the operator function $R_\lambda - R_\lambda^0$ is analytic in the region $\rho(L)$ with respect to the norm in $\sigma_1(H_1)$ where $\rho(L)$ is the resolvent set of L. Let $\{\psi_{mn}(x)\}_{m,n=1}^{\infty}$ be orthonormal eigenfunctions corresponding to eigenvalues$\{\lambda_{mn}(x)\}_{m,n=1}^{\infty}$ of L and let $B_{mn}^0 = (\cdot, \psi_{mn}^0)_{H_1}\psi_{mn}^0; B_{mn} = (\cdot, \psi_{mn})_{H_1}\psi_{mn}$.

The spectra of L and L_0 only consist of eigenvalues and its limit points. Hence, it is known that

$$R_\lambda^0 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{B_{mn}^0}{\mu_m - \lambda}; \quad R_\lambda = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{B_{mn}}{\lambda_{mn} - \lambda} \tag{2.1}$$

(see [13]).

By using the equalities (2.1) it can be seen that the series $\sum_{n=1}^{\infty}(\lambda_{pn} - \mu_p)(p = 0, 1, 2, \ldots)$ are convergent absolutely. Hence for every $\lambda \in \rho(L)$ we have

$$\operatorname{tr}(R_\lambda - R_\lambda^0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{\mu_m - \lambda}\right)$$

since $R_\lambda - R_\lambda^0 \in \sigma_1(H_1)$. Multiplying this equation by $\frac{\lambda}{2\pi\mathrm{i}}$ and integrating on the circle $|\lambda| = b_p = \mu_p + p$ $(p \geqslant 1)$ we find

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_{|\lambda|=b_p}\lambda\operatorname{tr}(R_\lambda - R_\lambda^0)\mathrm{d}\lambda &= \frac{1}{2\pi\mathrm{i}}\int_{|\lambda|=b_p}\lambda\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{\mu_m - \lambda}\right)\mathrm{d}\lambda \\ &\quad + \frac{1}{2\pi\mathrm{i}}\int_{|\lambda|=b_p}\lambda\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{\mu_m - \lambda}\right)\mathrm{d}\lambda\end{aligned} \tag{2.2}$$

On the other hand, when $m \leqslant p$ and $p \geqslant 1$, we have

$$\mu_m - \|Q\|_{H_1} \leqslant \lambda_{mn} \leqslant \mu_m + \|Q\|_{H_1} \leqslant \mu_p + p = b_p.$$

Thus we get

$$|\lambda_{mn}| < b_p, \quad m \leqslant p; \ p \geqslant 1; \ n = 1, 2, \ldots \tag{2.3}$$

When $m > p$ we have

$$\lambda_{mn} \geqslant \mu_m - \|Q\|_{H_1} \geqslant \mu_{p+1} - \|Q\|_{H_1} > \mu_p + p = b_p$$

or

$$|\lambda_{mn}| > b_p, \quad m > p; \ p \geqslant 1; \ n = 1, 2, \ldots \tag{2.4}$$

Taking into account the inequalities (2.3) and (2.4), the relation (2.2) becomes

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda \operatorname{tr}\left(R_\lambda - R_\lambda^0\right)d\lambda = \sum_{m=0}^{p}\sum_{n=1}^{\infty}(\mu_m - \lambda_{mn}). \tag{2.5}$$

By using $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0$, we find

$$R_\lambda - R_\lambda^0 = \sum_{j=1}^{N}(-1)^j R_\lambda^0 (QR_\lambda^0)^j + (-1)^{N+1} R_\lambda (QR_\lambda^0)^{N+1}$$

for a natural number N. If we substitute this equality in (2.5) we obtain

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty}(\mu_m - \lambda_{mn}) = \sum_{j=1}^{N}\frac{(-1)^j}{2\pi i}\int_{|\lambda|=b_p}\lambda \operatorname{tr}[R_\lambda^0 (QR_\lambda^0)^j]d\lambda + \frac{(-1)^{N+1}}{2\pi i}\int_{|\lambda|=b_p}\lambda \operatorname{tr}[R_\lambda (QR_\lambda^0)^{N+1}]d\lambda. \tag{2.6}$$

Now, we need to introduce the following notations:

$$M_{pj} = \frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_p}\lambda \operatorname{tr}[R_\lambda^0 (QR_\lambda^0)^j]d\lambda, \tag{2.7}$$

$$M_p^{(N)} = \frac{(-1)^N}{2\pi i}\int_{|\lambda|=b_p}\lambda \operatorname{tr}\left[R_\lambda (QR_\lambda^0)^{N+1}\right]d\lambda. \tag{2.8}$$

By virtue of (2.7) and (2.8) we can write (2.6) in the form

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty}(\lambda_{mn} - \mu_m) = \sum_{j=1}^{N} M_{pj} + M_p^{(N)}. \tag{2.9}$$

Since the operator function QR_λ^0 is analytic with respect to the norm in $\sigma_1(H_1)$ at every point $\lambda \neq \mu_m$ $(m = 0, 1, 2, \ldots)$ we have

$$M_{pj} = \frac{(-1)^j}{2\pi i j}\int_{|\lambda|=b_p}\operatorname{tr}(QR_\lambda^0)^j]d\lambda \quad (j = 1, 2, \ldots). \tag{2.10}$$

From now, our aim is to evaluate the expressions M_{pj} and $M_p^{(N)}$.

Theorem 2.1. *If $Q(x)$ satisfies condition* (3) *and the function $\|Q(x)\|_{\sigma_1(H_1)}$ is integrable on the interval $[0, \pi]$, then we have*

$$M_{p1} = \frac{p+1}{\pi}\int_0^\pi \operatorname{tr} Q(x)dx + \frac{1}{\pi}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\int_0^\pi (Q(x)\varphi_n, \varphi_n)\cos(2m+1)x\,dx.$$

Proof. For $j = 1$, from (2.10) and orthonormal basis of the space H_1 given in (1.1) we obtain

$$M_{p1} = -\frac{1}{2\pi i}\int_{|\lambda|=b_p}\operatorname{tr}(QR_\lambda^0)d\lambda = -\frac{1}{2\pi i}\int_{|\lambda|=b_p}\sum_{m=0}^{p}\sum_{n=1}^{\infty}(QR_\lambda^0\psi_{mn}^0, \psi_{mn}^0)_{H_1}d\lambda. \tag{2.11}$$

Let us first try to estimate $|(QR_\lambda^0\psi_{mn}^0, \psi_{mn}^0)_{H_1}|$:

$$\begin{aligned}\left|(QR_\lambda^0\psi_{mn}^0, \psi_{mn}^0)_{H_1}\right| &= \left|\int_0^\pi (QR_\lambda^0\psi_{mn}^0(x), \psi_{mn}^0(x))dx\right| \\ &= |\mu_m - \lambda|^{-1}\left|\int_0^\pi (Q(x)\sqrt{2\pi^{-1}}\cos\left(m+\frac{1}{2}\right)x\cdot\varphi_n, \sqrt{2\pi^{-1}}\cos\left(m+\frac{1}{2}\right)x\cdot\varphi_n)dx\right| \\ &\leqslant |\mu_m - \lambda|^{-1}\int_0^\pi |(Q(x)\varphi_n, \varphi_n)|dx \leqslant |\mu_m - \lambda|^{-1}\int_0^\pi \|Q(x)\varphi_n\|dx\end{aligned}$$

$$\leqslant \sqrt{\pi}|\mu_m-\lambda|^{-1}\left(\int_0^{\pi}\|Q(x)\varphi_n\|^2\mathrm{d}x\right)^{\frac{1}{2}}$$
$$=\sqrt{\pi}|\mu_m-\lambda|^{-1}\|Q(x)\varphi_n\|_{H_1}.$$

Then the condition (3) on $Q(x)$ will give that the series

$$\sum_{m=0}^{\infty}\alpha_m(\lambda);\quad \alpha_m(\lambda)=\sum_{n=1}^{\infty}(QR_\lambda^0\psi_{mn}^0,\psi_{mn}^0)_{H_1}\quad (m=0,1,2,\ldots)$$

converge absolutely and uniformly with respect to λ on the circle $|\lambda|=b_p$. So, from (2.11) one obtains

$$M_{p1}=\sum_{m=0}^{p}\sum_{n=1}^{\infty}(QR_\lambda^0\psi_{mn}^0,\psi_{mn}^0)_{H_1}\frac{1}{2\pi\mathrm{i}}\int_{|\lambda|=b_p}\frac{\mathrm{d}\lambda}{\lambda-\mu_m}.$$

Taking into account (1.1), (2.3) and (2.4) and this last equality we obtain

$$M_{p1}=\sum_{m=0}^{p}\sum_{n=1}^{\infty}(QR_\lambda^0\psi_{mn}^0,\psi_{mn}^0)_{H_1}=\frac{2}{\pi}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos^2\left(m+\frac{1}{2}\right)x\mathrm{d}x$$
$$=\frac{1}{\pi}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)(1+\cos(2m+1)x)\mathrm{d}x. \tag{2.12}$$

On the other hand, since

$$\left|\sum_{n=1}^{q}(Q(x)\varphi_n,\varphi_n)\right|\leqslant\sum_{n=1}^{\infty}|(Q(x)\varphi_n,\varphi_n)|\leqslant\|Q(x)\|_{H_1}\quad (q=1,2,\ldots)$$

and

$$\int_0^{\pi}\|Q(x)\|_{H_1}\mathrm{d}x<\infty$$

it follows that

$$\sum_{n=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\mathrm{d}x=\int_0^{\pi}\mathrm{tr}\,Q(x)\mathrm{d}x. \tag{2.13}$$

From (2.12) and (2.13) we conclude that

$$M_{p1}=\frac{p+1}{\pi}\int_0^{\pi}\mathrm{tr}Q(x)\mathrm{d}x+\frac{1}{\pi}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos(2m+1)x\,\mathrm{d}x$$

as desired.

Now, we will prove that

$$\lim_{p\to\infty}M_{pj}=0\quad (j=2,3,4,\ldots).$$

To do this, we first consider the case $j=2$. From (2.10) we have

$$M_{p2}=\frac{1}{4\pi\mathrm{i}}\int_{|\lambda|=b_p}\mathrm{tr}(QR_\lambda^0)^2]\mathrm{d}\lambda=\frac{1}{4\pi\mathrm{i}}\int_{|\lambda|=b_p}\left[\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left((QR_\lambda^0)^2\psi_{mn}^0,\psi_{mn}^0\right)_{H_1}\right]\mathrm{d}\lambda. \tag{2.14}$$

Moreover, since

$$QR_\lambda^0\psi_{mn}^0=\frac{Q\psi_{mn}^0}{\mu_m-\lambda}$$

and

$$(QR_\lambda^0)^2\psi_{mn}^0=(\mu_m-\lambda)^{-1}\left\{\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}(\mu_r-\lambda)^{-1}(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}Q\psi_{rq}^0\right\},$$

we transform (2.14) into

$$M_{p2}=\frac{1}{4\pi i}\int_{|\lambda|=b_p}\left[\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left(\frac{(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}(Q\psi_{rq}^0,\psi_{mn}^0)_{H_1}}{(\lambda-\mu_m)(\lambda-\mu_r)}\right)\right]d\lambda \tag{2.15}$$

It can be easily seen that we have

$$\int_{|\lambda|=b_p}\frac{d\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)}=0;\qquad m,r\leqslant p. \tag{2.16}$$

For m, $r>p$, we have

$$\int_{|\lambda|=b_p}\frac{d\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)}=0$$

and by using (2.15) and (2.16), we obtain

$$\begin{aligned}M_{p2}&=\frac{1}{2\pi i}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}(Q\psi_{rq}^0,\psi_{mn}^0)_{H_1}\int_{|\lambda|=b_p}\frac{d\lambda}{(\lambda-\mu_m)(\lambda-\mu_r)}\\&=\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}(Q\psi_{rq}^0,\psi_{mn}^0)_{H_1}\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{1}{\mu_m-\mu_r}\left[\frac{1}{\lambda-\mu_m}-\frac{1}{\lambda-\mu_r}\right]d\lambda\\&=\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}(\mu_m-\mu_r)^{-1}\left|(Q\psi_{mn}^0,\psi_{rq}^0)_{H_1}\right|^2.\end{aligned}$$

This means

$$\left|M_{p2}\right|\leqslant\sum_{r=p+1}^{\infty}(\mu_r-\mu_p)^{-1}\sum_{q=1}^{\infty}\left\|Q\psi_{rq}^0\right\|_{H_1}^2.$$

We can write

$$\begin{aligned}\sum_{q=1}^{\infty}\left\|Q\psi_{rq}^0\right\|_{H_1}^2&=\sum_{q=1}^{\infty}\int_0^{\pi}\left\|Q(x)\sqrt{2\pi^{-1}}\cos\left(r+\frac{1}{2}\right)x\cdot\varphi_q\right\|^2dx\leqslant\sum_{q=1}^{\infty}\int_0^{\pi}\left\|Q(x)\varphi_q\right\|dx=\sum_{q=1}^{\infty}\left\|Q(x)\varphi_q\right\|_{H_1}^2\\&<C\quad\text{(by the condition (3))}\ \ (C\text{ is a positive constant})\end{aligned}$$

This gives

$$\left|M_{p2}\right|<C\sum_{r=p+1}^{\infty}(\mu_r-\mu_p)^{-1}.$$

Moreover, it can be seen that

$$\sum_{r=p+1}^{\infty}(\mu_r-\mu_p)^{-1}<2p^{-\frac{1}{2}}. \tag{2.17}$$

Thus,

$$\lim_{p\to\infty}M_{p2}=0$$

since

$$\left|M_{p2}\right|<2Cp^{-\frac{1}{2}}.$$

By a similar way, we can prove that

$$\lim_{p\to\infty} M_{p3} = 0.$$

Finally, we want to show that, for $j \geqslant 4$, we have

$$\lim_{p\to\infty} M_{pj} = 0.$$

By using

$$\left\|QR_{\lambda}^{0}\right\|_{\sigma_1(H_1)} \leqslant \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left\|QR_{\lambda}^{0}\psi_{mn}^{0}\right\|_{H_1}$$

we obtain

$$\left\|QR_{\lambda}^{0}\right\|_{\sigma_1(H_1)} < C_1\sum_{m=0}^{\infty}|\mu_m-\lambda|^{-1} \tag{2.18}$$

where C_1 is a positive constant. We get

$$\begin{aligned}\sum_{m=0}^{\infty}|\mu_m-\lambda|^{-1} &= \sum_{m=0}^{p}|\mu_m-\lambda|^{-1}+\sum_{m=p+1}^{\infty}|\mu_m-\lambda|^{-1} \leqslant \sum_{m=0}^{p}(|\lambda|-\mu_m)^{-1}+\sum_{m=p+1}^{\infty}(\mu_m-|\lambda|)^{-1}\\ &= \sum_{m=0}^{p}(\mu_p+p-\mu_m)^{-1}+\sum_{m=p+1}^{\infty}(\mu_m-\mu_p-p)^{-1} < \sum_{m=0}^{p}p^{-1}+\sum_{m=p+1}^{\infty}\left[\frac{1}{2}(\mu_m-\mu_p)+\frac{1}{2}(\mu_m-\mu_p)-p\right]^{-1}\\ &< 2+\sum_{m=p+1}^{\infty}\frac{2}{\mu_m-\mu_p} < 2+\sum_{m=p+1}^{\infty}\frac{2}{m^2-p^2}.\end{aligned}$$

Hence by using (2.17), we have

$$\sum_{m=0}^{\infty}|\mu_m-\lambda|^{-1} < 6.$$

Thus, from (2.18) we obtain

$$\left\|QR_{\lambda}^{0}\right\|_{\sigma_1(H_1)} < C_2; \quad |\lambda| = b_p = \mu_p + p, \tag{2.19}$$

where C_2 is a positive constant.

Now, let us estimate $\|R_{\lambda}^{0}\|_{H_1}$ on the circle $|\lambda| = b_p = \mu_p + p$: For $m \leqslant p$ we have

$$|\mu_m-\lambda| \geqslant |\lambda|-\mu_m \geqslant p^2+p-m$$

and for $m \geqslant p+1$ we have

$$|\mu_m-\lambda| \geqslant \mu_m-|\lambda| = \mu_m-\mu_p-p \geqslant (p+1)^2-p^2-p > p.$$

This means that

$$|\mu_m-\lambda|^{-1} \leqslant p^{-1}; \quad |\lambda| = b_p = \mu_p + p$$

and

$$\left\|R_{\lambda}^{0}\right\|_{H_1} \leqslant p^{-1} \tag{2.20}$$

since

$$\left\|R_{\lambda}^{0}\right\|_{H_1} = \max_{m}\left\{|\mu_m-\lambda|^{-1}\right\}.$$

From (2.10), (2.19) and (2.20) we find

$$\begin{aligned}|M_{pj}| &= \frac{1}{2\pi j}\left|\int_{|\lambda|=b_p} \operatorname{tr}\left[(QR_\lambda^0)^j\right]\mathrm{d}\lambda\right| \leqslant \frac{1}{2\pi j}\int_{|\lambda|=b_p} \|(QR_\lambda^0)^j\|_{\sigma_1(H_1)}|\mathrm{d}\lambda| \\ &\leqslant \frac{1}{2\pi j}\int_{|\lambda|=b_p} \|(QR_\lambda^0)\|_{\sigma_1(H_1)}\|(QR_\lambda^0)^{j-1}\|_{H_1}|\mathrm{d}\lambda| \leqslant (2\pi j)^{-1}\int_{|\lambda|=b_p} \|(QR_\lambda^0)\|_{\sigma_1(H_1)}\|Q\|_{H_1}^{j-1}\|R_\lambda^0\|_{H_1}^{j-1}|\mathrm{d}\lambda| \\ &\leqslant (2\pi j)^{-1}C_2\int_{|\lambda|=b_p} p^{1-j}|\mathrm{d}\lambda| \leqslant C_3 p^{3-j}, \quad (C_3>0)\end{aligned}$$

So, we conclude that

$$\lim_{p\to\infty} M_{pj} = 0 \quad \text{for } j \geqslant 4.$$

By using $|\lambda_{mn}-\mu_m|<1$ ($m=0,1,2,\ldots$; $n=1,2,3,\ldots$), we can show in a similar way that

$$\|R_\lambda\|_{H_1} \leqslant C_4 p^{-1} \quad \text{for } C_4>0. \tag{2.21}$$

From (2.8), (2.19), (2.20) and (2.21) we obtain

$$\begin{aligned}|M_p^{(N)}| &= \frac{1}{2\pi}\left|\int_{|\lambda|=b_p} \lambda\operatorname{tr}\left[R_\lambda(QR_\lambda^0)^{N+1}\right]\mathrm{d}\lambda\right| \leqslant \int_{|\lambda|=b_p} |\lambda||\operatorname{tr}\left[R_\lambda(QR_\lambda^0)^{N+1}\right]|\mathrm{d}\lambda| \\ &\leqslant b_p\int_{|\lambda|=b_p} \|R_\lambda(QR_\lambda^0)^{N+1}\|_{\sigma_1(H_1)}|\mathrm{d}\lambda| \leqslant b_p\int_{|\lambda|=b_p} \|R_\lambda\|_{H_1}\|(QR_\lambda^0)^{N+1}\|_{\sigma_1(H_1)}|\mathrm{d}\lambda| \\ &\leqslant C_4 b_p p^{-1}\int_{|\lambda|=b_p} \|QR_\lambda^0\|_{H_1}^N\|QR_\lambda^0\|_{\sigma_1(H_1)}|\mathrm{d}\lambda| \leqslant C_4 b_p p^{-1}\|Q\|^N p^{-N}C_2 2\pi b_p \leqslant C_5 p^{3-N} \quad (C_5>0).\end{aligned}$$

This means that we have

$$\lim_{p\to\infty} M_p^{(N)} = 0; \quad N \geqslant 4 \tag{2.22}$$

We give our main result in the following theorem. □

Theorem 2.2. *Consider the operator function $Q(x)$ satisfying the conditions* (1)–(4)*. Then the formula for the regularized trace of L is in the form*

$$\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn}-\mu_m) - \frac{1}{\pi}\int_0^\pi \operatorname{tr}Q(x)\mathrm{d}x\right] = \frac{1}{4}[\operatorname{tr}Q(0) - \operatorname{tr}Q(\pi)].$$

Proof. From (2.9), (2.22) and Theorem 2.1 we find

$$\lim_{p\to\infty}\left[\sum_{m=0}^{p}\sum_{n=1}^{\infty}(\lambda_{mn}-\mu_m) - \frac{p+1}{\pi}\int_0^\pi \operatorname{tr}Q(x)\mathrm{d}x\right] = \frac{1}{\pi}\lim_{p\to\infty}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\int_0^\pi (Q(x)\varphi_n,\varphi_n)\cos(2m+1)x\,\mathrm{d}x.$$

This equality can be written as

$$\lim_{p\to\infty}\left[\sum_{m=0}^{p}\sum_{n=1}^{\infty}(\lambda_{mn}-\mu_m) - \sum_{m=0}^{p}\frac{1}{\pi}\int_0^\pi \operatorname{tr}Q(x)\mathrm{d}x\right] = \frac{1}{\pi}\lim_{p\to\infty}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\int_0^\pi (Q(x)\varphi_n,\varphi_n)\cos(2m+1)x\,\mathrm{d}x$$

or

$$\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn}-\mu_m) - \frac{1}{\pi}\int_0^\pi \operatorname{tr}Q(x)\mathrm{d}x\right] = \frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\int_0^\pi (Q(x)\varphi_n,\varphi_n)\cos(2m+1)x\,\mathrm{d}x. \tag{2.23}$$

Let us put

$$I = \frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos(2m+1)x\,\mathrm{d}x.$$

By definition of $Q(x)$ it is not difficult to show that

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos(2m+1)x\,\mathrm{d}x\right| < \infty. \tag{2.24}$$

Hence we obtain

$$\begin{aligned}
I &= \frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos(2m+1)x\,\mathrm{d}x\\
&= \frac{1}{2\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left[\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos mx\,\mathrm{d}x-(-1)^m\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos mx\,\mathrm{d}x\right]\\
&= \frac{1}{4}\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos mx\,\mathrm{d}x\right]\cos m0+\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\mathrm{d}x\right\}\\
&\quad-\frac{1}{4}\sum_{n=1}^{\infty}\left\{\sum_{m=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\cos mx\mathrm{d}x\right]\cos m\pi+\frac{1}{\pi}\int_0^{\pi}(Q(x)\varphi_n,\varphi_n)\mathrm{d}x\right\}.
\end{aligned}$$

The terms appear in $\{\cdots\}$ above are the values at the points 0 and π, respectively of the Fourier series of the function $(Q(x)\varphi_n,\varphi_n)$, which have continuous derivative of second order, with respect to $\{\cos mx\}_{m=0}^{\infty}$ in $[0,\pi]$.

Hence, we obtain

$$I = \frac{1}{4}\sum_{n=1}^{\infty}(Q(0)\varphi_n,\varphi_n)-\frac{1}{4}\sum_{n=1}^{\infty}(Q(\pi)\varphi_n,\varphi_n) = \frac{1}{4}[\mathrm{tr}Q(0)-\mathrm{tr}Q(\pi)]$$

and this proves our theorem. □

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