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Two parameter deformation of grassmann matrix group and supergroup

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Abstract

The two parameter quantum deformation of 2x2 Grassmann matrices, Gr(2), and supermatrices, Gr(1|1), are presented. Gr(2) whose matrix elements are all Grassmannian variables is called the superdual of the genel linear group GL(2), and Gr(1|1) whose diagonal matrix elements are Grassmannian variables is called the superdual of the supergroup GL(1|1) whose nondiagonal elements are Grassmannian. Noncentral dual superdeterminant for Grassmann supermatrices belonging to $\mathrm{Gr}_{p,q}(1|1)$ is constructed. As with the 2x2 quantum matrices, the relations satisfied by the matrix elements of the Grassmann matrices are expressed in terms of an $\widehat{R}$-matrix. The properties of nth power of a Grassmann supermatrix are given as an Appendix.

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I. INTRODUCTION

Quantum groups are a generalization of the concept of groups. More precisely, a quantum group is a deformation of a group that, for particular values of the deformation parameter, coincides with the group. The theory and applications of quantum groups have attracted a lot of attention among mathematicians and physicists. The main physical motivation for quantum groups is that when nonlinear physical systems which are classically completely integrable are quantized, th classical symmetry group should be replaced by the corresponding quantum group.[1] On the other hand, most of the difficulties involving the divergences of quantum field theories which lie at the heart of all interactions require supersymmetry and thus the introduction of supergroups.[2] The algebraic structure underlying quantum groups extends the theory of the supergroups.[3–8] In Ref. 9, the q-analog, $\mathrm{Gr}_q(1|1)$, of the dual supermatrices $\mathrm{Gr}(1|1)$ is presented. $\mathrm{Gr}_q(1|1)$ is the superdual of the quantum group $\mathrm{GL}_q(1|1)$ and the properties of the quantum dual supermatrices are discussed. In this paper we present a two parameter deformation, $\mathrm{Gr}_{p,q}(2)$ and $\mathrm{Gr}_{p,q}(1|1)$, of the Grassmann matrices and supermatrices, respectively and give an $\widehat{R}$-matrix for this deformation.

We will say that Grassmann matrices are the dual matrices in GL(2) and Grassmann supermatrices are the dual supermatrices in GL(1|1). To study the two parameter extension of the Grassmann matrices and supermatrices, we follow the approach of Manin[3] in Sec. II. In the following section we get an $\widehat{R}$-matrix which gives the relations between the matrix elements of a dual matrix in GL(1|1). The properties of the n-th power of a dual supermatrix which are more compact than the single deformation parameter case are given in Appendix.

II. THE QUANTUM GRASSMANN MATRIX GROUP $\mathbf{Gr}_{p,q}(2)$

Before discussing the two parameter deformation of the dual matrices in the general linear group GL(2), we give some notations and useful formulas.

A. Notations

Consider 2x2 matrices with Grassmannian entries. We will say that such matrices form the Grassmann matrix group and denote it by Gr(2). Explicitly, a Grassmannian 2x2 matrix $\widehat{A}$ is of the form

$$\widehat{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2.1}$$

where all entries are Grassmannian.

Since the matrix elements of $\widehat{A}$ are all Grassmannian, for the conventional tensor products

$$\widehat{A}_1 = \widehat{A} \otimes I \quad \text{and} \quad \widehat{A}_2 = I \otimes \widehat{A} \tag{2.2}$$

one can write (no-grading)

$$(\widehat{A}_1)^{ij}{}_{kl} = \widehat{A}^i{}_k \delta^j{}_l, \tag{2.3a}$$

$$(\widehat{A}_2)^{ij}{}_{kl} = \delta^i{}_k \widehat{A}^j{}_l \tag{2.3b}$$

where δ denotes the Kronecker delta.

B. Two parameter deformation of Gr(2)

The one parameter deformation of Grassmann matrices was given by Corrigan et al.[4] In this section, we will give a two parameter deformation Grassmann matrices, i.e., of $Gr(2)$. Let $R_p[2|0]$ be a quantum vector space which is two dimensional. The coordinates of a vector $V = (x, y)^T \in R_p[2|0]$ satisfy the bilinear product relation

$$xy - pyx = 0. \tag{2.4}$$

We consider a dual quantum vector space $R_q[0|2]$, the generators of which are Grassmannian. The coordinates of a (dual) vector $\widehat{V} = (\xi, \eta)^T \in R_q[0|2]$ satisfy the relations

$$\xi^2 = 0 = \eta^2, \ \eta\xi + q\xi\eta = 0 \tag{2.5}$$

as introduced in Ref. 3.

Now we want to define a two parameter deformation of the algebra of functions on the Grassmann matrix group Gr(2) as an associative algebra with unit, generated by the generators α, β, γ, and δ. For this, we considering linear transformations $\widehat{A}$ with the following properties:

$$\widehat{A} : R_p[2|0] \longrightarrow R_q[0|2], \tag{2.6a}$$

$$\widehat{A} : R_q[0|2] \longrightarrow R_p[2|0]. \tag{2.6b}$$

We assume that the matrix elements of $\widehat{A}$ commute with the coordinates of $R_p[2|0]$ and anti-commute with the coordinates of $R_q[0|2]$. Then the endomorphisms in (2.6) impose the following (p, q)-anti-commutation relations among the matrix elements of $\widehat{A}$:

$$\alpha\beta + p^{-1}\beta\alpha = 0, \qquad \alpha\gamma + q^{-1}\gamma\alpha = 0,$$

$$\gamma\delta + p^{-1}\delta\gamma = 0, \qquad \beta\delta + q^{-1}\delta\beta = 0, \tag{2.7}$$
$$\alpha\delta + \delta\alpha = 0, \qquad \alpha^2 = \beta^2 = \gamma^2 = \delta^2 = 0,$$
$$\beta\gamma + pq^{-1}\gamma\beta = (p - q^{-1})\delta\alpha$$

where p and q are non-zero complex numbers and $pq \pm 1 \neq 0$.

Since the entries of $\widehat{A}$ are all Grassmannian, a proper inverse cannot exist. However, the left and right inverses of $\widehat{A}$ can be constructed. Let

$$\Delta_L = \beta\gamma + q^{-1}\delta\alpha, \tag{2.8a}$$

$$\Delta_R = \gamma\beta - p^{-1}\alpha\delta. \tag{2.8b}$$

Then at least the formally, the left and right inverses of $\widehat{A}$ become

$$\widehat{A}_L^{-1} = \begin{pmatrix} q^{-1}\delta & \beta \\ -pq^{-1}\gamma & -p\alpha \end{pmatrix}, \tag{2.9}$$

$$\widehat{A}_R^{-1} = \begin{pmatrix} -q\delta & \beta \\ -qp^{-1}\gamma & p^{-1}\alpha \end{pmatrix}. \tag{2.10}$$

Indeed, it is easy to show that

$$\widehat{A}_L^{-1}\widehat{A} = \Delta_L I, \tag{2.11a}$$

$$\widehat{A}\widehat{A}_R^{-1} = \Delta_R I \tag{2.11b}$$

where I is the 2x2 unit matrix. In this case, Δ_L may be considered as a left quantum (dual) determinant and Δ_R as a right quantum (dual) determinant. Note that, one can write

$$\Delta_L \widehat{A}_R^{-1} = \widehat{A}_L^{-1}\Delta_R \tag{2.12}$$

using (2.8-10) and associativity of the algebra (2.7).

The algebra (2.7) is associative under multiplication and the relations in (2.7) may be also expressed in a tensor product form

$$\widehat{R}(1)\widehat{A}_1\widehat{A}_2 = -\widehat{A}_2\widehat{A}_1\widehat{R}(1) \tag{2.13}$$

where

$$\begin{aligned} \widehat{R}(x) \;=\; & (p + q^{-1}) \sum_i e_i^i \otimes e_i^i + 2x \sum_{i \neq j} (pq^{-1})^{i-1} e_i^i \otimes e_j^j + \\ & (p - q^{-1}) \left(\sum_{i>j} - \sum_{i<j} \right) e_j^i \otimes e_i^j \end{aligned}$$

Here the elements of the matrix $e^k{}_l$ are

$$\left(e^k{}_l\right)^i_j = \delta^i_k \delta^j_l. \tag{2.15}$$

The explicit form of $\widehat{R}(x)$ is

$$\widehat{R}(x) = \begin{pmatrix} p+q^{-1} & 0 & 0 & 0 \\ 0 & 2x & q^{-1}-p & 0 \\ 0 & p-q^{-1} & 2xpq^{-1} & 0 \\ 0 & 0 & 0 & p+q^{-1} \end{pmatrix}. \tag{2.16}$$

In terms of the matrix elements Eq. (2.13) is of the form

$$\widehat{R}^{ij}{}_{kl}\widehat{A}^k{}_m\widehat{A}^l{}_n = -\widehat{A}^i{}_l\widehat{A}^i{}_k\widehat{R}^{kl}{}_{ij}. \tag{2.17}$$

Finally, we note that the algebra (2.7) and the $\widehat{R}$-matrix in (2.16) with $p = q$ and $x = -1$ was given in Ref. 4 (Sec. III).

III. TWO PARAMETER DEFORMATION OF THE GRASSMANN MATRIX SUPERGROUP

In this section, we consider 2x2 supermatrices whose diagonal elements are Grassmannian. We remark that the supergroup GL(1|1) whose nondiagonal elements are Grassmannian is (p, q) deformed in Refs. 7 and 8. We will say that such supermatrices form the Grassmann supermatrix group and denote it by Gr(1|1). Explicity, a Grassmann 2x2 supermatrix $\widehat{\mathcal{A}}$ is of the form

$$\widehat{\mathcal{A}} = \begin{pmatrix} \alpha & b \\ c & \delta \end{pmatrix} \tag{3.1}$$

with two odd (greek letters) and two even (latin letters) matrix elements. Even matrix elements commute with everything and odd matrix elements anti-commute among themselves.

We begin with Manin's approach.[3] To do this, we consider the endomorphisms of a two-dimensional quantum superplane and its dual, denoted by $R_p[1|1]$ and $R^*_q[1|1]$, respectively.

$$U = \begin{pmatrix} x \\ \xi \end{pmatrix} \in R_p[1|1] \iff x\xi - p\xi x = 0,\ \xi^2 = 0, \tag{3.2}$$

and its dual

$$\widehat{U} = \begin{pmatrix} \eta \\ y \end{pmatrix} \in R^*_q[1|1] \iff \eta^2 = 0,\ \eta y - q^{-1} y\eta = 0. \tag{3.3}$$

Suppose that the matrix elements of $\widehat{\mathcal{A}}$ (anti-)commute with the coordinates of $R_p[1|1]$ and $R_q^*[1|1]$. Then, the endomorphisms

$$\widehat{\mathcal{A}} : R_p[1|1] \longrightarrow R_q^*[1|1], \tag{3.4a}$$

$$\widehat{\mathcal{A}} : R_q^*[1|1] \longrightarrow R_p[1|1] \tag{3.4b}$$

impose the following bilinear product relations among the generators of $\widehat{\mathcal{A}}$:

$$\alpha b = p^{-1} b \alpha, \qquad \alpha c = q^{-1} c \alpha, \tag{3.5a}$$

$$\delta b = p^{-1} b \delta, \qquad \delta c = q^{-1} c \delta, \tag{3.5b}$$

$$\alpha\delta + \delta\alpha = 0, \qquad \alpha^2 = 0 = \delta^2, \tag{3.5c}$$

$$bc = pq^{-1} cb + (p - q^{-1})\delta\alpha \tag{3.5d}$$

where p and q are non-zero complex numbers and $pq \pm 1 \neq 0$. These relations may be considered as a two parameter deformation of a Grassmann superalgebra on four elements (α, b, c, δ) where α and δ are Grassmannian elements. This deformed algebra denoted by $\mathrm{Gr}_{p,q}(1|1)$. For $p = q$, one obtains the one parameter deformation of the generators of $\widehat{\mathcal{A}}$ that was given in Ref. 9.

The inverse of $\widehat{\mathcal{A}}$ can be found as in Ref. 9 and it is of the form

$$\widehat{\mathcal{A}}^{-1} = \begin{pmatrix} -c^{-1}\delta b^{-1} & c^{-1} + c^{-1}\delta b^{-1}\alpha c^{-1} \\ b^{-1} + b^{-1}\alpha c^{-1}\delta b^{-1} & -b^{-1}\alpha c^{-1} \end{pmatrix} \tag{3.6}$$

provided that b and c are invertible. It is easy to verify that this is the proper right and left inverse of $\widehat{\mathcal{A}}$, i.e.

$$\widehat{\mathcal{A}}\widehat{\mathcal{A}}^{-1} = I = \widehat{\mathcal{A}}^{-1}\widehat{\mathcal{A}}.$$

Let

$$\widehat{\mathcal{A}}^{-1} = \begin{pmatrix} \alpha' & b' \\ c' & \delta' \end{pmatrix}.$$

Then, the matrix elements of $\widehat{\mathcal{A}}^{-1}$ satisfy the following relations

$$\begin{gathered} \alpha' b' = p b' \alpha', \qquad \alpha' c' = q c' \alpha', \\ \delta' b' = p b' \delta', \qquad \delta' c' = q c' \delta', \\ \alpha'\delta' + \delta'\alpha' = 0, \qquad \alpha'^2 = 0 = \delta'^2, \\ b'c' = qp^{-1} c'b' + (q - p^{-1})\delta'\alpha'. \end{gathered} \tag{3.7}$$

Therefore, the matrix elements of $\widehat{\mathcal{A}}^{-1}$ satisfy the (p^{-1}, q^{-1})-commutation relations while the matrix elements of $\widehat{\mathcal{A}}$ satisfy the (p, q)-commutation relations.

The quantum (dual) superdeterminant of $\widehat{\mathcal{A}}$ is defined as

$$s\widehat{D}_{p,q}(\widehat{\mathcal{A}}) = \widehat{D} = c^{-1}b - c^{-1}\alpha c^{-1}\delta = pq^{-1}(bc^{-1} - \alpha c^{-1}\delta c^{-1}) \tag{3.8}$$

which for $p = q$ is the same as $s\widehat{D}_q(\widehat{\mathcal{A}})$ in Ref. 9. The factor pq^{-1} in (3.8) appeared because of the relation (3.5d). Note that the second equality in (3.8) is obtained by using the relation

$$bc^{-1} = qp^{-1}c^{-1}b - (q - p^{-1})c^{-1}\delta\alpha c^{-1} \tag{3.9}$$

which in turn is obtained from equation (3.5d).

In general $\widehat{D}$, the quantum (dual) superdeterminant of $\widehat{\mathcal{A}}$, is not central but obeys the following commutation relations

$$\widehat{D}\alpha = pq^{-1}\alpha\widehat{D}, \quad \widehat{D}\delta = pq^{-1}\delta\widehat{D},$$

$$\widehat{D}b = pq^{-1}b\widehat{D}, \quad \widehat{D}c = pq^{-1}c\widehat{D}. \tag{3.10}$$

It is interesting that the quantum (dual) superdeterminant $\widehat{D}$ is not central while the quantum superdeterminant of a matrix $GL_{p,q}(1|1)$ is 7. Hovewer, it becomes central for $p = q$ as noted down in Ref. 9.

Before passing to the next section, we remark that the interesting point in the construction of (3.6) is the fact that the dual superdeterminant $\widehat{D}$ is not necessarily central.

IV. THE $\widehat{R}$-MATRIX

In this section, we give an $\widehat{R}$-matrix to obtain the relations (3.5). The algebra (3.5) is associative under multiplication and the relation (3.5) may be expressed in terms of a graded $\widehat{R}$-matrix condition, as with the quantum supermatrix. To this end, we use the tensoring convention

$$(\widehat{\mathcal{A}}_1)^{ij}{}_{kl} = (\widehat{\mathcal{A}} \otimes I)^{ij}{}_{kl} = (-1)^{k(j+l)}\widehat{\mathcal{A}}^i{}_k\delta^j{}_l = \widehat{\mathcal{A}}^i{}_k\delta^j{}_l, \tag{4.1a}$$

$$(\widehat{\mathcal{A}}_2)^{ij}{}_{kl} = (I \otimes \widehat{\mathcal{A}})^{ij}{}_{kl} = (-1)^{i(j+l)}\widehat{\mathcal{A}}^j{}_l\delta^i{}_k. \tag{4.1a}$$

The explicit form of $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$ is

$$\widehat{\mathcal{A}}_1 = \begin{pmatrix} \alpha & 0 & b & 0 \\ 0 & \alpha & 0 & b \\ c & 0 & \delta & 0 \\ 0 & c & 0 & \delta \end{pmatrix}, \tag{4.2a}$$

$$\widehat{\mathcal{A}}_2 = \begin{pmatrix} -\alpha & -b & 0 & 0 \\ -c & -\delta & 0 & 0 \\ 0 & 0 & -\alpha & b \\ 0 & 0 & c & -\delta \end{pmatrix}. \tag{4.2b}$$

Then the associative algebra (3.5) is equivalent to equation

$$\widehat{R}(-1)\widehat{\mathcal{A}}_1\widehat{\mathcal{A}}_2 = -\widehat{\mathcal{A}}_2\widehat{\mathcal{A}}_1\widehat{R}(-1) \tag{4.3}$$

where

$$\widehat{R}(-1) = \begin{pmatrix} p+q^{-1} & 0 & 0 & 0 \\ 0 & -2 & q^{-1}-p & 0 \\ 0 & p-q^{-1} & -2pq^{-1} & 0 \\ 0 & 0 & 0 & p+q^{-1} \end{pmatrix}. \tag{4.4}$$

This $\widehat{R}(-1)$-matrix obtained from (2.14) with $x=-1$. Here a 4x4 matrix in the form (4.2a) is labeled in the following way

$$M = \begin{pmatrix} {M^{11}}_{11} & {M^{11}}_{12} & {M^{11}}_{21} & {M^{11}}_{22} \\ {M^{12}}_{11} & {M^{12}}_{12} & {M^{12}}_{21} & {M^{12}}_{22} \\ {M^{21}}_{11} & {M^{21}}_{12} & {M^{21}}_{21} & {M^{21}}_{22} \\ {M^{22}}_{11} & {M^{22}}_{12} & {M^{22}}_{21} & {M^{22}}_{22} \end{pmatrix} \tag{4.5}$$

similar to Ref. 6.

We have given the (p,q)-commutation relations which satisfied by the matrix elements of a Grassmannian matrix and a Grassmannian supermatrix, i.e. we made a two parameter deformation of the Grassmann matrix group Gr(2) and the supermatrix group Gr(1|1). We obtained the Grassmannian quantum superdeterminant of a Grassmannian quantum supermatrix (p,q)-deformed case. Hovewer, it reduces to the case discussed in Ref. 9 for $p=q$. We have given an $\widehat{R}$-matrix which by use of a tensor product gives the (p,q)-commutation relations between the matrix elements of a Grassmannian supermatrix.

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APPENDIX: THE PROPERTIES OF THE nth POWER OF GRASSMANN SUPERMARICES

Here we will discuss the properties of the nth power of a Grassmann supermatrix. First we note that the product of two Grassmann supermatrices is not a Grassmann supermatrix, i.e., the matrix elements of a product $M=$

$\widehat{M}\widehat{M}'$ do not satisfy (3.5). Hovewer, $\widehat{M}\widehat{M}' \in GL_{p,q}(1|1)$ if $\widehat{M}$ and $\widehat{M}'$ are two Grassmann supermatrices and (b, c) $((\alpha, \delta))$ pairwise commute (anti-commute) with (b', c') $((\alpha', \delta'))$. So, we must consider the matrix elements of $\widehat{M}^n$ with respect to even and odd values of n. Let the $(2n-1)$-th power of $\widehat{M}$ be

$$\widehat{M}^{2n-1} = \begin{pmatrix} A_{2n-1} & B_{2n-1} \\ & \\ C_{2n-1} & D_{2n-1} \end{pmatrix}, \ n \geq 1. \tag{A1}$$

After some algebra, one obtains

$$\begin{aligned} A_{2n-1} &= \{< n >_{pq} \alpha + p < n-1 >_{pq} \delta\} (bc)^{n-1}, \\ B_{2n-1} &= \{bc + p < n-1 >_{p^2q^2} \alpha\delta\} (bc)^{n-2} b, \\ C_{2n-1} &= \{cb + q < n-1 >_{p^2q^2} \delta\alpha\} (cb)^{n-2} c, \\ D_{2n-1} &= \{< n >_{pq} \delta + q < n-1 >_{pq} \alpha\} (cb)^{n-1}, \end{aligned} \tag{A2}$$

where

$$< N >_{pq} = \frac{1 - p^N q^N}{1 - pq}. \tag{A3}$$

Now it is easy to show the following relations are satisfied:

$$\begin{aligned} A_{2n-1}B_{2n-1} &= p^{-(2n-1)} B_{2n-1}A_{2n-1} \\ A_{2n-1}C_{2n-1} &= p^{-(2n-1)} C_{2n-1}A_{2n-1} \\ D_{2n-1}B_{2n-1} &= q^{-(2n-1)} B_{2n-1}D_{2n-1} \\ D_{2n-1}C_{2n-1} &= q^{-(2n-1)} C_{2n-1}D_{2n-1}, \\ A_{2n-1}D_{2n-1} + D_{2n-1}A_{2n-1} &= 0, \\ A_{2n-1}^2 = 0 &= D_{2n-1}^2, \\ B_{2n-1}C_{2n-1} = p^{2n-1}q^{-(2n-1)} C_{2n-1}B_{2n-1} &+ \left(p^{2n-1} - q^{-(2n-1)}\right) A_{2n-1}D_{2n-1}. \end{aligned} \tag{A4}$$

Thus, $\widehat{M}^{2n-1}$ is a Grassmann supermatrix with deformation parameters p^{2n-1} and q^{2n-1}, i.e. $\widehat{M}^{2n-1} \in Gr_{p^{2n-1},q^{2n-1}}(1|1)$.

Similarly, if we write the matrix $\widehat{M}^{2n}$, the (2n)-th power of $\widehat{M}$ as

$$\widehat{M}^{2n} = \begin{pmatrix} A_{2n} & B_{2n} \\ & \\ C_{2n} & D_{2n} \end{pmatrix}, \ n \geq 1 \tag{A5}$$

where

$$A_{2n} = \left\{ bc + p\frac{1-pq}{1+pq} < n >_{pq} < n-1 >_{pq} \alpha\delta \right\} (bc)^{n-1},$$

$$B_{2n} =< n >_{pq} \{\alpha + p\delta\} (bc)^{n-1} b, \qquad (A6)$$
$$C_{2n} =< n >_{pq} \{\delta + q\alpha\} (cb)^{n-1} c,$$
$$D_{2n} = \left\{ cb + q\frac{1-pq}{1+pq} < n >_{pq} < n-1 >_{pq} \delta\alpha \right\} (cb)^{n-1},$$

then the elements of $\widehat{M}^{2n}$ obey the following relations

$$A_{2n}B_{2n} = p^{2n} B_{2n} A_{2n}, \quad A_{2n}C_{2n} = p^{2n} C_{2n} A_{2n}$$
$$D_{2n}B_{2n} = q^{2n} B_{2n} D_{2n}, \quad D_{2n}C_{2n} = q^{2n} C_{2n} D_{2n}, \qquad (A7)$$
$$B_{2n}C_{2n} + p^n q^{-n} C_{2n} B_{2n} = 0,$$
$$B_{2n}^2 = 0 = C_{2n}^2,$$
$$A_{2n}D_{2n} - D_{2n}A_{2n} = (p^{2n} - q^{-2n}) C_{2n} B_{2n}.$$

Thus the matrix $\widehat{M}^{2n}$ is a supermatrix in the form of

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \qquad (A8)$$

with the deformation parameters p^{2n} and q^{2n}. The n-th power of such a supermatrix and the relations between the matrix elements of T^n can be found in Ref. 10 (Sec. 3).

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