Differential Equations

Definition: An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables is said to be a "differential equation".

Generally we consider the one dependent and one independent form

$$F(x, y, y', y'', \dots, y^n) = 0 \quad \text{or}$$

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0 \quad n \in \mathbb{Z}^+$$

Classification of Differential Equations

1. By Type

If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, then it's said to be an "ordinary D.E". Otherwise, it's called "partial D.E".

Examples

$$\left.\begin{array}{ll} \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = te^t \Rightarrow & \begin{array}{l} y \to \text{dependent} \\ t \to \text{independent} \end{array} \\[2ex] y^{IV} - 2y' = x + 4 \to & \begin{array}{l} y \to \text{dependent} \\ x \to \text{indepent} \end{array} \\[2ex] \dfrac{dx}{dt} + \dfrac{dy}{dt} = 2x + y \Rightarrow & \begin{array}{l} x, y \to \text{dependent} \\ t \to \text{independent} \end{array} \end{array}\right\} \text{Ordinary D.E}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xy + 1 \rightarrow \begin{array}{l} u \rightarrow \text{dependent} \\ x, y \rightarrow \text{independent} \end{array}$$

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} - 2u = 0 \rightarrow \begin{array}{l} u \rightarrow \text{dependent} \\ x, y \rightarrow \text{independent} \end{array}$$

Partial D.E

2. By Order

The order of a D.E is the order of the highest derivative in the equation.

$y^{IV} - y'' = x$ forth order ODE

$y'' - (y')^2 = y$ second order ODE

$\frac{\partial^3 u}{\partial x^2 \partial y} = u$ third order PDE

First order ODEs are mostly written in the forms

$y' = f(x, y)$ (explicit form)

$F(x, y, y') = 0$ (implicit form)

$m(x, y)dx + N(x, y)dy = 0$ (differential form) $(y' = \frac{dy}{dx})$

3. By Linearity

The degree of a D.E is the degree of the highest derivative appearing in the equation.

$\frac{d^2 y}{dx^2} + 4y = 0$ → second order, first degree

$(y''')^2 + (y')^3 = 4y$ → third order, second degree

If the degree of each dependent term is 1, then D.E is linear. Otherwise, it's called nonlinear D.E.

$y'' - 2y' + y = x^2$ → second order linear ODE

$(y')^3 + y = 1$ → first order nonlinear ODE

4. By Coefficients

A linear ODE of order n is in the form

$$a_0(x) y^n + a_1(x) y^{n-1} + \dots + a_n y = f(x) \quad [a_0(x) \neq 0]$$

★ If all $\underset{(i=1,\dots,n)}{a_i(x)}$ are constants, then it's called "ODE with constant coefficients."

★ If at least one coefficient is variable, then it's called "ODE with variable coefficients."

$xy'' + 4y = 0$ D.E with variable coefficients.

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$ D.E with constant coefficiets.

Examples

★ $\frac{dy}{dx} + x^2 y = xe^x$ first order linear ODE with variable coefficients

★ $(y'')^3 + 4y = \sin x$ second order nonlinear ODE with constant coefficients.

★ $yy''' + 6y' = 0$ third order nonlinear ODE with constant coeffients.

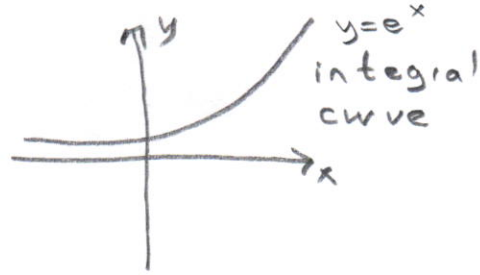
★ $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = z$ second order linear PDE with constant coefficients.

Solution of an ODE

A solution to an ODE on an interval I is any function $y=f(x)$ which is defined on I and satisfies the ODE. The graph of $y=f(x)$ is called as integral (solution) curves.

For example, $y=e^x$ is the solution of $y''-2y'+y=0$

$$\left.\begin{aligned} y&=e^x \\ y'&=e^x \\ y''&=e^x \end{aligned}\right\} e^x-2e^x+e^x=0 \quad \checkmark$$



Explicit and Implicit Solutions

A relation $G(x,y)=0$ is called implicit solution.

A relation $y=f(x)$ is called explicit solution.

★ $x^2y^2+xy=4$ implicit solution

★ $y=x^2+\sin x+4$ explicit solution

Mainly, we have 3 kinds of solutions:

1. General Solution

Consider the nth order ODE

$$F(x,y,y',\ldots,y^n)=0$$

and the following family of function with n-parameter

$$F(x,y,c_1,c_2,\ldots,c_n)=0$$

where $c_1,c_2,\ldots,c_n$ are arbitrary constants.