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Comment on the differential calculus on the quantum exterior plane

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Abstract

We give a two-parameter quantum deformation of the exterior plane and its differential calculus without the use of any R-matrix and relate it to the differential calculus with the R-matrix. We prove that there are two types of solutions of the Yang-Baxter equation whose symmetry group is $GL_{p,q}(2)$. We also give a two-parameter deformation of the fermionic oscillator algebra.

Quantum groups are a generalization of the concept of groups. During the past few years, these new mathematical objects have found wide interest among theoretical physicsts and mathematicians. More precisely, the quantum group is an example of a Hopf algebra which is related to noncommutative geometry.[1] After Wess-Zumino introduced the differential calculus on the quantum (hyper)plane, the quantum plane was generalized to the supersymmetric quantum (super)plane by some authors.[2–5]

In this letter we shall give a differential calculus on the quantum exterior plane whose symmetry group is $GL_{p,q}(2)$ and obtain two R-matrices. They are both solutions of the Yang-Baxter equation.

Let us begin with the quantum exterior (dual) plane, defined as the polynomial ring generated by coordinates θ, ϕ which satisfy[6]

$$\theta\phi + p^{-1}\phi\theta = 0 \tag{1}$$

$$\theta^2 = 0 = \phi^2 \tag{2}$$

where p is a complex deformation parameter. To develop the differential calculus on the p-exterior plane in terms of the coordinates satisfying (1), (2), we shall make the following ansatz for the commutation relations of the coordinates with their differentials. Let the differentials of coordinates be denoted by

$$\Theta = d\theta \qquad \Phi = d\phi. \tag{3}$$

In Ref. 1, Wess and Zumino have interpreted θ and ϕ as the differentials of coordinates of the quantum plane, respectively. Recall that the quantum plane is defined as the polynomial ring generated by coordinates x, y obeying the relation

$$xy = qyx.$$

As an alternative to this interpration, x and y can be identified with the differentials of θ and ϕ as in Ref. 6. The definition (3) is closely related to this approach. We assume that

$$\theta\Theta = A\Theta\theta \qquad \theta\Phi = F_{11}\Phi\theta + F_{12}\Theta\phi$$

$$\phi\Phi = B\Phi\phi \qquad \phi\Theta = F_{21}\Theta\phi + F_{22}\Phi\theta. \tag{4}$$

Then we find, from (2),

$$A = 1 \qquad B = 1. \tag{5}$$

The consistency condition

$$d(\theta\phi + p^{-1}\phi\theta) = 0$$

$$(\theta\phi + p^{-1}\phi\theta)\Theta = 0 \qquad (\theta\phi + p^{-1}\phi\theta)\Phi = 0 \tag{6}$$

gives

$$F_{22} + pF_{11} = 1 \qquad F_{21} + pF_{12} = p \qquad F_{12}F_{22} = 0. \tag{7}$$

We define the exterior derivative d obeying the condition:

$$\mathsf{d}^2 = 0 \tag{8}$$

and the graded Leibniz rule

$$\mathsf{d}(fg) = (\mathsf{d}f)g + (-1)^{\hat{f}} f(\mathsf{d}g) \tag{9}$$

where $\hat{f} = 0$ for even variables and $\hat{f} = 1$ for odd variables. Applying the exterior differential d on second and fourth relations of eq.(4) and using (8), (9) one gets

$$F_{11}F_{21} = 1 - F_{12} - F_{22}. \tag{10}$$

We now assume that the commutation relation of differentials has the form

$$\Theta\Phi = q\Phi\Theta \tag{11}$$

where q is another complex deformation parameter. Then we have

$$F_{11} = q(1 - F_{12}) \qquad F_{21} = q^{-1}(1 - F_{22}). \tag{12}$$

The system (7), (10), (12) admits two solutions:

Type I

$$\begin{aligned} A = 1 \qquad F_{11} = q \qquad F_{21} = p \\ B = 1 \qquad F_{12} = 0 \qquad F_{22} = 1 - pq \end{aligned} \tag{13}$$

and the relations (4) take the form

$$\begin{aligned} \theta\Theta = \Theta\theta \qquad \theta\Phi = q\Phi\theta \\ \phi\Phi = \Phi\phi \qquad \phi\Theta = p\Theta\phi + (1 - pq)\Phi\theta. \end{aligned} \tag{14}$$

Type II

$$\begin{aligned} A = 1 \qquad F_{11} = p^{-1} \qquad F_{21} = q^{-1} \\ B = 1 \qquad F_{12} = 1 - p^{-1}q^{-1} \qquad F_{22} = 0 \end{aligned} \tag{15}$$

and in this case the relations (4) take the form

$$\begin{aligned} \theta\Theta = \Theta\theta \qquad \theta\Phi = p^{-1}\Phi\theta - (1 - p^{-1}q^{-1}\Theta\phi \\ \phi\Phi = \Phi\phi \qquad \phi\Theta = q^{-1}\Theta\phi. \end{aligned} \tag{16}$$

To complete the differential geometric scheme we introduce derivatives of the quantum exterior plane in the standard way

$$\mathsf{d} = \Theta\partial_\theta + \Phi\partial_\phi. \tag{17}$$

Multiplying this expression from the right by θf and ϕf, respectively, and using the graded Leibniz rule for partial derivatives

$$\partial_i(fg) = (\partial_i f)g + (-1)^{\hat{f}} f(\partial_i g) \tag{18}$$

one finds

$$\begin{aligned} \partial_\theta\theta &= 1 - \theta\partial_\theta - F_{12}\phi\partial_\phi \qquad \partial_\theta\phi = -F_{21}\phi\partial_\theta \\ \partial_\phi\phi &= 1 - \phi\partial_\phi - F_{22}\theta\partial_\theta \qquad \partial_\phi\theta = -F_{11}\theta\partial_\phi. \end{aligned} \tag{19}$$

These commutation relations, for type I, are

$$\partial_\theta\theta = 1 - \theta\partial_\theta \qquad \partial_\theta\phi = -p\phi\partial_\theta$$

$$\partial_\phi\phi = 1 - \phi\partial_\phi + (pq-1)\theta\partial_\theta \qquad \partial_\phi\theta = -q\theta\partial_\phi \tag{20a}$$

and for type II, are

$$\partial_\theta\theta = 1 - \theta\partial_\theta + (p^{-1}q^{-1}-1)\phi\partial_\phi \qquad \partial_\theta\phi = -q^{-1}\phi\partial_\theta$$

$$\partial_\phi\phi = 1 - \phi\partial_\phi \qquad \partial_\phi\theta = -p^{-1}\theta\partial_\phi. \tag{20b}$$

The commutation relations between the derivatives can be easily obtained by using that $\mathsf{d}^2 = 0$. So it follows that

$$0 = \mathsf{d}^2 = \Theta^2\partial_\theta^2 + \Phi\Theta(\partial_\theta\partial_\phi + q\partial_\phi\partial_\theta) + \Phi^2\partial_\phi^2$$

which says that

$$\partial_\theta\partial_\phi + q\partial_\phi\partial_\theta = 0 \qquad \partial_\theta^2 = 0 = \partial_\phi^2. \tag{21}$$

Finally to find the commutation rules between the differentials and derivatives we shall assume that they have the following form

$$\begin{aligned} \partial_\theta\Theta &= A_{11}\Theta\partial_\theta + A_{12}\Phi\partial_\phi \\ \partial_\theta\Phi &= A_{21}\Phi\partial_\theta + A_{22}\Theta\partial_\phi \\ \partial_\phi\Theta &= B_{11}\Theta\partial_\phi + B_{12}\Phi\partial_\theta \\ \partial_\phi\Phi &= B_{21}\Phi\partial_\phi + B_{22}\Theta\partial_\theta. \end{aligned} \tag{22}$$

From the fact that

$$\partial_i(\theta^j\Theta^k) = \delta_j^i\delta_l^k\Theta^k \tag{23}$$

where $\partial_1 = \partial_\theta$, $\theta^1 = \theta$, $\Theta^1 = \Theta$, etc. we have

$$A_{11} = 1 \qquad A_{22} = 0 \qquad B_{21} = 1 \qquad B_{12} = 0$$

$$F_{11}A_{21} + F_{12}A_{12} = 1 \qquad F_{21}A_{12} + F_{22}A_{21} = 0$$
$$F_{21}B_{11} + F_{22}B_{22} = 1 \qquad F_{11}B_{22} + F_{12}B_{11} = 0. \tag{24}$$

Thus there are two types of solutions.

Type I

$$\partial_\theta\Theta = \Theta\partial_\theta + (1 - p^{-1}q^{-1})\Phi\partial_\phi \qquad \partial_\theta\Phi = q^{-1}\Phi\partial_\theta \tag{25a}$$
$$\partial_\phi\Theta = p^{-1}\Theta\partial_\phi \qquad \partial_\phi\Phi = \Phi\partial_\phi$$

and

Type II

$$\partial_\theta\Theta = \Theta\partial_\theta \qquad \partial_\theta\Phi = p\Phi\partial_\theta \tag{25b}$$
$$\partial_\phi\Theta = q\Theta\partial_\phi \qquad \partial_\phi\Phi = \Phi\partial_\phi + (1 - pq)\Theta\partial_\theta.$$

We now should compute the R-matrix for these two types of solutions satisfying the Yang-Baxter equations

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \qquad \hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{26}$$

where

$$\hat{R}^{ij}_{kl} = R^{ji}_{kl}. \tag{27}$$

From the definition of R matrix for the Yang-Baxter equation

$$\theta^i\Theta^j = R^{ji}_{kl}\Theta^k\theta^l \tag{28}$$

we have

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & F_{21} & F_{22} & 0 \\ 0 & F_{12} & F_{11} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (R^{ij}{}_{kl}). \tag{29}$$

The R matrix for type I is

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 1 - pq & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{30a}$$

and for type II is

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 1-p^{-1}q^{-1} & p^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{30b}$$

It can be checked that both matrices in (30) satisfy the Yang-Baxter equation (26). Note that the matrix in (30a) becomes the transposition of the inverse of the matrix in (30b) for $p = q$.

We formulate the differential calculus, with the R-matrix, as follows:

Type I, II

$$\begin{gathered} \theta^i\theta^j = -\frac{1}{pq}\hat{R}^{ij}{}_{kl}\theta^k\theta^l \qquad \theta^i\theta^j = -pq\hat{R}^{ij}{}_{kl}\theta^k\theta^l \\ \theta^i\Theta^j = \hat{R}^{ij}{}_{kl}\Theta^k\theta^l \\ \partial_i\theta^j = \delta^j{}_i - \hat{R}^{jk}{}_{il}\theta^l\partial_k \\ \partial_i\Theta^j = (\hat{R}^{-1})^{kj}{}_{li}\Theta^l\partial_k \\ \partial_i\partial_j = -\frac{1}{pq}\hat{R}^{kl}{}_{ji}\partial_l\partial_k \qquad \partial_i\partial_j = -pq\hat{R}^{kl}{}_{ji}\partial_l\partial_k. \end{gathered} \tag{31}$$

To deform the group $GL(2)$, as two-parametric, we can use both R-matrices above, too. The equation

$$\hat{R}T_1T_2 = T_1T_2\hat{R} \tag{32}$$

gives[8] the (p, q)-commutation relations between the matrix elements of any matrix T in $GL_{p,q}(2)$, where $T_1 = T \otimes I$ and $T_2 = I \otimes T$ in the standard notation. One can verify that the relations (1), (2) and (11) are preserved under the transformations

$$\begin{gathered} \theta \longrightarrow a\theta + b\phi \qquad \Theta \longrightarrow a\Theta + b\Phi \\ \phi \longrightarrow c\theta + d\phi \qquad \Phi \longrightarrow c\Theta + d\Phi \end{gathered} \tag{33}$$

provided (32) satisfied. Here, it assummed that the coordinates and differentials commute with the matrix elements a, b, c and d. In the same sense the relations (14) [or (16)] are preserved under the action of the quantum matrix $T \in GL_{p,q}(2)$, and so. Consequently, two-parameter calculus is covariant under the action of the quantum group $GL_{p,q}(2)$.

We finally obtain a two-parameter deformation of fermionic oscillator algebra. For this, we assume that

$$p = \bar{q}. \tag{34}$$

We then introduce two fermionic oscillators, B_1 and B_2, and make identification

$$\begin{aligned} \theta &\longrightarrow B_1^+ \qquad \partial_\theta \longrightarrow B_1 \\ \phi &\longrightarrow B_2^+ \qquad \partial_\phi \longrightarrow B_2 \end{aligned} \tag{35}$$

we construct the (p,q)-deformed fermionic oscillator algebra as follows:

$$\begin{aligned} B_1 B_2 + q B_2 B_1 = 0 \qquad & B_1^2 = 0 = B_1^{+2} \\ B_1^+ B_2^+ + p^{-1} B_2^+ B_1^+ = 0 \qquad & B_2^2 = 0 = B_2^{+2} \end{aligned} \tag{36}$$

Type I

$$\begin{aligned} B_1 B_2^+ + p B_2^+ B_1 = 0 \qquad & B_2 B_1^+ + q B_1^+ B_2 = 0 \\ B_1 B_1^+ + B_1^+ B_1 = 1 \qquad & B_2 B_2^+ + B_2^+ B_2 = 1 + (pq-1) B_1^+ B_1 \end{aligned} \tag{37a}$$

Type II

$$\begin{aligned} B_1 B_2^+ + q^{-1} B_2^+ B_1 = 0 \qquad & B_2 B_1^+ + p^{-1} B_1^+ B_2 = 0 \\ B_1 B_1^+ + B_1^+ B_1 = 1 + (p^{-1}q^{-1} - 1) B_2^+ B_2 \qquad & B_2 B_2^+ + B_2^+ B_2 = 1. \end{aligned} \tag{37b}$$

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