4) Exact Differential Equations:

Consider the DE:

\[ M(x,y)\,dx + N(x,y)\,dy = 0 \quad (1) \]

where \( M \) and \( N \) have continuous first partial derivatives at all points \((x,y)\) in a rectangular domain \( D \). 

- Exactness test:

If 
\[ \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \quad (2) \quad \text{for all} \quad (x,y) \in D, \]

then the DE (1) is exact in \( D \).

If (2) holds, then (1) is exact. If (2) does not hold, then (1) is not exact.

Necessary and sufficient condition that (1) be exact in \( D \) is that (2) holds for all \((x,y) \in D \).

Solution method: let the solution of an exact DE be the function \( u(x,y) = k \) (\( k \) : constant).

The total differential of \( u \) is

\[ du(x,y) = \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy = 0 = M(x,y)\,dx + N(x,y)\,dy \]

\[ \frac{\partial u}{\partial x} = M(x,y) \quad (3) \quad \frac{\partial u}{\partial y} = N(x,y) \quad (4) \]

Thus, we aim to find a function \( u(x,y) \) which satisfies (3) and (4).

Let us assume that \( u \) satisfies (3). Then with integrating (3), we obtain

\[ u(x,y) = \int M(x,y)\,dx + \phi(y) \quad (5) \]
where \( \int M(x,y) \) indicates a partial integration with respect to \( x \), holding \( y \) constant, and \( \phi \) is an arbitrary function of \( y \) only. This \( \phi(y) \) is needed in (5) so that \( u(x,y) \) given by (5) will represent all solutions of (3). It corresponds to a constant of integration in the one-variable case. Differentiating (5) partially with respect to \( y \), we obtain,

\[
\frac{\partial u(x,y)}{\partial y} = \frac{\partial}{\partial y} \int M(x,y) \partial x + \frac{\partial \phi(y)}{\partial y}
\]

Now, if (6) is to be satisfied, we must have

\[
N(x,y) = \frac{\partial}{\partial y} \int M(x,y) \partial x + \frac{\partial \phi(y)}{\partial y} \quad \text{(6)}
\]

and hence

\[
\frac{\partial \phi(y)}{\partial y} = N(x,y) - \frac{\partial}{\partial y} \int M(x,y) \partial x.
\]

Since \( \phi \) is a function of \( y \) only, the derivative \( \frac{\partial \phi}{\partial y} \) must also be independent of \( x \) (mean is a function of \( y \) only too).

That is, in order for (6) to hold,

\[
N(x,y) - \frac{\partial}{\partial y} \int M(x,y) \partial x = \eta \quad \text{(7)}
\]

must be independent of \( x \). So, we shall show that

\[
\frac{\partial}{\partial y} N(x,y) - \frac{\partial}{\partial y} \int M(x,y) \partial y = 0 \quad \text{w.r.t. } x \text{ is equal to zero.}
\]
We at once have
\[
\frac{\partial}{\partial x} \left[ N(x,y) - \frac{\partial}{\partial y} \int M(x,y) \, dx \right] = \frac{\partial N(x,y)}{\partial x} - \frac{\partial}{\partial x} \int M(x,y) \, dx
\]
\[
= \frac{\partial N(x,y)}{\partial x} - \frac{\partial^2}{\partial x \partial y} \int M(x,y) \, dx
\]
\[
= \frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y} = 0
\]
by using (2); these terms equal, so the result is 0.

Thus we may write \( \varphi(y) = \int \left[ N(x,y) - \frac{\partial M(x,y)}{\partial x} \right] dy \)

Substituting this into Equation (5)
\[
u(x,y) = \int M(x,y) \, dx + \int \left[ N(x,y) - \frac{\partial M(x,y)}{\partial x} \right] dy
\]

(8) is the solution of the DE (1). However, in solving exact DEs it is neither necessary nor desirable to use (8). Instead one can obtain \( u(x,y) \) by proceeding as in the solution method.

Example: Solve DE \( (3x^2 + 4xy) \, dx + (2x^2 + 2y) \, dy = 0 \).

Our first duty is to determine whether or not the given equation is exact.

\[ M(x,y) = 3x^2 + 4xy \quad , \quad N(x,y) = 2x^2 + 2y \]
\[ \frac{\partial M}{\partial y} = 4x \quad (x \text{ constant}) \quad , \quad \frac{\partial N}{\partial x} = 4x \quad (y \text{ constant}) \]
\[ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 4 \Rightarrow \text{So the DE is exact. Thus we must find } u(x,y) \text{ such that}, \]
\[\frac{\partial u(x, y)}{\partial x} = M(x, y) = 3x^2 + uy \]

and

\[\frac{\partial u(x, y)}{\partial y} = N(x, y) = 2x^2 + 2y\]

From the first of these:

\[u(x, y) = \int M(x, y) \, dx + \varphi(y)\]

\[u(x, y) = \int (3x^2 + uy) \, dx + \varphi(y)\]

\[u(x, y) = x^3 + 2x^2y + \varphi(y)\]

\[\frac{\partial u}{\partial y} = N(x, y) = 2x^2 + 2y = 2y^2 + \frac{d\varphi(y)}{dy}\]

\[\int \frac{d\varphi(y)}{dy} = 2y\]

\[\varphi(y) = y^2 + c_0\]

So the solution of the given DE is:

\[u(x, y) = x^3 + 2x^2y + y^2 + c_0\]

or we can write:

\[x^3 + 2x^2y + y^2 + c_0 = c_1 \implies x^3 + 2x^2y + y^2 + c_0 - c_1 = 0\]

\[x^3 + 2x^2y + y^2 + c = 0\]
Same Example: \( (3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0 \)

\[
\begin{align*}
\frac{\partial M}{\partial y} &= 4x \\
\frac{\partial N}{\partial x} &= 4x \
\Rightarrow \text{DE is exact.}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial x} &= 3x^2 + 4xy \\
\frac{\partial u}{\partial y} &= 2x^2 + 2y
\end{align*}
\]

From the second of these:

\[
\frac{\partial u}{\partial y} = 2x^2 + 2y
\]

\[
\int \frac{\partial u}{\partial y} \, dy = \int (2x^2 + 2y) \, dy
\]

\[u(x, y) = 2x^2y + y^2 + \phi(x)\]

\[
\frac{\partial u}{\partial x} = (xy \, y' + \frac{d\phi(x)}{dx}) = 3x^2 + 4xy
\]

\[
\phi(x) = \int 3x^2 \, dx + C
\]

\[
\phi(x) = x^3 + C
\]

\[u(x, y) = 2x^2y + y^2 + x^3 + C \quad \text{general solution.}
\]

Example: \( (x + y\cos x)\,dx + (y + \sin y)\,dy = 0 \) find the G.S.

\[
\begin{align*}
\frac{\partial M}{\partial y} &= \cos x \\
\frac{\partial N}{\partial x} &= \cos x \
\frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x} \Rightarrow \text{Exact DE.}
\end{align*}
\]

So we can find \( u = u(x, y) \) such that

\[
\frac{\partial u}{\partial x} = (x + y\cos x) \text{ and } \frac{\partial u}{\partial y} = (y + \sin y)
\]
\[ \frac{\partial u}{\partial x} = x + \gamma \cos x \]

\[ u = \int (x + \gamma \cos x) \, dx + \varphi(y) \]

\[ u = \frac{x^2}{2} + \gamma \sin x + \varphi(y) \]

\[ \frac{\partial u}{\partial y} = \sin x + \frac{d\varphi(y)}{dy} = \gamma + \sin x \]

Thus \[ \frac{d\varphi(y)}{dy} = \gamma \]

\[ \varphi(y) = \frac{y^2}{2} + C \]

\[ u(x, y) = \frac{x^2}{2} + \gamma \sin x + \frac{y^2}{2} + C = k \]

\[ \frac{x^2 + \gamma \sin x + \frac{y^2}{2} + C = 0}{\text{Linear:}} \]

\[ \frac{\partial u}{\partial y} = \gamma + \sin x \]

\[ u = \int (\gamma + \sin x) \, dy + g(x) \]

\[ u = \frac{x^2}{2} + \gamma \sin x + g(x) \]

\[ \frac{\partial u}{\partial x} = \gamma \cos x + \frac{dg(x)}{dx} = x + \gamma \cos x \]

\[ \frac{dg(x)}{dx} = x \]

\[ g(x) = \frac{x^2}{2} + C \]

\[ u = \frac{x^2}{2} + \gamma \sin x + \frac{x^2}{2} + C \]

\[ \gamma^2 + \gamma \sin x + \frac{x^2}{2} = C \]
Example: \( (x^2e^y + \frac{x}{y} - 3y^2 + 1)\,dy + (2xe^y + 1ny + \frac{1}{\sqrt{1-x^2}})\,dx \)

Find the O.S. \( M(x,y) = 2xe^y + 1ny + \frac{1}{\sqrt{1-x^2}} \) \( N(x,y) = x^2e^y + \frac{x}{y} - 3y^2 + 1 \)

\( \frac{\partial M}{\partial y} = 2xe^y + \frac{1}{y} \) \( \frac{\partial N}{\partial x} = 2xe^y + \frac{1}{y} \) \( \triangleright \) exact.

We can find a function \( u(x,y) \) such that \( \frac{\partial u}{\partial x} = M = 2xe^y + 1ny + \frac{1}{\sqrt{1-x^2}} \) and \( \frac{\partial u}{\partial y} = N = x^2e^y + \frac{x}{y} - 3y^2 + 1 \)

\[ u(x,y) = \int \left( 2xe^y + 1ny + \frac{1}{\sqrt{1-x^2}} \right) \,dx + \psi(y) \]

\[ u(x,y) = x^2e^y + xny + \arcsin x + \psi(y) \]

\( \frac{\partial u}{\partial y} = x^2e^y + \frac{x}{y} + \frac{d\psi}{dy} = x^2e^y + \frac{x}{y} - 3y^2 + 1 \)

\( \Rightarrow \) \( \frac{d\psi}{dy} = -3y^2 + 1 \) \( \Rightarrow \) \( \psi(y) = \int (-3y^2 + 1) \,dy \)

\[ \psi(y) = -y^3 + y + C \]

\[ u(x,y) = x^2e^y + xny + \arcsin x - y^3 + y + C \]
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