*fractal and fractional*

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# Lie Algebra-Based Modeling of Nonlinear Macroeconomic Dynamics Under Fractal Structures

**Melike Bildirici [1,*], Ramazan Tekercioglu [2] and Yasemen Uçan [2,*]**

[1] Department of Economics, Yıldız Technical University, 34220 Istanbul, Türkiye
[2] Department of Mathematical Engineering., Yıldız Technical University, 34220 Istanbul, Türkiye; tramazan@yildiz.edu.tr
* Correspondence: bildiri@yildiz.edu.tr (M.B.); ucan@yildiz.edu.tr (Y.U.)

**Abstract**

Regression methods are widely used to investigate macroeconomic relationships; however, they are generally estimated without first examining whether the underlying variables exhibit fractal structures, persistence, and chaotic dynamics. Although nonlinear regression models relax the assumption of linearity, they rarely account for the complex geometric, long-memory, and dynamical properties that characterize macroeconomic time series. Motivated by this limitation, this study proposes a fractal-oriented Lie regression framework that integrates fractional persistence and Lie algebra to model nonlinear macroeconomic interactions within a unified analytical structure. For Türkiye, the empirical analysis employs monthly data on inflation, interest rates, exchange rates and oil prices covering the period 2000M1–2026M1, encompassing major economic crises and structural breaks. Prior to model estimation, the dynamical characteristics of the variables are examined using entropy measures, long-range dependency analysis, Lyapunov exponents and attractors. The results reveal persistent fractal structures, significant fractional dependence and chaotic behavior, indicating that macroeconomic variables evolve within a complex nonlinear dynamical system rather than around a conventional equilibrium. Based on these results, the variables are represented within a Lie algebra framework in which nonlinear transformation matrices preserve the underlying geometric structure while simultaneously capturing both self-dynamics and cross-variable interactions. The proposed Lie regression model demonstrates substantial improvements over standard regression methods in both model adequacy and forecasting performance. Oil prices emerge as the dominant transmitter of shocks by generating pronounced asymmetric effects on inflation, exchange rates and overall macroeconomic stability. The model achieves remarkable forecasting accuracy by reducing RMSE, MAE, and MAPE from 18.58, 13.61, and 69.92 under a standard regression model to 0.27, 0.22 and 16.4, respectively. Finally, the estimated Lie transformation matrix is employed as a policy-simulation mechanism to evaluate the transmission of alternative oil-price shocks. Scenarios based on 5%, 10%, and 20% increases in oil prices quantify the resulting adjustments in inflation, interest rates, and exchange rates by providing forward-looking assessments of macroeconomic vulnerability. The proposed framework extends standard regression analysis by explicitly incorporating fractional persistence and chaotic dynamics into a Lie algebra representation, thereby offering a more accurate and theoretically consistent approach for modeling complex macroeconomic systems.

**Keywords:** fractal; Lie method; chaos; inflation; interest rate

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Academic Editors: Carlo Cattani and Haci Mehmet Baskonus

Received: 18 June 2026
Revised: 14 July 2026
Accepted: 15 July 2026
Published: 20 July 2026

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## 1. Introduction

Fractal structures and Lie regression models represent distinct but conceptually related approaches for analyzing complex nonlinear systems. Fractality refers to self-similarity, long-memory behavior, and irregular fluctuations commonly observed in economic and financial time series. These properties show that economic variables may exhibit persistent dependence across different time horizons and nontrivial scaling behavior that cannot be adequately captured by linear frameworks. Empirical evidence from entropy-based complexity measures and Lyapunov exponents further suggests that such systems may exhibit nonlinear and potentially chaotic dynamics. Lie regression is a nonlinear modeling approach grounded in symmetry analysis and invariant transformations aimed at identifying functional relationships that remain stable under continuous transformation groups.

Although fractality, entropy-based complexity, and chaotic dynamics originate from different theoretical perspectives, they share a common implication: the presence of invariance structures. Lie regression does not directly measure these properties, nor do these properties imply the existence of Lie symmetries in a strict sense. However, the coexistence of these features motivates the use of symmetry-based modeling frameworks. In this context, Lie regression provides a complementary tool capable of capturing invariant functional relationships embedded in complex dynamic systems.

Early contributions [1–5] introduced Lie-based formulations in financial dynamics, while subsequent studies extended these approaches to $S^1$ and $S^2$ manifolds in short-rate models [6,7]. More recent work has applied Lie methods to volatility modeling, oil-price dynamics and correlation flows using hybrid machine learning frameworks such as Lie–LSTM–OLS models [8]. In addition [9], stochastic differential equation methods formulated on Lie algebras and special orthogonal groups have been used to construct dynamic correlation structures.

More recent studies have extended these frameworks to nonlinear assets such as cryptocurrencies and commodities by highlighting the importance of accounting for fractal and chaotic characteristics in correlation dynamics [10]. In Reference [11], they developed a theory and methodology for error-measured non-parametric regression on the Lie group. Their methodology based on harmonic analysis has been demonstrated for real data samples. Reference [12] introduced geodesic and polynomial regression equations on Riemannian manifolds. Reference [13], by building on the work in Reference [12], modeled rigid body motion with the SE(3) Lie group by expressing polynomial regression in a parametric framework. In Reference [14], Porikli modeled the linear regression model for object motion tracking on Lie groups. As a result of the work, Porikli showed that the method based on Affine motions can be generalized to any matrix Lie group.

As seen in the literature, regression models based on Lie groups have generally been used in the field of robotics. However, this model has not been developed for variables in the field of economics. In this study, we aim to close this gap in the literature by developing the method described in Reference [14] for the field of economics.

In this context, we extend the Lie group framework by integrating both rotation and translation components within a unified regression-based structure. The proposed method allows the model to better reflect nonlinear, fractal, and chaotic transitions that are not adequately captured by regression methods. Interest rate, inflation rate, oil prices, and exchange rates are selected as key macro-financial variables due to their central roles in global economic dynamics. These variables are highly sensitive to global risk conditions, monetary policy changes, and geopolitical developments, and they frequently exhibit nonlinear adjustment processes, volatility clustering and persistent deviations from equilibrium. Such features make them suitable candidates for fractal, entropy-based, and chaotic analysis. For Türkiye, the dataset covers the monthly period 2000M1–2026M1, which

includes multiple episodes of global financial stress, commodity market fluctuations and exchange rate instability, thereby providing a rich environment for examining nonlinear macro-financial dynamics.

The empirical contribution of this paper is threefold. First, the paper integrates fractal analysis, entropy-based complexity measures and Lyapunov exponents within a unified framework to characterize the nonlinear and potentially chaotic structure variables. Second, it proposes a two-stage methodology in which the dynamic properties of the data are first identified through complexity measures and then modeled using a Lie symmetry-based regression framework capable of capturing invariant nonlinear relationships. Third, by linking empirically detected fractal and chaotic structures with a symmetry-based econometric specification, the paper develops a coherent methodological bridge between complexity economics and regression modeling by offering a structurally consistent approach to macro-financial interdependence.

Building on this method, the paper will contribute to the broader literature on Lie-based modeling methods. Developments in mathematical physics and differential geometry have established Lie groups as a fundamental tool for analyzing nonlinear dynamical systems characterized by continuous symmetries. Lie group techniques have frequently been applied to systems that follow partial differential equations and stochastic dynamics by providing a rigorous framework for studying invariance properties. Unlike standard OLS methods, the proposed method captures the geometric transformation. This distinction is crucial because macroeconomic dynamics are rarely governed by isolated linear relationships; they evolve through multidimensional and interconnected structures. Within this framework, a diagonal matrix reflects only the self-dynamics of each variable by covering its effect on itself, whereas a full transformation matrix with off-diagonal elements reveals cross-variable interactions and spillover mechanisms. By incorporating these geometric interdependencies, the model provides a richer representation of systemic economic dynamics beyond the limitations of standard OLS and NLS methods.

The rest of this paper is arranged as follows: Section 2 covers the modeling approach, Section 3 details the data, and Section 4 discusses the empirical findings. Section 5 discusses the main results, while the final section presents the conclusion.

## 2. Modeling

In order to reformulate the regression equation within a financial modeling framework, this paper utilizes the geometric and algebraic structures of the Lie group $SE(2)$, which is generated by transformations preserving the symmetries and translations of the regression equation, and its corresponding Lie algebra $se(2)$. By exploiting the group–algebra correspondence, the invariant properties of the underlying dynamical system are incorporated into the model structure.

In this paper, the regression equation $Y = XB + \varepsilon$ is reformulated for the finance problem using the geometric and algebraic properties between the Lie group $SE(2)$ and the Lie algebra $se(2)$ formed by transformations that leave symmetry and translations invariant. That is, the vectorial expression for the transformation that leaves the rotations and translations invariant in $\mathbb{R}^2$ is given by: $\overline{y} = A\overline{X} + \overline{P}$, $\det A \neq 0$, $P = (P_1, P_2) \in \mathbb{R}^2$. The group formed by the given coordinate transformations, where $(A, P)$ and $(B, Q)$ are the elements, is defined by the group operation $(A, P)x(B, Q) = (AB, AQ + P)$ semi-direct product.

The following definitions are given, where the Lie group elements are denoted by the uppercase letter A and its Lie algebra element by the lowercase letter a.

**Definition 1.** *The $SE(2)$ Lie group is the semi-direct product of the Lie group SO(2) orthogonal rotation and $P \in \mathbb{R}^2$ translation group, and is denoted as follows:*

$$SE(2) = \left\{ A|A = \begin{pmatrix} R(\theta) & P \\ 0_{1x2} & 1 \end{pmatrix} \in R^{3x3}, \quad R^T R = I_2 \right\}$$

*where* $R \in SO(2)$ *and* $\det R = 1$, $P \in \mathbb{R}^2$ *with group operation* $A_1 x A_2 = \begin{pmatrix} R(\theta_1 + \theta_2) & P_1 + R(\theta_1)P_2 \\ 0_{1x2} & 1 \end{pmatrix}$.

**Definition 2.** *The Lie algebra $se(2)$ is defined by*

$$se(2) = \left\{ a|a = \begin{pmatrix} \overline{\omega} & v \\ 0_{1x2} & 0 \end{pmatrix} \in R^{3x3}, \, v \in \mathbb{R}^2, \, \omega \in \mathbb{R} \right\}$$

*where* $\overline{\omega} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \in so(2)$ *and the base of the Lie algebra* $se(2)$ *with* $E_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

The mapping between this group and its algebra is given as follows:

$$\begin{gathered} exp : g \to G, \\ a \to A, \\ exp(a) = \sum_{r=0}^{\infty} \frac{1}{r!} a^r = I_3 + \frac{sinw}{w} a + \frac{1 - cosw}{w^2} a^2, \\ log(A) = \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} (A - I)^r \end{gathered} \tag{1}$$

In this model, the regression function $\beta : \mathbb{R}^3 \to SE(2)$ predicts element A of the $SE(2)$ group for the feature vector $f$ such that $A = \beta(f)$. This model is schematically shown in Figure 1.

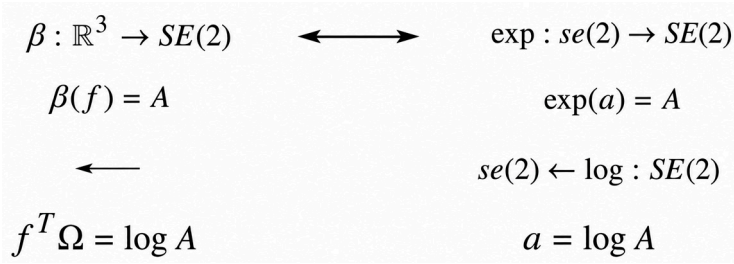



**Figure 1.** Schematic representation of the model.

As seen in this model, $\beta(f) = f^T \Omega = A$, where $\Omega$ is the regression coefficient matrix.

## 3. Data and Variable Construction

The empirical analysis focuses on a multivariate macroeconomic system consisting of interest rate, inflation, oil prices, and the exchange rate. This paper uses a historical dataset containing monthly data from 2000M1 to 2026M1 in Türkiye. Table 1 provides the nomenclatures table.

**Table 1.** Nomenclatures.

| Data | Abbreviations | Definition | Taken from |
|---|---|---|---|
| Exchange rates | EX | USD/TL | https://fred.stlouisfed.org (accessed on 21 May 2026) |
| Inflation | INF | CPI inflation | |
| Oil prices | OP | Brent crude | |
| Interest rates | IntR | Policy rate | Central Bank |

Table 2 presents descriptive statistics for the variables.

**Table 2.** Descriptive statistics.

| | | STD | Kurtosis | JB (Jarque–Bera) |
|---|---|---|---|---|
| EX | Level | 7.9535 | 2.7185 | 1150.9417 |
| | Ln | 1.6441 | −0.4986 | 17.7386 |
| INF | Level | 31.8259 | 0.8485 | 52.9994 |
| | Ln | 1.0254 | 0.2765 | 40.6549 |
| OP | Level | 32.3201 | 0.325 | 22.3819 |
| | Ln | 0.6701 | −0.4442 | 28.5421 |
| IntR | Level | 20.7755 | 0.4711 | 40.8705 |
| | Ln | 0.7148 | −0.0239 | 33.5901 |

### *3.1. Empirical Stages*

The results were obtained from five stages.

#### 3.1.1. Nonlinearity Testing

The BDS (Brock–Dechert–Scheinkman) test was evaluated. This test evaluates whether residual dependence structures deviate from independent and identically distributed processes by confirming the presence of hidden nonlinear structures not captured by linear models.

#### 3.1.2. Long-Term Dependence and Fractionality

The second stage investigates whether macroeconomic variables exhibit long-memory and fractal persistence. Long-run dependence is examined using the Lo's modified R/S test and Hurst–Mandelbrot R/S test.

In addition, the fractional d parameter is estimated using both the Phillips and Geweke & Porter & Hudak (GPH) methods. These complementary estimators allow for a more reliable characterization of fractional integration and persistence properties in macroeconomic time series.

#### 3.1.3. Entropy, Lyapunov Exponents and Complexity

Rule of thumb: The sign and magnitude of the Lyapunov exponent are used as diagnostic indicators of system dynamics; positive values indicate sensitive dependence on initial conditions and chaotic behavior, while non-positive values suggest regular or non-chaotic dynamics. This classification is used as a pre-condition for subsequent dependence analysis, ensuring that nonlinear interaction structures are evaluated under appropriate dynamical regimes.

#### 3.1.4. Attractors

In the third stage, phase-space reconstruction methods are employed to identify the underlying attractor structure of the macroeconomic system. The reconstructed attractor is provided with a geometric representation of the system's evolution The existence of

a coherent attractor structure supports the interpretation of macroeconomic variables as evolving on a nonlinear manifold.

#### 3.1.5. Modeling of Lie Regression

Given evidence of long-memory behavior, nonlinear dependence and attractor-based dynamics, the final stage implements the Lie algebra-based regression framework. This model represents macroeconomic variables as a state vector and estimates their evolution through transformation matrices derived from local Lie algebra projections. The resulting structure captures both self-dynamics and cross-variable spillovers within geometric framework, consistent with the underlying fractal and nonlinear empirical evidence.

Figure 2 shows the flow chart.

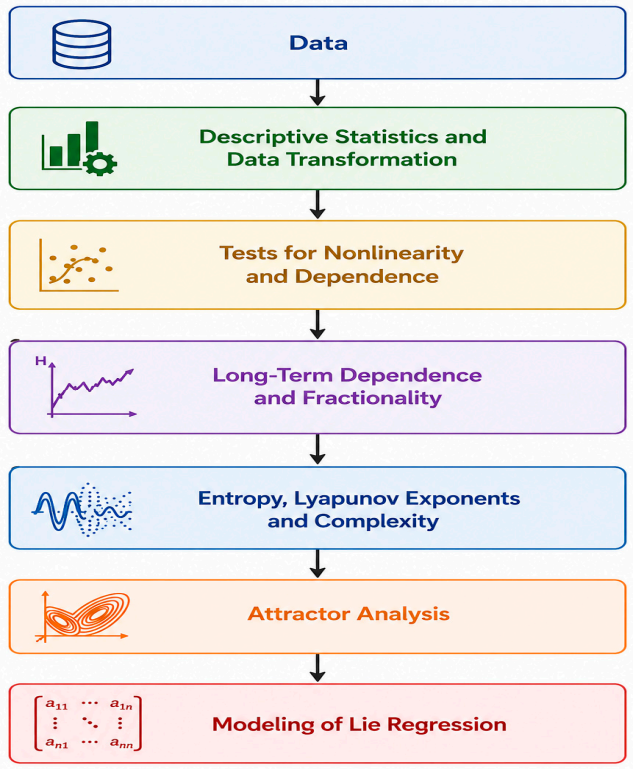



**Figure 2.** Flow chart.

## 4. Results

### *4.1. Nonlinearity Testing*

The BDS test rejects the null hypothesis of independent and identically distributed residuals by suggesting the presence of remaining dependence and nonlinear structure that is not captured by the linear specification. BDS test results are reported in Table 3.

**Table 3.** BDS test results.

| | EX | | INF | | OP | | IntR | |
|---|---|---|---|---|---|---|---|---|
| **Dimension** | **Level** | **Ln** | **Level** | **Ln** | **Level** | **Ln** | **Level** | **Ln** |
| m = 2 | 22.972 | 44.8056 | 82.210 | 289.0351 | 143.9849 | 140.1762 | 164.064 | 422.5443 |
| m = 3 | 22.661 | 54.3191 | 104.28 | 375.7289 | 187.8869 | 178.7123 | 210.7301 | 561.7564 |
| m = 4 | 22.452 | 67.5087 | 136.58 | 505.4782 | 254.3412 | 235.1733 | 279.645 | 775.2193 |
| m = 5 | 22.663 | 87.157 | 187.36 | 711.3143 | 360.6685 | 323.6184 | 388.4215 | 1122.7906 |
| m = 6 | 23.257 | 116.398 | 267.15 | 1041.403 | 533.1761 | 462.808 | 560.5198 | 1694.5104 |

### *4.2. Fractionality and Long-Term Dependence*

The results for long memory and fractionality were obtained by employing Lo's test, the d parameter (Phillips and GPH) test and the Hurst–Mandelbrot R/S method. The results are given in Table 4.

**Table 4.** R/S tests and fractional results.

| | Level | | | |
|---|---|---|---|---|
| **Statistics** | **EX** | **IntR** | **INF** | **OP** |
| Hurst–Mandelbrot's R/S | 22.54 | 23.41 | 20.09 | 19.36 |
| Lo's Modified R/S | 2.85 | 3.18 | 2.89 | 2.78 |
| GPH d parameter p, $H_0$: d = 0 | 0.94 | 0.858 | 0.942 | 0.848 |
| Philips d parameter p, $H_0$: d = 0 | 0.997 | 0.831 | 0.942 | 0.876 |
| First Difference | | | | |
| Hurst–Mandelbrot's R/S | 16.129 | 9.709 | 9.804 | 14.286 |
| Lo's Modified R/S | 1.11 | 2.33 | 0.346 | 1.08 |
| GPH d parameter p, $H_0$: d = 0 | 1.09 | 2.21 | 0.345 | 1.07 |
| Philips d parameter p, $H_0$: d = 0 | 0.074 | 0.12 | 0.062 | 0.025 |

The tests were conducted for the series in Table 4 The paper used four complementary long-memory tests, and each of them captured persistence from a different methodological perspective. Hurst–Mandelbrot's R/S and Lo's Modified R/S tests measured long-range dependence, whereas GPH (Geweke–Porter-Hudak) and Phillips fractional integration tests estimated the fractional parameter (d). While the Hurst-based methods focused on fractal scaling and persistence, the d-based methods focused on fractional integration and stochastic memory structure. Using all four tests provided robust evidence by cross-validating long-memory behavior across both fractal and econometric methods.

### *4.3. Chaos and Entropy Tests*

Following the fractionality tests, the focus shifted to examining the chaotic properties. The Lyapunov exponents (λ) results are given in Table 5 All variables have chaotic dynamics.

**Table 5.** Lyapunov exponent results.

| | **Level** | **Ln** |
|---|---|---|
| EX | 0.2338 | 0.2146 |
| INF | 0.3277 | 0.3216 |
| OP | 0.3062 | 0.3063 |
| IntR | 0.2143 | 0.2243 |

Entropy results are reported in Table 6.

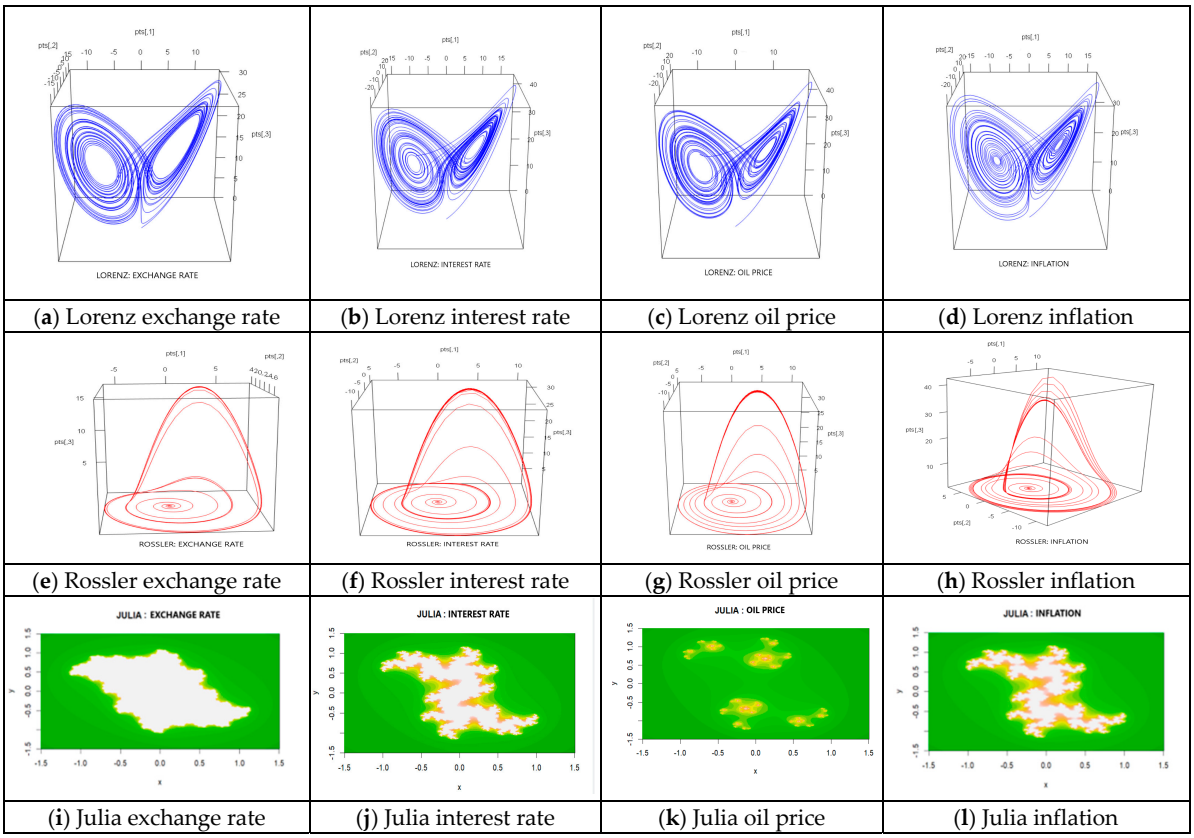
**Table 6.** Tsallis, Renyi and Shannon entropy.

| | | Shannon | 1/SE | Tsallis q = 1 | Renyi q = 1 | Tsallis q = 2 | Renyi q = 2 |
|---|---|---|---|---|---|---|---|
| EX | Level | 0.973 | 1.0277 | 1.0277 | 0.973 | 0.4015 | 0.5133 |
| | Ln | 1.9772 | 0.5058 | 0.5058 | 1.9772 | 0.8209 | 1.7198 |
| INF | Level | 1.6984 | 0.5888 | 0.5888 | 1.6984 | 0.7049 | 1.2206 |
| | Ln | 2.0741 | 0.4821 | 0.4821 | 2.0741 | 0.8503 | 1.8991 |
| OP | Level | 2.1623 | 0.4625 | 0.4625 | 2.1623 | 0.8745 | 2.0754 |
| | Ln | 2.1871 | 0.4572 | 0.4572 | 2.1871 | 0.8772 | 2.0969 |
| IntR | Level | 1.9569 | 0.511 | 0.511 | 1.9569 | 0.8135 | 1.6795 |
| | Ln | 2.1499 | 0.4651 | 0.4651 | 2.1499 | 0.8717 | 2.0534 |

Entropy results showed that the series exhibits a level of complexity.

### *4.4. Attractors*

Most importantly, the results presented in Figure 3 provided the theoretical foundation for the proposed chaotic regression method. If macroeconomic variables evolve within fractal and chaotic structures, then relationships among them cannot be fully represented by constant linear coefficients. By incorporating Lyapunov-based measures, the proposed Lie regression model is capable of capturing amplification impacts and nonlinear interactions that remain largely hidden within traditional regression methods.




(**a**) Lorenz exchange rate (**b**) Lorenz interest rate (**c**) Lorenz oil price (**d**) Lorenz inflation

(**e**) Rossler exchange rate (**f**) Rossler interest rate (**g**) Rossler oil price (**h**) Rossler inflation

(**i**) Julia exchange rate (**j**) Julia interest rate (**k**) Julia oil price (**l**) Julia inflation

**Figure 3.** Lorenz, Rossler and Julia results.

### *4.5. Lie Regression Method*

We assume that the economy evolves via a state-dependent transformation

$$x(t) = A_t x_{t-1} \tag{2}$$

where $A_t$ is a transformation matrix belonging to a matrix Lie group. This formulation allows the economic system to exhibit nonlinear, path-dependent dynamics.

Direct estimation from the transformation manifold is infeasible due to non-commutativity and curvature. Therefore, we employ the logarithmic map to project transformations from the Lie group G onto its associated Lie algebra $g$, a linear tangent space around the identity.

$$a_t = \log(A_t),\ a_t \in g \tag{3}$$

The Lie algebra representation captures local structural changes in a linear space, enabling tractable estimation while preserving the intrinsic geometry of the transformation process. $a_t$ represents the local structural adjustment governing the macroeconomic system at time $t$.

The original transformation is recovered via the exponential map:

$$A_t = \exp(a_t) \tag{4}$$

This log–exp formulation ensures that estimated transformations respect the underlying manifold structure and allows for path-dependent accumulation of small shocks into large structural shifts.

Let $f_t^T$ denote a vector of explanatory variables. The transformation regression is specified as

$$a_t = f_t^T \Omega \tag{5}$$

where $\Omega$ is a matrix of regression coefficients estimated via regularized least squares. This specification implies that macroeconomic shocks do not directly affect variable levels but instead influence the structural transformation governing the system's evolution.

Macroeconomic transformations are inherently non-commutative; the order in which shocks occur matters. The Lie group framework explicitly accommodates this feature, as

$$A_t A_{t-1} \neq A_{t-1} A_t \tag{6}$$

Consequently, structural effects depend on the sequence of shocks rather than their marginal impact, a property absent from conventional linear models.

In this paper, to mitigate overfitting and ensure numerical stability, ridge regularization is employed. The estimated transformation matrices are then reconstructed using the exponential map and applied recursively to generate the dynamic macroeconomic path. This approach allows each period to be characterized by a distinct transformation by accommodating gradual evolution as well as abrupt regime shifts.

In this stage, the estimated Lie transformation matrix is employed to evaluate the effects of alternative oil-price shock scenarios. Unlike standard OLS methods, this method covers the geometric transformation of the economic system and reveals the multidimensional transmission of shocks. A diagonal matrix reflects only the self-dynamics of each variable by representing its effect on itself (e.g., oil $\rightarrow$ oil). By contrast, a full transformation matrix containing off-diagonal elements captures cross-variable interactions and spillover effects such as the transmission from oil prices to inflation.

Specifically, the impacts of 5%, 10%, and 20% increases in oil prices on inflation, interest rates, and exchange rates are quantified through the rotational structure of the proposed Lie regression framework.

The results of tests for A matrices are given in Table 7.

**Table 7.** A matrices.

| | 5% | | | 10% | | | 20% | | |
|---|---|---|---|---|---|---|---|---|---|
| | **INF** | **IntR** | **EX** | **INF** | **IntR** | **EX** | **INF** | **IntR** | **EX** |
| INF | 0.268 | 0.052117 | 0.0364 | 0.379 | −0.030 | 0.047660 | 0.87 | 0.36 | 0.48 |
| IntR | 0.0488 | 0.1678 | 0.008064 | 0.026 | 0.0139 | −0.019513 | 0.05 | 1.09 | 0.07 |
| EX | 0.4433 | 0.0877 | 1.364563 | 0.442 | −0.079 | −0.368612 | 0.25 | 0.17 | 1.01 |
| OP | 0.2781 | 0.1093 | 0.101112 | 0.3105 | 0.0291 | 0.005444 | 0.42 | 0.25 | 0.15 |

The coefficient estimates associated with the OP variable are not reported because the corresponding column entries in the estimated matrix were not statistically significant. Accordingly, Matrix A presents only the coefficients whose associated *t*-statistics are statistically significant. Therefore, all reported coefficients satisfy conventional statistical significance criteria. The absence of statistically significant coefficients for the OP variable is also consistent with theoretical expectations. The existing literature generally suggests that the variables included in the model do not exert a systematic impact on international oil prices. Furthermore, since Türkiye is a net oil-importing country, its macroeconomic conditions are not expected to have a measurable impact on global crude oil prices. Consequently, the lack of significant effects on the OP equation is both statistically and theoretically justified.

Oil prices act as a key driving and transmitting force within the system by functioning as both a catalyst and a shock propagator. Increases in oil prices exert upward pressure on both inflation and exchange rates, while simultaneously dampening economic growth. The empirical results show that a 20% increase in oil prices generates a substantial macroeconomic response, exerting an additional 42% upward pressure on inflation while simultaneously leading to a 25% depreciation in the exchange rate. A strong positive feedback mechanism is identified between exchange rates and inflation by implying that currency shocks are rapidly and directly transmitted into domestic price levels. Within this interconnected structure, oil prices emerge as a key driving (or "forcing") variable by simultaneously amplifying inflationary and exchange rate pressures.

Comparison with Standard Models

For benchmarking purposes, the proposed Lie Regression model was compared with OLS and NLS regression models. All models were estimated under the same dataset, and their statistical properties and out-of sample forecasting performances were systematically evaluated to provide a comprehensive assessment of the proposed method. Unlike standard OLS- and NLS-based methods, the proposed method covers the underlying nonlinear transformation structure of the economic system by offering a more comprehensive representation of dynamic interdependencies.

The results of the tests for residuals are given in Table 8.

**Table 8.** Kurtosis, JB and Ramsey RESET tests for residuals.

| | **Lie Method** | **OLS Regression** | **NLS Regression** |
|---|---|---|---|
| Kurtosis | 3.007 | 8.745 | 8.003 |
| JB | 3.41 | 777.43 | 734.19 |
| F, RESET | 1.502 | 24.65 | 21.92 |

For kurtosis statistics, the 3.007 result is close to 3, as shown in Table 8 The distribution of residuals is quite close to the normal distribution. In terms of kurtosis, the residuals

can be considered as normally distributed. The JB value is below the critical value. The assumption of normality was provided generally. For Ramsey RESET testing, there are no obvious model specification errors in the models.

Excessive kurtosis in OLS and NLS residuals indicates that the model is unable to explain extreme values and shocks. This indicates that non-linearity and structural breaks are not captured, especially in financial/macro series. In the Lie model, the residual distribution is very close to normal, so the model seems to have absorbed the systematic structure better. On the OLS side, a high JB value is a strong indicator of model misspecification and missing nonlinear structures. In the context of the RESET test, a low value in the Lie model implies that the functional form is sufficient and the specification is correct, while a high values in OLS and NLS indicate missing nonlinear terms and incorrect functional structure (in the context of the problems of the OLS method, variable variance is one of the notable assumption breakdowns. Butterfly dispersing residuals are also a special case of changing variance. See detailed discussions for this problem [15,16]).

According to our results, the problem is not only the error size, but also the wrong model structure. Overall, residual diagnostic tests indicate that the Lie regression framework provides a more adequate specification by effectively capturing nonlinear and fractal structures that remain unaccounted for in the standard regression model.

In the second stage, forecast success was analyzed to evaluate the success of the model, as shown in Table 9. For out-of-sample forecasting results, RMSE (Root Mean Squared Error), MAE (Mean Absolute Error) and MAPE (Mean Absolute Percentage Error) results showed the importance of the results for forecast accuracy.

**Table 9.** Forecast results.

| | **One Month Ahead** | |
|---|---|---|
| **RMSE** | **MAE** | **MAPE** |
| | **Lie Method** | |
| 0.27 | 0.22 | 16.4 |
| | **OLS Regression Method** | |
| 18.58 | 13.61 | 69.92 |
| | **NLS Regression Method** | |
| 15.42 | 9.87 | 58.13 |

The forecast success of the Lie regression model was compared with the forecast success of the OLS regression method. It is observed that the OLS regression yields substantially higher values across all three performance metrics.

Lastly, we compared the methods according to the performance criteria in Table 10.

**Table 10.** Comparison of methods.

| **Method** | **Primary Objective** | **Forecasting** | **Structural Interpretation** | **Rotation/Translation Operators** | **Policy Simulation** |
|---|---|---|---|---|---|
| OLS | Linear regression | Limited | ✓ | ✗ | Limited |
| NLS | Linear regression | Limited | ✗ | ✗ | Limited |
| Proposed Lie Regression | Geometric structural transformation | Secondary | ✓ | ✓ | ✓ |

Accordingly, the main contribution of this paper lies in shifting the role of Lie algebra from a descriptive geometric tool to an operative econometric mechanism that directly generates regression outcomes and enables policy-relevant shock simulation. This repre-

sents a substantive methodological advancement beyond existing Lie-based applications in economics and related fields.

## 5. Discussion

A growing body of the literature has recently incorporated Lie algebra-based methods into economic and financial modeling. Reference [7] employs a Lie framework to analyze interest rate dynamics, while Reference [10] proposes an SO(3)-based Lie structure to estimate time-dependent correlation flows among chaotic, fractal and entropy-driven variables across multiple frequencies. Similarly, Reference [14] introduces a Lie algebra-based regression approach in computer vision by demonstrating that modeling transformations directly in Lie algebra space improves estimation accuracy by preserving geometric structure in nonlinear motion dynamics. The proposed methodology generates an explicit transformation matrix that can be directly employed for policy simulation. Once estimated, the matrix quantifies how a given percentage change in oil prices propagates simultaneously to inflation, exchange rates, and interest rates without requiring separate scenario-specific estimations.

Despite these important advances, a common limitation across these studies is that the Lie algebra structure is primarily used as a representation or estimation tool rather than as an active transformation mechanism for dynamic regression and shock propagation. In particular, the translational and rotational properties embedded in Lie group structures are not explicitly exploited to model the propagation of shocks or the evolution of dependent variables within a regression method.

In contrast, the present paper extends the literature by explicitly utilizing the transformation (rotation–translation) structure of Lie algebra within a regression context. Rather than treating Lie algebra solely as a geometric embedding space, the proposed framework employs the estimated Lie transformation matrix as a dynamic operator that governs the propagation of macroeconomic shocks across variables. This allows the model to capture not only correlation structures but also the directional transmission and amplification of disturbances within a nonlinear and fractal economic system.

The results provide strong evidence that the selected macroeconomic variables evolve within a complex dynamic environment characterized by nonlinear dependence, fractal persistence and chaotic behavior. The entropy measures, nonlinear dependence tests and positive Lyapunov exponents collectively indicate that inflation, interest rates, exchange rates and oil prices do not follow simple equilibrium-based dynamics. These variables exhibited long-memory properties, self-similar structures and sensitivity to initial conditions, suggesting that macroeconomic systems operate through interconnected nonlinear mechanisms.

These results have important methodological implications. The proposed Lie regression method should therefore not be interpreted merely as a statistical improvement. It reflects the ability of Lie transformations to preserve and represent the underlying geometric structure of a system governed by nonlinear, fractal and chaotic interactions. By projecting economic dynamics onto a Lie algebra method, the model captured both self-dynamics and cross-variable interactions while maintaining the intrinsic transformation properties of the system. This characteristic enables the model to accommodate complex adjustment mechanisms that are difficult to represent within conventional regression structures.

The forecasting comparison further supports this interpretation. The proposed Lie regression model substantially outperforms the standard OLS regression approach according to all evaluation criteria. The remarkably lower RMSE, MAE and MAPE values indicate that a considerable portion of the information contained in macroeconomic variables is embedded in nonlinear and fractal structures that remain unaccounted for in traditional regression models. The results suggested that forecasting performance improves signifi-

cantly when the complexity characteristics of the system are incorporated directly into the modeling rather than treated as residual noise.

Another important contribution of the paper is the use of the estimated Lie transformation matrix as a policy simulation mechanism. Unlike traditional regression models that primarily provide parameter estimates, the proposed method allows the transmission of shocks to be traced throughout the entire macroeconomic system. By employing the rotational structure generated by the Lie transformation matrix, the effects of alternative oil-price shock scenarios by including 5%, 10%, and 20% increases can be quantified and propagated through inflation, interest rates, and exchange rates. This feature provides a dynamic representation of how shocks spread across interconnected macroeconomic variables within a nonlinear and fractal environment.

The simulation results are particularly relevant for policymakers, central banks, financial institutions and market analysts. Since oil prices constitute a major external source of macroeconomic disturbances, the ability to evaluate the consequences of alternative oil-price scenarios before they fully materialize offers an important decision-support mechanism. The method contributes not only to forecasting accuracy but also to risk assessment, stress testing and policy design under conditions of uncertainty and complexity.

Overall, the results indicate that macroeconomic systems should not be viewed as collections of isolated linear relationships. The proposed Lie regression method provides a more theoretically consistent and empirically effective representation of economic reality than conventional regression approaches. The paper contributes to the growing literature on complexity economics by demonstrating that the integration of fractal analysis, chaos theory and Lie algebra-based modeling can provide a richer understanding of macroeconomic interactions and improve both forecasting and policy evaluation.

## 6. Conclusions and Policy Recommendations

This paper developed a Lie regression method to analyze macroeconomic variables characterized by fractal persistence and chaotic dynamics. The proposed method is designed to capture the nonlinear and complex evolution of key macroeconomic indicators, specifically inflation, interest rates, exchange rates and oil prices, over the period 2000M1–2026M1.

The empirical results provided consistent evidence of long-memory behavior and chaotic structures. Fractional and long-range dependence and fractionality tests (Lo's modified R/S test and Hurst–Mandelbrot R/S test GPH, and Phillips) confirmed persistent dependence, while entropy-based measures further indicated high levels of system complexity. The presence of chaos is supported by positive Lyapunov exponents, and nonlinear dependence is provided by the BDS test. Within this empirically validated nonlinear regime, the proposed Lie algebra method provides a geometrically consistent representation of macroeconomic dynamics. The estimated transformation matrix captured both self-dynamics and cross-variable spillovers by revealing oil prices as the central transmission channel of macroeconomic shocks.

The main contribution of this paper lies in integrating fractal persistence, chaotic dynamics and Lie algebraic transformation geometry into a unified macroeconomic modeling method. Unlike regression or nonlinear regression models, the proposed method contributes to the literature by demonstrating that macroeconomic dynamics cannot be fully understood through standard OLS and NLS methods alone. Unlike standard methods, the proposed method covers the geometric transformation of the economic system by allowing for a more comprehensive representation of shock transmission mechanisms. The distinction between diagonal and full transformation matrices is particularly informative:

while diagonal matrices reflect only self-dynamics, full matrices with off-diagonal elements reveal the complex spillover effects across variables.

The forecasting performance results strongly support the superiority of the proposed technique. Compared with OLS and NLS regression, the Lie regression method yields substantially lower RMSE, MAE and MAPE values, thereby demonstrating significant improvements in predictive accuracy. This evidence indicated that a substantial portion of macroeconomic information is embedded in nonlinear and fractal structures that are not captured by traditional regression methods.

Although the proposed method introduces additional computational complexity and requires careful model specification, it provides a more realistic representation of macroeconomic dynamics characterized by fractality, nonlinearity and chaotic evolution.

The results of this paper also have important implications for macroeconomic policy design. First, the presence of fractality and long-memory behavior implies that macroeconomic shocks are persistent and path-dependent, suggesting that short-term and isolated policy interventions may be insufficient. Second, the identification of chaotic dynamics indicates that the system is highly sensitive to initial conditions; therefore, small policy deviations or external shocks, particularly oil-price fluctuations, may generate disproportionately large macroeconomic effects, thereby highlighting the need for system-wide stability-oriented policy design. Third, oil prices are identified as the dominant transmission channel of macroeconomic instability, implying that energy price volatility should be treated as a core systemic risk factor in macroeconomic policy frameworks. Fourth, the strong exchange rate–inflation feedback mechanism suggests that conventional linear policy rules may be inadequate under high-volatility regimes due to nonlinear and regime-dependent interactions. Finally, the estimated Lie transformation matrix provides a policy-simulation mechanism. Using its rotational structure, the model allows the effects of alternative oil-price shock scenarios (5%, 10%, and 20% increases) to be traced through inflation, interest rates, and exchange rates. This feature enables a forward-looking assessment of macroeconomic vulnerability and shock transmission under nonlinear and fractal system dynamics.

Overall, the findings suggest that macroeconomic systems should be understood as interconnected fractal and chaotic structures rather than collections of isolated linear relationships. The superior performance of the proposed Lie regression method indicates that its effectiveness stems from its ability to model economic interactions in a manner consistent with their inherently nonlinear, fractal, and attractor-driven nature. Consequently, Lie-based modeling provides a promising alternative paradigm.

### *6.1. Policy Implications*

The proposed Lie regression framework has important implications for economic policy because it explicitly quantifies how oil-price shocks propagate through inflation, interest rates, and exchange rates within a unified transformation structure. Rather than estimating separate econometric models for different policy scenarios, the estimated Lie transformation matrix directly evaluates the magnitude and direction of macroeconomic adjustments generated by alternative oil-price shocks. This provides policymakers with a coherent analytical framework for assessing the transmission of external shocks across interconnected macroeconomic variables.

For central banks, the proposed model supports monetary policy by quantifying the inflationary and exchange-rate pressures associated with alternative oil-price scenarios, thereby improving short-term policy assessment. Fiscal authorities and ministries of finance may use the estimated transformation matrix to evaluate the potential macroeconomic consequences of energy-price fluctuations when designing stabilization policies and budgetary

strategies. Energy policymakers can assess the economy-wide effects of oil-price movements and identify policies that reduce vulnerability to external energy shocks. In addition, financial regulators and financial institutions may employ the framework for macroeconomic stress testing and scenario analysis by evaluating the system-wide transmission of external disturbances.

Unlike standard regression models, which primarily estimate statistical relationships among variables, the proposed methodology provides an interpretable transformation mechanism that explicitly captures the propagation of shocks through rotation and translation operators. Consequently, it serves not only as an econometric model but also as a practical policy-simulation tool for evaluating macroeconomic vulnerability under alternative oil-price scenarios in complex and nonlinear economic environments.

### *6.2. Limitations and Future Research*

Despite its methodological contributions, this study has several limitations. First, the empirical application focuses on a single-country macroeconomic system; consequently, the generalizability of the results to other economies with different institutional structures should be interpreted with caution. Second, the analysis considers four key macroeconomic variables—inflation, interest rates, exchange rates, and oil prices—and therefore does not incorporate additional factors such as fiscal policy variables, financial market indicators, or global uncertainty measures that may also influence nonlinear macroeconomic dynamics. Third, the proposed framework is developed within the SE(2) Lie algebra, and higher-dimensional Lie groups may provide greater flexibility for representing more complex economic systems.

These limitations provide promising directions for future research. Future studies may also investigate higher-dimensional Lie groups and alternative geometric representations for modeling large-scale economic systems.

**Author Contributions:** M.B.: conceptualization; methodology; software; validation; formal analysis; investigation; resources; data curation; writing—original draft preparation; writing—review and editing; visualization; supervision; project administration. R.T.: conceptualization; methodology; software; validation; investigation; formal analysis; resources; data curation; writing—original draft preparation; writing—review and editing; funding acquisition; visualization. Y.U.: methodology; software; investigation; formal analysis; data curation; writing—review and editing; funding acquisition; visualization. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** The data presented in this study are openly available at https://fred.stlouisfed.org (accessed on 21 May 2026).

**Acknowledgments:** This study was supported by the Yildiz Technical University Scientific Research Projects Coordination Unit, Türkiye, under project number FCD-2026-7815.

**Conflicts of Interest:** The authors declare no conflicts of interest.

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