An application on intuitionistic fuzzy weakly 2-absorbing ideals of lattices

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Abstract

In this study, we investigate the notion of an intuitionistic fuzzy prime ideal and intuitionistic fuzzy weakly 2-absorbing ideal of a lattice as an expansion of a notion of a weakly prime ideal. Then, we characterize the concept of intuitionistic fuzzy weakly 2-absorbing ideals and intuitionistic fuzzy weakly primary ideals and give some results. Finally, we suggest the concept of an intuitionistic fuzzy weakly 2-absorbing ideals in a product of lattices.

Keywords: lattice; intuitionistic fuzzy ideals; intuitionistic fuzzy ideals 2-absorbing ideals; intuitionistic fuzzy 2-absorbing primary ideals

1 Introduction

Zadeh's introduction of fuzzy set theory in 1965 [38] has had a widespread impact across numerous fields. Building on this foundation, various scholars, including Yuan *et al.* [13], have applied these concepts to group theory, particularly examining fuzzy ideals and congruences in distributive lattices. Lattice theory, a core area of algebra, plays a crucial role in both mathematical and applied sciences, offering a framework for analyzing ordered sets. Its intersection with fuzzy logic has propelled progress in understanding algebraic structures in contexts involving uncertainty. Ajmal *et al.* [1] expanded the application of fuzzy sets within lattice theory, while Šešelja and Tepavčević [31] made a notable contribution by developing a foundational approach to representing lattices via fuzzy sets, which significantly advanced information sciences. Zhang *et al.* [39] expanded this work by introducing fuzzy complete lattices, shedding new light on the structural aspects of fuzzy sets. In another contribution, Kehagias [22] explored the lattice structure of fuzzy intervals, adding to the ongoing discussions in the field. Additionally, Klement and Mesiar [23] critically examined the claims of novelty surrounding L-fuzzy sets and their isomorphic lattices.

In 1986, Atanassov [24] expanded the realm of fuzzy set theory by introducing intuitionistic fuzzy sets, which have proven highly effective in managing vagueness. These sets generalize

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traditional fuzzy sets by incorporating a hesitation margin, calculated as one minus the sum of the membership and non-membership degrees. This addition enriches the representation of intuitionistic fuzzy sets and broadens their applicability across various domains. The inclusion of degrees of membership, non-membership, and hesitation enhances the expressiveness and practical utility of knowledge and semantic representations within intuitionistic fuzzy set theory, as elaborated in [3, 5].

In 1989, Biswas [12] introduced the concept of intuitionistic fuzzy subgroups and examined their properties. This line of research was furthered by Banerjee *et al.* [10] in 2003, who explored intuitionistic fuzzy subgroups and ideals. Hur *et al.* [18–20] also contributed by studying the properties of intuitionistic fuzzy subgroupoids, subgroups, and subrings.

The interaction between intuitionistic fuzzy sets and lattice structures has been an active area of research, yielding significant insights and advancements in fuzzy logic and its applications. Yon *et al.* [37] introduced intuitionistic fuzzy sublattices, filters, and ideals, investigating their characteristics. Additionally, Hur *et al.* [25] applied intuitionistic fuzzy sets to lattice congruences. Thomas and Nair [34] made a notable contribution by studying rough intuitionistic fuzzy sets within a lattice framework. Tripathy *et al.* [35] furthered this exploration by delving into intuitionistic fuzzy lattices and intuitionistic fuzzy Boolean algebras. Mahmood *et al.* [27] significantly enriched the discourse on lattice-ordered intuitionistic fuzzy sets. Boudaoud *et al.* [14] examined principal intuitionistic fuzzy ideals and filters on lattices, while Bashir *et al.* [11] explored the algebraic and lattice structures of type-2 intuitionistic fuzzy sets.

Prime ideals and their generalizations play a crucial role in commutative ring theory. Badawi [7] initially defined 2-absorbing ideals as an extension of prime ideals. Badawi [8] introduced the idea of weakly 2-absorbing ideals within commutative rings. A proper ideal I of a commutative ring R is considered weakly 2-absorbing if, for any elements $a, b, c \in R$, and given $0 \neq abc \in I$, it follows that either $ab \in I$, $ac \in I$, or $bc \in I$. Subsequent studies by Badawi *et al.* [2, 9] explored n-absorbing ideals, with a specific focus on 2-absorbing ideals in commutative rings. For additional information on 2-absorbing ideals, see [29, 30].

The concept of 2-absorbing elements has attracted considerable attention in lattice theory due to its algebraic significance and applications. Jayaram *et al.* [21] investigated 2-absorbing and weakly 2-absorbing elements in multiplicative lattices. Atani and Bazari [6] extended this discussion to 2-absorbing filters of lattices. Wasadikar and Gaikwad [36] advanced the study by exploring 2-absorbing and weakly 2-absorbing ideals in product lattices. Teshale Amare [33] examined weakly 2-absorbing ideals in almost distributive lattices. Additionally, Darani [15] introduced L-fuzzy 2-absorbing ideals, which were further studied by Darani and Hashempoor [16] in semirings.

While the concept of 2-absorbing has been adapted to lattice structures, the exploration of weakly 2-absorbing primary ideals within intuitionistic fuzzy environments remains relatively unexplored. Investigating the approximation of intuitionistic fuzzy weakly 2-absorbing primary ideals in lattices presents a novel and rich field of study. Therefore, it is both prudent and necessary to develop new approaches to understanding these ideals.

This study focuses on the foundational aspects of intuitionistic fuzzy weakly 2-absorbing primary ideals in lattices and analyzes their properties. We introduce the notions of an intuitionistic fuzzy weakly 2-absorbing ideal and a 2-absorbing intuitionistic fuzzy weakly ideal within a lattice L. Additionally, we define a primary intuitionistic fuzzy weakly ideal and examine the radical of an intuitionistic fuzzy weakly ideal in L. Various properties of these intuitionistic fuzzy weakly ideals are established. Furthermore, we extend and investigate these concepts within the context of a product of lattices.

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2 Preliminaries

In this section, we will discuss the concepts of lattice, intuitionistic fuzzy set, and intuitionistic fuzzy ideal, which we will use in explaining the other sections. Additionally, we will denote $L = (L, \wedge, \vee)$ as a lattice, where $0, 1 \in L$. Here, 0 is the least element of L, and 1 is the greatest element of L.

DEFINITION 2.1
[17]. A partially ordered set $(L; \leq)$ is a called lattice if for every $a, b \in L$, $a \wedge b = inf\{a, b\}$ and $a \vee b = sup\{a, b\}$ are defined, where $\wedge$ denotes the meet operation and $\vee$ denotes the join operation.

DEFINITION 2.2
[26]. Let X be a nonempty set. An intuitionistic fuzzy set (*IFS*) A of X is an object of the following form $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$, where the functions $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ define the degree of membership and degree of non-membership of the element, $x \in X, 0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

DEFINITION 2.3
[4] Let X be a nonempty set and A is *IFS* of X. Let $X^* = X \times [0, 1] \times [0, 1]$. Obviously, $A \subseteq X^*$ and we can define the characteristic function $\Omega_* : X^* \to \{0, 1\}$ by

$$\Omega_A(< x, a, b >) = \begin{cases} 1, \text{ if } \mu_A(x) = a, \nu_A(x) = b \text{ and } < x, a, b > \in A \\ 0, \textit{otherwise}. \end{cases}$$

DEFINITION 2.4
[26]. If $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ and $B = \{< x, \mu_B(x), \nu_B(x) > | x \in X\}$ are any two *IFS* of X, then

(1) $A \subseteq B \iff \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x), \forall x \in X$;
(2) $A = B \iff \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x), \forall x \in X$.
(3) $A^{\complement} = \{< x, \nu_A(x), \mu_A(x) > | x \in X\}$;
(4) $A \cup B = \{< x, \mu_{A \cup B}(x), \nu_{A \cup B}(x) > | x \in X\}$,where
(4) $\mu_{A \cup B}(x) = max\{\mu_A(x), \mu_B(x)\}$ and $\nu_{A \cup B}(x) = min\{\nu_A(x), \nu_B(x)\}$; and
(5) $A \cap B = \{< x, \mu_{A \cap B}(x), \nu_{A \cap B}(x) > | x \in X\}$,where
(5) $\mu_{A \cap B}(x) = min\{\mu_A(x), \mu_B(x)\}$ and $\nu_{A \cap B}(x) = max\{\nu_A(x), \nu_B(x)\}$.

DEFINITION 2.5
[26]. *IFS* A of L is called an intuitionistic fuzzy ideal (*IFI*) of L if the following conditions are satisfied for all $x, y \in L$:

$$(i)\mu_A(x \vee y) \geq min\{\mu_A(x), \mu_A(y)\}; (ii)\mu_A(x \wedge y) \geq max\{\mu_A(x), \mu_A(y)\};$$
$$(iii)\nu_A(x \vee y) \leq max\{\nu_A(x), \nu_A(y)\}; (iv)\nu_A(x \wedge y) \leq min\{\nu_A(x), \nu_A(y)\}.$$

DEFINITION 2.6
[32]. Let X be a set, $A \in IFS(X)$ and let $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. Then the set, $A_{(\alpha,\beta)} = \{x | \mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta, x \in L\}$ is called a (α, β)-the level subset of A.

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3 Intuitionistic fuzzy weakly prime ideals of a lattice

In this section, we will show that if I is a weakly prime ideal, then characteristic function $< \chi_I, \chi_{I^c} >$ is an intuitionistic fuzzy weakly prime ideal, and vice versa. Additionally, we will explain the relationship between intuitionistic fuzzy prime ideal and intuitionistic fuzzy weakly prime ideal. Let's start with the definition of an ideal of a lattice L.

DEFINITION 3.1
[28] A nonempty subset I of a lattice L is called an ideal, if for $a, b \in L$, the following conditions holds. (i)If $a, b \in I$, then $a \vee b \in I$ and (ii) if $a \leq b$ and $b \in I$, then $a \in I$. A proper ideal I (i.e. $I \neq L$) is called a prime ideal (*PI*), if $a \wedge b \in I$ implies that either $a \in I$ or $b \in I$.

DEFINITION 3.2
[28] A proper ideal I of a lattice L is called a weakly prime ideal (*WPI*), if for $a, b \in L, a \wedge b \neq 0, a \wedge b \in I$ implies that either $a \in I$ or $b \in I$.

DEFINITION 3.3
A proper *IFI* $< \mu, \nu >$ of a lattice L is called an intuitionistic fuzzy prime ideal (*IFPI*), if for all $a, b \in L$, $\mu(a \wedge b) \leq \mu(a) \vee \mu(b)$ and $\nu(a \wedge b) \geq \nu(a) \wedge \nu(b)$.

DEFINITION 3.4
A proper *IFI* $< \mu, \nu >$ of a lattice L is called an intuitionistic fuzzy weakly prime ideal (*IFWPI*), if $a, b \in L, a \wedge b \neq 0$, $\mu(a \wedge b) \leq \mu(a) \vee \mu(b)$ and $\nu(a \wedge b) \geq \nu(a) \wedge \nu(b)$.

THEOREM 3.5
An ideal I of a lattice L is a *WPI* if and only if $< \chi_I, \chi_{I^c} >$ is an *IFWPI* of L.

PROOF. Suppose that I is a *WPI* of L. Let $a, b \in L$ be such that $a \wedge b \neq 0$. If $a \wedge b \in I$, then as I is *WPI*, either $a \in I$ or $b \in I$. Thus, in this case,

$$\chi_I(a \wedge b) \leq \chi_I(a) \vee \chi_I(b) \text{ and}$$
$$\chi_{I^c}(a \wedge b) \geq \chi_{I^c}(a) \wedge \chi_{I^c}(b).$$

If $a \wedge b \notin I$, then clearly, $a \notin I$, $b \notin I$. Thus, in this case,

$$\chi_I(a \wedge b) \leq \chi_I(a) \vee \chi_I(b) \text{ and}$$
$$\chi_{I^c}(a \wedge b) \geq \chi_{I^c}(a) \wedge \chi_{I^c}(b).$$

Hence, $A =< \chi_I, \chi_{I^c} >$ is an *IFWPI* of L.

Conversely, suppose that $A =< \chi_I, \chi_{I^c} >$ is an *IFWPI* of L. Let $a, b \in L$ be such that $a \wedge b \neq 0$. Suppose that $a \wedge b \in I$, but $a \notin I$ and $b \notin I$. This implies that $\chi_I(a \wedge b) = 1$ and $\chi_I(a) = \chi_I(b) = 0$. Then,

$$\chi_I(a \wedge b) \not\leq \chi_I(a) \vee \chi_I(b) \text{ and}$$
$$\chi_{I^c}(a \wedge b) \not\geq \chi_{I^c}(a) \wedge \chi_{I^c}(b)$$

a contradiction, as $A =< \chi_I, \chi_{I^c} >$ is an *IFWPI*. □

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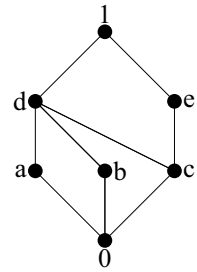


Figure 1. A representative from the family of 7-element lattices.

EXAMPLE 3.6
Consider the lattice L shown in Figure 1. We note that the ideal $I = (c]$ is not a *WPI* of L, as $d \wedge e = c \in I$ but neither $d \in I$ nor $e \in I$.

We have $d \wedge e = c \in I$. Hence, $\chi_I(d \wedge e) = 1$ and $\chi_I(d) = \chi_I(e) = 0$, $\chi_{I^{\complement}}(d \wedge e) = 0$ and $\chi_{I^{\complement}}(d) = \chi_{I^{\complement}}(e) = 1$. Hence, $< \chi_I, \chi_{I^{\complement}} >$ is not an *IFWPI* of L.

THEOREM 3.7
An ideal I of a lattice L is a *PI* if and only if $< \chi_I, \chi_{I^{\complement}} >$ is an *IFPI* of L.

The proof of the following lemma follows from the definition of an *IFPI*.

LEMMA 3.8
If $< \mu, \nu >$ is an *IFPI* of L, then $< \mu, \nu >$ is an *IFWPI* of L.

EXAMPLE 3.9
Consider the lattice L shown in Figure 1. We note that the ideal $I = (a]$ is *WPI* of L but is not a *PI*, as $b \wedge c = 0 \in I$ but neither $b \in I$ nor $c \in I$. Hence, by Theorem 3.1, $< \chi_I, \chi_{I^{\complement}} >$ is an *IFWPI* and by Theorem 3.2, $< \chi_I, \chi_{I^{\complement}} >$ is not *IFPI*.

4 Intuitionistic fuzzy weakly 2-absorbing ideals

In this section, we will introduce the definition of intuitionistic fuzzy weakly 2-absorbing. Then, we will examine the relationships between the weakly 2-absorbing ideal and intuitionistic fuzzy weakly 2-absorbing ideal.

DEFINITION 4.1
[28] An ideal I of L is called a weakly 2-absorbing ideal (*W2AI*), if for $a, b, c \in L, a \wedge b \wedge c \neq 0$, $a \wedge b \wedge c \in I$ implies that either $a \wedge b \in I$ or $b \wedge c \in I$ or $c \wedge a \in I$.

DEFINITION 4.2
A proper *IFI* $< \mu, \nu >$ of a lattice L is called an intuitionistic fuzzy 2-absorbing ideal (*IF2AI*) of L, if for all $a, b, c \in L$,

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a) \text{ and}$$

$$\nu(a \wedge b \wedge c) \geq \nu(a \wedge b) \wedge \nu(b \wedge c) \wedge \nu(c \wedge a).$$

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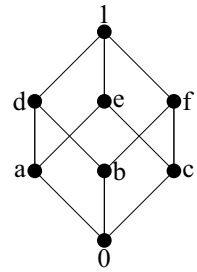


Figure 2. Lattice whose trivial ideal is W2AI but not 2AI.

LEMMA 4.3
Let I be an ideal of L. Then I is a 2-absorbing ideal of L if and only if $A =< \chi_I, \chi_{I^c} >$ is an *IF*2*AI* of L.

DEFINITION 4.4
A proper $IFI < \mu, \nu >$ of a lattice L is called an intuitionistic fuzzy weakly 2-absorbing ideal (*IFW*2*AI*) of L, if for all $a, b, c \in L, a \wedge b \wedge c \neq 0$,

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a) \text{ and}$$
$$\nu(a \wedge b \wedge c) \geq \nu(a \wedge b) \wedge \nu(b \wedge c) \wedge \nu(c \wedge a).$$

LEMMA 4.5
Let I be an ideal of L. Then I is a *W*2*AI* of L if and only if $A =< \chi_I, \chi_{I^c} >$ is an *IFW*2*AI* of L.

PROOF. Suppose that I is a *W*2*AI* of L. Let $a, b, c \in L$ be such that $a \wedge b \wedge c \neq 0$. If $a \wedge b \wedge c \in I$, then as I is *W*2*AI*, either $a \wedge b \in I$ or $b \wedge c \in I$ or $c \wedge a \in I$. Thus, in this case,

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_I(b \wedge c) \vee \chi_I(c \wedge a) \text{ and}$$
$$\chi_{I^c}(a \wedge b \wedge c) \geq \chi_{I^c}(a \wedge b) \wedge \chi_{I^c}(b \wedge c) \wedge \chi_{I^c}(c \wedge a).$$

If $a \wedge b \wedge c \notin I$, then clearly, $a \wedge b \notin I$, $b \wedge c \notin I$ and $c \wedge a \notin I$. Thus, in this case,

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_I(b \wedge c) \vee \chi_I(c \wedge a) \text{ and}$$
$$\chi_{I^c}(a \wedge b \wedge c) \geq \chi_{I^c}(a \wedge b) \wedge \chi_{I^c}(b \wedge c) \wedge \chi_{I^c}(c \wedge a).$$

Hence, $A =< \chi_I, \chi_{I^c} >$ is an *IFW*2*AI* of L.

Conversely, suppose that $A =< \chi_I, \chi_{I^c} >$ is an *IFW*2*AI* of L. Let $a, b, c \in L$ be such that $a \wedge b \wedge c \neq 0$. Suppose that $a \wedge b \wedge c \in I$, but $a \wedge b \notin I$, $b \wedge c \notin I$ and $c \wedge a \notin I$. This implies that $\chi_I(a \wedge b \wedge c) = 1$ and $\chi_I(a \wedge b) = \chi_I(b \wedge c) = \chi_I(c \wedge a) = 0$. Then,

$$\chi_I(a \wedge b \wedge c) \not\leq \chi_I(a \wedge b) \vee \chi_I(b \wedge c) \vee \chi_I(c \wedge a) \text{ and}$$
$$\chi_{I^c}(a \wedge b \wedge c) \not\geq \chi_{I^c}(a \wedge b) \wedge \chi_{I^c}(b \wedge c) \wedge \chi_{I^c}(c \wedge a)$$

a contradiction, as $A =< \chi_I, \chi_{I^c} >$ is an *IFW*2*AI*. □

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EXAMPLE 4.6
Consider the lattice shown in Figure 2.

The ideal $I = (0]$ is trivially a $W2AI$. It is not 2-absorbing as $d \wedge e \wedge f = 0 \in I$ but neither $d \wedge e = a \in I$, nor $e \wedge f = c \in I$ nor $f \wedge d = b \in I$. It follows from Lemma 4.2 that $< \chi_I, \chi_{I^c} >$ is an $IFW2AI$ and by Lemma 4.1, $< \chi_I, \chi_{I^c} >$ is not an $IF2AI$.

LEMMA 4.7
An IFI $A =< \mu_A, \nu_A >$ of L is an $IFW2AI$ if and only if for each $t, s \in [0, 1]$, the level set $A_{(t,s)}$ is a $W2AI$ of L.

PROOF. Let $A =< \mu_A, \nu_A >$ be an $IFW2AI$ of L. Let $t, s \in [0, 1]$. Let $a, b, c \in L$ be such that $a \wedge b \wedge c \neq 0$ and $a \wedge b \wedge c \in A_{(t,s)}$. Then $t \leq \mu_A(a \wedge b \wedge c)$ and $s \geq \nu_A(a \wedge b \wedge c)$. Since $A =< \mu_A, \nu_A >$ is an $IFW2AI$,

$$t \leq \mu_A(a \wedge b \wedge c) \leq \mu_A(a \wedge b) \vee \mu_A(b \wedge c) \vee \mu_A(c \wedge a) \text{ and}$$
$$s \geq \nu_A(a \wedge b \wedge c) \geq \nu_A(a \wedge b) \wedge \nu_A(b \wedge c) \wedge \nu_A(c \wedge a),$$

then either

$$t \leq \mu_A(a \wedge b) \text{ or } t \leq \mu_A(b \wedge c) \text{ or } t \leq \mu_A(c \wedge a) \text{ and}$$
$$s \geq \nu_A(a \wedge b) \text{ or } s \geq \nu_A(b \wedge c) \text{ or } s \geq \nu_A(c \wedge a).$$

Thus, either $a \wedge b$ or $b \wedge c$ or $c \wedge a \in A_{(t,s)}$. So, $A_{(t,s)}$ is a $W2AI$ of L.

In contrast, $A_{(t,s)}$ be a $W2AI$ of L, for each $t, s \in [0, 1]$. Let $a, b, c \in L$ and $t = \mu_A(a \wedge b \wedge c)$, $s = \nu_A(a \wedge b \wedge c)$. Then $a \wedge b \wedge c \in A_{(t,s)}$. Since $A_{(t,s)}$ is a $W2AI$ of L, either $a \wedge b$ or $b \wedge c$ or $c \wedge a \in A_{(t,s)}$. This implies that

$$t = \mu_A(a \wedge b \wedge c) \leq \mu_A(a \wedge b) \vee \mu_A(b \wedge c) \vee \mu_A(c \wedge a) \text{ and}$$
$$s = \nu_A(a \wedge b \wedge c) \geq \nu_A(a \wedge b) \wedge \nu_A(b \wedge c) \wedge \nu_A(c \wedge a).$$

Thus, $A =< \mu_A, \nu_A >$ is an $IFW2AI$ of L. □

LEMMA 4.8
Let $A =< \mu_A, \nu_A >$ be an $IFWPI$ of L. Then $A =< \mu_A, \nu_A >$ is an $IFW2AI$ of L.

PROOF. Let $A =< \mu_A, \nu_A >$ be an $IFWPI$ of L. Then for all $a, b \in L, a \wedge b \neq 0$, $\mu_A(a \wedge b) \leq \mu_A(a) \vee \mu_A(b)$ and $\nu_A(a \wedge b) \geq \nu_A(a) \wedge \nu_A(b)$. Hence, for all $a, b, c \in L$, we have

$$\mu_A(a \wedge b \wedge c) \leq \mu_A(a \wedge b) \vee \mu_A(c)$$
$$\mu_A(a \wedge b \wedge c) \leq \mu_A(b \wedge c) \vee \mu_A(a)$$
$$\mu_A(a \wedge b \wedge c) \leq \mu_A(c \wedge a) \vee \mu_A(b) \text{ and}$$
$$\nu_A(a \wedge b \wedge c) \geq \nu_A(a \wedge b) \wedge \nu_A(c)$$
$$\nu_A(a \wedge b \wedge c) \geq \nu_A(b \wedge c) \wedge \nu_A(a)$$
$$\nu_A(a \wedge b \wedge c) \geq \nu_A(c \wedge a) \wedge \nu_A(b).$$

Hence,

$$\mu_A(a \wedge b \wedge c) \leq \mu_A(a \wedge b) \vee \mu_A(c) \vee \mu_A(b \wedge c) \vee \mu_A(a) \vee \mu_A(c \wedge a) \vee \mu_A(b),$$
$$\nu_A(a \wedge b \wedge c) \geq \nu_A(a \wedge b) \wedge \nu_A(c) \wedge \nu_A(b \wedge c) \wedge \nu_A(a) \wedge \nu_A(c \wedge a) \wedge \nu_A(b).$$

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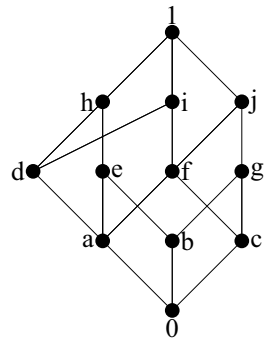


Figure 3. Lattice with the connections established by a,d,e,f elements.

By the definition of an *IFI*, it follows that for any $x, y \in L$, $\mu(x) \leq \mu(x \wedge y)$ and $\nu(x) \geq \nu(x \wedge y)$. Hence, we can write

$$\mu_A(a \wedge b \wedge c) \leq \mu_A(a \wedge b) \vee \mu_A(b \wedge c) \vee \mu_A(c \wedge a),$$
$$\nu_A(a \wedge b \wedge c) \geq \nu_A(a \wedge b) \wedge \nu_A(b \wedge c) \wedge \nu_A(c \wedge a).$$

Thus, $A =< \mu_A, \nu_A >$ is an *IFW2AI* of L. □

EXAMPLE 4.9
Consider the lattice shown in Figure 3. We note $I = (a]$ is a *W2AI* but it is neither prime nor *WPI* as $d \wedge e = a \in I$ but $d \notin I, e \notin I$. Hence, by Lemma 4.2, $< \chi_I, \chi_{I^{\complement}} >$ is *IFW2AI* and by Theorem 3.1, $< \chi_I, \chi_{I^{\complement}} >$ is not *IFWPI*.

LEMMA 4.10
The intersection of any two distinct *IFWPI*s of L is an *IFW2AI* of L.

PROOF. Let $A =< \mu_A, \nu_A >, B =< \mu_B, \nu_B >$ be two distinct *IFWPI*s of L. We know that $A \cap B =< \mu_{A\cap B}, \nu_{A\cap B} >$ where $\mu_{A\cap B} = \mu_A \wedge \mu_B$ and $\nu_{A\cap B} = \nu_A \vee \nu_B$. So, for any $a \in L$,

$$\mu_{A\cap B}(a) = \mu_A(a) \wedge \mu_B(a) \text{ and } \nu_{A\cap B}(a) = \nu_A(a) \vee \nu_B(a).$$

Let $a, b, c \in L, a \wedge b \wedge c \neq 0$. We have

$$\mu_{A\cap B}(a \wedge b \wedge c) = \mu_A(a \wedge b \wedge c) \wedge \mu_B(a \wedge b \wedge c) \text{ and}$$
$$\nu_{A\cap B}(a \wedge b \wedge c) = \nu_A(a \wedge b \wedge c) \vee \nu_B(a \wedge b \wedge c).$$

Since every *IFWPI* is *IFW2AI*, we get

$$\mu_{A\cap B}(a \wedge b \wedge c) \leq [\mu_A(a \wedge b) \vee \mu_A(b \wedge c) \vee \mu_A(c \wedge a)] \wedge [\mu_B(a \wedge b) \vee \mu_B(b \wedge c) \vee \mu_B(c \wedge a)] \text{ and}$$
$$\nu_{A\cap B}(a \wedge b \wedge c) \geq [\nu_A(a \wedge b) \wedge \nu_A(b \wedge c) \wedge \nu_A(c \wedge a)] \vee [\nu_B(a \wedge b) \wedge \nu_B(b \wedge c) \wedge \nu_B(c \wedge a)].$$

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Since A and B are *IFWPI*s, we can write

$$\mu_A(a \wedge b) \vee \mu_A(b \wedge c) \vee \mu_A(c \wedge a) \leq \mu_A(a) \vee \mu_A(b) \vee \mu_A(c)$$
$$\mu_B(a \wedge b) \vee \mu_B(b \wedge c) \vee \mu_B(c \wedge a) \leq \mu_B(a) \vee \mu_B(b) \vee \mu_B(c)$$
$$\nu_A(a \wedge b) \wedge \nu_A(b \wedge c) \wedge \nu_A(c \wedge a) \geq \nu_A(a) \wedge \nu_A(b) \wedge \nu_A(c)$$
$$\nu_B(a \wedge b) \wedge \nu_B(b \wedge c) \wedge \nu_B(c \wedge a) \geq \nu_B(a) \wedge \nu_B(b) \wedge \nu_B(c).$$

Hence, we can write

$$\begin{aligned}\mu_{A\cap B}(a \wedge b \wedge c) &\leq [\mu_A(a) \vee \mu_A(b) \vee \mu_A(c)] \wedge [\mu_B(a) \vee \mu_B(b) \vee \mu_B(c)] \\ &= [\mu_A(a) \wedge \mu_B(a)] \vee [\mu_A(a) \wedge \mu_B(b)] \vee [\mu_A(a) \wedge \mu_B(c)] \\ &\vee [\mu_A(b) \wedge \mu_B(a)] \vee [\mu_A(b) \wedge \mu_B(b)] \vee [\mu_A(b) \wedge \mu_B(c)] \\ &\vee [\mu_A(c) \wedge \mu_B(a)] \vee [\mu_A(c) \wedge \mu_B(b)] \vee [\mu_A(c) \wedge \mu_B(c)] \text{ and} \\ \nu_{A\cap B}(a \wedge b \wedge c) &\geq [\nu_A(a) \wedge \nu_A(b) \wedge \nu_A(c)] \vee [\nu_B(a) \wedge \nu_B(b) \wedge \nu_B(c)] \\ &= [\nu_A(a) \vee \nu_B(a)] \wedge [\nu_A(a) \vee \nu_B(b)] \wedge [\nu_A(a) \vee \nu_B(c)] \\ &\wedge [\nu_A(b) \vee \nu_B(a)] \wedge [\nu_A(b) \vee \nu_B(b)] \wedge [\nu_A(b) \vee \nu_B(c)] \\ &\wedge [\nu_A(c) \vee \nu_B(a) \wedge [\nu_A(c) \vee \nu_B(b)] \wedge [\nu_A(c) \vee \nu_B(c)].\end{aligned}$$

For any *IFI* D, we have $\mu_D(x) \leq \mu_D(x \wedge y)$ and $\nu_D(x) \geq \nu_D(x \wedge y)$, for all $x, y \in L$. This implies

$$\mu_A(a) \wedge \mu_B(b) \leq \mu_A(a \wedge b) \wedge \mu_B(a \wedge b) = \mu_{A\cap B}(a \wedge b) \text{ and}$$
$$\nu_A(a) \vee \nu_B(b) \geq \nu_A(a \wedge b) \vee \nu_B(a \wedge b) = \nu_{A\cap B}(a \wedge b).$$

So, we can write

$$\begin{aligned}\mu_{A\cap B}(a \wedge b \wedge c) &\leq \mu_{A\cap B}(a) \vee \mu_{A\cap B}(a \wedge b) \vee \mu_{A\cap B}(a \wedge c) \\ &\vee \mu_{A\cap B}(b \wedge a) \vee \mu_{A\cap B}(b) \vee \mu_{A\cap B}(b \wedge c) \\ &\vee \mu_{A\cap B}(c \wedge a) \vee \mu_{A\cap B}(c \wedge b) \vee \mu_{A\cap B}(c) \text{ and} \\ \nu_{A\cap B}(a \wedge b \wedge c) &\geq \nu_{A\cap B}(a) \wedge \nu_{A\cap B}(a \wedge b) \wedge \nu_{A\cap B}(a \wedge c) \\ &\wedge \nu_{A\cap B}(b \wedge a) \wedge \nu_{A\cap B}(b) \wedge \nu_{A\cap B}(b \wedge c) \\ &\wedge \nu_{A\cap B}(c \wedge a) \wedge \nu_{A\cap B}(c \wedge b) \wedge \nu_{A\cap B}(c).\end{aligned}$$

Since $A \cap B$ is an *IFI*, for all $x, y \in L$, we have $\mu_{A\cap B}(x) \leq \mu_{A\cap B}(x \wedge y)$ and $\nu_{A\cap B}(x) \geq \nu_{A\cap B}(x \wedge y)$. So we can write

$$\mu_{A\cap B}(a \wedge b \wedge c) \leq \mu_{A\cap B}(a \wedge b) \vee \mu_{A\cap B}(b \wedge c) \vee \mu_{A\cap B}(c \wedge a) \text{ and}$$
$$\nu_{A\cap B}(a \wedge b \wedge c) \geq \nu_{A\cap B}(a) \wedge \nu_{A\cap B}(a \wedge b) \wedge \nu_{A\cap B}(b \wedge c) \wedge \nu_{A\cap B}(c \wedge a).$$

Thus, $A \cap B$ is an *IFW2AI* of L. □

EXAMPLE 4.11

Consider the lattice shown in Figure 4. We know that for any two ideals I, J of L, $< \chi_I, \chi_{I^c} > \cap < \chi_J, \chi_{J^c} > = < \chi_I \cap \chi_J, \chi_{I^c} \cup \chi_{J^c} > = < \chi_{I\cap J}, \chi_{I^c \cup J^c} >$. We note that the ideals $I = (d]$ and $J = (f]$

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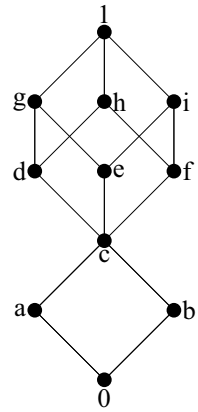


Figure 4. Lattice that should be considered for c,d,e,f elements installed for W2AI.

re not *WPI* and $I \cap J = (c]$ is not a weakly 2-absorbing. By the Theorem 3.1, $< \chi_I, \chi_{I^{\complement}} >$ and $< \chi_J, \chi_{J^{\complement}} >$ are not *IFWPI* and by Lemma 4.2, $< \chi_{(c]}, \chi_{(c]^{\complement}} >$ is not *IFW2AI*.

5 Intuitionistic fuzzy weakly primary ideals

In this section, we will introduce the definition of intuitionistic fuzzy weakly primary ideal. Then, we will discuss the relationship between an intuitionistic fuzzy weakly primary ideal and weakly primary ideal I, as well as the concepts explained in the previous sections. Additionally, we will introduce another concept, intuitionistic fuzzy weakly 2-absorbing primary ideal, and examine its relationship with the previously mentioned concepts.

DEFINITION 5.1
[28] Let L be a lattice with 0. An ideal I of L is called a weakly primary ideal (*WPRI*), if for $a, b \in L, a \wedge b \neq 0, a \wedge b \in I$ implies that either $a \in I$ or $b \in \sqrt{I}$, where $\sqrt{I}$ denotes the radical of I (i.e. the intersection of all *PI*s containing I). If there does not exist a *PI* containing an ideal I in a lattice L then we define $\sqrt{I} = L$.

DEFINITION 5.2
[28] Let I be an ideal of a lattice L. We define the weakly prime radical of I as the intersection of all *WPI*s containing I. We denote it by $\sqrt{_wI}$. If there does not exist any *WPI* containing I, we define $\sqrt{_wI}$ as L.

EXAMPLE 5.3
[28] Consider the lattice shown in Figure 5. We note that $I = (a]$ is not a *WPI* and so is not *PI*. The ideal $J = (e]$ and $K = (d]$ are a *WPI* and not a *PI*. We note that the weakly prime radical of I i.e., $\sqrt{_wI} = J \cap K = I$. Since there is no proper *PI* containing I, the prime radical of I,i.e.,$\sqrt{I} = L$. Thus, the concepts of the prime radical and the weakly prime radical of an ideal are different.

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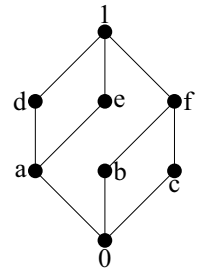


Figure 5. A representative from the family of 8-element lattices.

DEFINITION 5.4
Let $A =< \mu_A, \nu_A >$ be an *IFI* of L. We define the intuitionistic fuzzy weakly radical of A as the intersection of all *IFWPI*s containing A and we denoted it by $\sqrt{_wA}$. We note that for an *IFI* A of L always $A \subseteq \sqrt{_wA}$. It can shown that for an ideal I of L, $\sqrt{_w\chi_I} = \chi_{\sqrt{_wI}}$.

DEFINITION 5.5
A proper *IFI* $A =< \mu, \nu >$ of a lattice L is called an intuitionistic fuzzy weakly primary ideal (*IFWPRI*) of L, if for $a, b \in L, a \wedge b \neq 0$, $\mu(a \wedge b) \leq \mu(a) \vee \sqrt{_w\mu}(b)$ and $\nu(a \wedge b) \geq \nu(a) \wedge \sqrt{_w\nu}(b)$.

LEMMA 5.6
Let I be a proper ideal of L. Then I is a *WPRI* of L if and only if $A =< \chi_I, \chi_{I^c} >$ is an *IFWPRI* of L.

PROOF. Suppose that I is a *WPRI* of L. Let $a, b \in L, a \wedge b \neq 0$. (i)If $a \wedge b \in I$, then as I is a *WPRI* of L, either $a \in I$ or $b \in \sqrt{_wI}$. Hence,

$$\chi_I(a \wedge b) \leq \chi_I(a) \vee \sqrt{_w\chi_I}(b) \text{ and } \chi_{I^c}(a \wedge b) \geq \chi_{I^c}(a) \wedge \sqrt{_w\chi_{I^c}}(b).$$

(ii) If $a \wedge b \notin I$, then clearly $a \notin I$ and $b \notin I$. In this case, also

$$\chi_I(a \wedge b) \leq \chi_I(a) \vee \sqrt{_w\chi_I}(b) \text{ and } \chi_{I^c}(a \wedge b) \geq \chi_{I^c}(a) \wedge \sqrt{_w\chi_{I^c}}(b).$$

Thus, $A =< \chi_I, \chi_{I^c} >$ is an *IFWPRI* of L.

Conversely, suppose that $A =< \chi_I, \chi_{I^c} >$ is an *IFWPRI* of L. Let $a \wedge b \in I, a \wedge b \neq 0$. Then

$$\chi_I(a \wedge b) \leq \chi_I(a) \vee \sqrt{_w\chi_I}(b) \text{ and } \chi_{I^c}(a \wedge b) \geq \chi_{I^c}(a) \wedge \sqrt{_w\chi_{I^c}}(b),$$

implies that either $\chi_I(a) = 1$ or $\sqrt{_w\chi_I}(b) = 1$ and $\chi_{I^c}(a) = 0$ or $\sqrt{_w\chi_{I^c}}(b) = 0$. Thus, either $a \in I$ or $b \in \sqrt{_wI}$. □

LEMMA 5.7
If $A =< \mu, \nu >$ is an *IFWPI* of L, then A is an *IFWPRI* of L.

PROOF. Let $A =< \mu, \nu >$ be an *IFWPI* of L. For all $a, b \in L, a \wedge b \neq 0$,

$$\mu(a \wedge b) \leq \mu(a) \vee \mu(b) \text{ and } \nu(a \wedge b) \geq \nu(a) \wedge \nu(b).$$

Since $A \subseteq \sqrt{_wA}$,

$$\mu(a \wedge b) \leq \mu(a) \vee \sqrt{_w\mu}(b) \text{ and } \nu(a \wedge b) \geq \nu(a) \wedge \sqrt{_w\nu}(b).$$

we get the result. □

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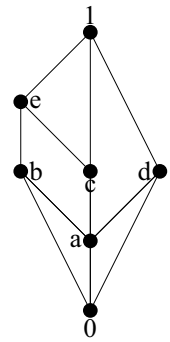


Figure 6. Lattice containing the ideal that is not WPI but is radical WPI.

EXAMPLE 5.8
Consider the ideal $I = (b]$ of the lattice shown in Figure 5. We noted that $I = (b]$ is not a *WPI*, as $c \wedge d = a$ but neither $c \in I$ nor $d \in I$.Ideal $J = (e]$ is *WPI*.The J is the only *WPI* that contains the I. So $\sqrt{_wI} = J$. We know that for any ideal K of L, $\sqrt{_w} < \chi_K, \overline{\chi_{K^{\complement}}} > = < \chi_{\sqrt{_wK}}, \chi_{\overline{\sqrt{_wK}}^{\complement}} >$. Since J is a *WPI*, $< \chi_J, \chi_{J^{\complement}} >$ is a *IFWPI* and so $< \chi_I, \chi_{I^{\complement}} >$ is a *IFWPRI*. We have $\chi_{I^{\complement}}(c \wedge d) = \chi_{I^{\complement}}(a) = 0$ but $\chi_{I^{\complement}}(c) \wedge \chi_{I^{\complement}}(d) = 1$ as $c, d \notin I$. Thus, $< \chi_I, \chi_{I^{\complement}} >$ is not *IFWPI*.

THEOREM 5.9
Let $A = < \mu, \nu >$ be an *IFI* of L. Then A is *IFWPRI* if and only if the level set $A_{(t,s)}$ is a *WPRI* of L.

PROOF. Suppose that $A = < \mu, \nu >$ be an *IFWPRI* of L. Let $a, b \in L, a \wedge b \neq 0$ be such that $a \wedge b \in A_{(t,s)}$ and $a \notin A_{(t,s)}$,$b \notin \sqrt{A_{(t,s)}}$. Then we have

$$t \leq \mu(a \wedge b), t > \mu(a), t > \mu(b)$$

$$s \geq \nu(a \wedge b), s < \nu(a), s < \nu(b).$$

Since A is *IFWPRI*, we have

$$\mu(a \wedge b) \leq \mu(a) \vee \sqrt{_w\mu}(b) \text{ and}$$

$$\nu(a \wedge b) \geq \nu(a) \wedge \sqrt{_w\nu}(b).$$

Thus, we get $t < t$ and $s > s$, which is not possible. Hence, $A_{(t,s)}$ is a *WPRI* of L.

Conversely, suppose that $A_{(t,s)}$ is a *WPRI* of L. Let $a, b \in L, a \wedge b \neq 0$ be such that

$$\mu(a \wedge b) \not\leq \mu(a) \vee \sqrt{_w\mu}(b) \text{ and}$$

$$\nu(a \wedge b) \not\geq \nu(a) \wedge \sqrt{_w\nu}(b).$$

Let $\mu(a \wedge b) = t$ and $\nu(a \wedge b) = s$. Then $t > \mu(a)$,$t > \sqrt{_w\mu}(b)$, $s < \nu(a)$,$s < \sqrt{_w\nu}(b)$. Since $A_{(t,s)}$ is a *WPRI*, $a \wedge b \in A_{(t,s)}$ implies that either $a \in A_{(t,s)}$ or $b \in \sqrt{_wA_{(t,s)}}$,i.e either $\mu(a) \geq t$ or $\sqrt{_w\mu} \geq t$ and $\nu(a) \leq s$ or $\sqrt{_w\nu} \leq s$, a contradiction. □

DEFINITION 5.10
A proper intuitionistic fuzzy ideal $A = < \mu, \nu >$ of a lattice L is called an intuitionistic fuzzy weakly 2-absorbing primary ideal (*IFW2APRI*) of L, if for $a, b, c \in L, a \wedge b \wedge c \neq 0$,=10

$$\mu(a \wedge b \wedge c) \leq \mu(a) \vee \sqrt{_w\mu}(b) \vee \sqrt{_w\mu}(c) \text{ and}$$

$$\nu(a \wedge b \wedge c) \geq \nu(a) \wedge \sqrt{_w\nu}(b) \wedge \sqrt{_w\nu}(c).$$

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LEMMA 5.11
A proper ideal I of L is a weakly 2-absorbing primary ideal (*W2APRI*), if and only if $A =< \chi_I, \chi_{I^c} >$ is an *IF2AWPRI* of L.

PROOF. Suppose that I is a *W2APRI* of L. Let $a, b, c \in L, a \wedge b \wedge c \neq 0$. If $a \wedge b \wedge c \in I$, then $\chi_I(a \wedge b \wedge c) = 1$ and $\chi_{I^c}(a \wedge b \wedge c) = 0$. As I is *W2APRI*, we have either

$$a \wedge b \in I \text{ or } b \wedge c \in \sqrt{_wI} \text{ or } c \wedge a \in \sqrt{_wI}.$$

Hence, either

$$\chi_I(a \wedge b) = 1 \text{ or } \chi_{\sqrt{_wI}}(b \wedge c) = 1 \text{ or } \chi_{\sqrt{_wI}}(c \wedge a) = 1 \text{ and}$$

$$\chi_{I^c}(a \wedge b) = 0 \text{ or } \chi_{\sqrt{_wI^c}}(b \wedge c) = 0 \text{ or } \chi_{\sqrt{_wI^c}}(c \wedge a) = 0.$$

Thus,

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_{\sqrt{_wI}}(b \wedge c) \vee \chi_{\sqrt{_wI}}(c \wedge a)$$

$$\chi_{I^c}(a \wedge b \wedge c) \geq \chi_{I^c}(a \wedge b) \wedge \chi_{\sqrt{_wI^c}}(b \wedge c) \wedge \chi_{\sqrt{_wI^c}}(c \wedge a).$$

If $a \wedge b \wedge c \notin I$, then $\chi_I(a \wedge b \wedge c) = 0$ and $\chi_{I^c}(a \wedge b \wedge c) = 1$. Clearly, $a \wedge b \notin I$, hence

$$\chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_{\sqrt{_wI}}(b \wedge c) \vee \chi_{\sqrt{_wI}}(c \wedge a)$$

$$\chi_{I^c}(a \wedge b \wedge c) \geq \chi_{I^c}(a \wedge b) \wedge \chi_{\sqrt{_wI^c}}(b \wedge c) \wedge \chi_{\sqrt{_wI^c}}(c \wedge a).$$

Thus, A is an *IFW2APRI*.

Conversely, suppose that A is an *IFW2APRI*. Let $a \wedge b \wedge c \in I, a \wedge b \wedge c \neq 0$. Then $\chi_I(a \wedge b \wedge c) = 1$ and $\chi_{I^c}(a \wedge b \wedge c) = 0$. Suppose that $a \wedge b \notin I, b \wedge c \notin \sqrt{_wI}$ and $c \wedge a \notin \sqrt{_wI}$. Since A is an *IFW2APRI*, we have

$$1 = \chi_I(a \wedge b \wedge c) \leq \chi_I(a \wedge b) \vee \chi_{\sqrt{_wI}}(b \wedge c) \vee \chi_{\sqrt{_wI}}(c \wedge a)$$

$$0 = \chi_{I^c}(a \wedge b \wedge c) \geq \chi_{I^c}(a \wedge b) \wedge \chi_{\sqrt{_wI^c}}(b \wedge c) \wedge \chi_{\sqrt{_wI^c}}(c \wedge a).$$

Since each of $\chi_I(a \wedge b)$, $\chi_{\sqrt{_wI}}(b \wedge c)$, $\chi_{\sqrt{_wI}}(c \wedge a)$ belongs to $[0, 1]$, at least one of these numbers must be 1. This implies that either

$$a \wedge b \in I \text{ or } b \wedge c \in \sqrt{_wI} \text{ or } c \wedge a \in \sqrt{_wI}.$$

Thus, I is a *W2APRI*. □

LEMMA 5.12
If $A =< \mu, \nu >$ is an *IFWPRI* of L, then $A =< \mu, \nu >$ is an *IFW2APRI* of L.

PROOF. Let $A =< \mu, \nu >$ be an *IFWPRI* of L. Let $a, b, c \in L, a \wedge b \wedge c \neq 0$. As A is an *IFWPRI*, we have

$$\begin{aligned}
\mu(a \wedge b \wedge c) = \mu(a \wedge b \wedge b \wedge c) &\leq \mu(a \wedge b) \vee \sqrt{_w\mu}(b \wedge c) \\
&\leq \mu(a \wedge b) \vee \sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(c \wedge a) \\
\nu(a \wedge b \wedge c) = \nu(a \wedge b \wedge b \wedge c) &\geq \nu(a \wedge b) \wedge \sqrt{_w\nu}(b \wedge c) \\
&\geq \nu(a \wedge b) \wedge \sqrt{_w\nu}(b \wedge c) \wedge \sqrt{_w\nu}(c \wedge a)
\end{aligned}$$

Thus, A is an *IFW2APRI*. □

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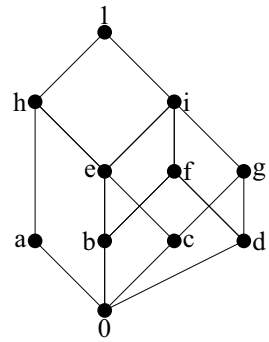


Figure 7. A representative from the family of 11-element lattices.

EXAMPLE 5.13
Consider the ideal $I = (d]$ of the lattice shown in Figure 7. We note that the ideals $(i] = \{0, b, c, d, e, f, g, i\}$ is the only *PI* containing I and $(g] = \{0, c, d, g\}$,$(f] = \{0, b, d, f\}$ are the only *WPI*s of L containing I. Hence, $\sqrt{I}_w = (i] \cap (g] \cap (f] = (d] = I$. We note that I is a *W2APRI* as for any $x, y, z \in L$, $x \wedge y \wedge z \in I$ implies that either $x \wedge y \in I$ or $y \wedge z \in \sqrt{_wI}$ or $z \wedge x \in \sqrt{_wI}$. Hence, by Lemma 5.3, $< \chi_I, \chi_{I^c} >$ is an *IFW2APRI* of L. We note that $\chi_I(f \wedge g) = 1$ and $\chi_{I^c}(f \wedge g) = 0$ but $\chi_I(g) = 0$ as well as $\chi_{\sqrt{_wI}}(f) = 0$. Thus, $\chi_I(f \wedge g) \nleq \chi_I(f) \vee \chi_{\sqrt{_wI}}(g)$. Hence, $< \chi_I, \chi_{I^c} >$ is not an *IFWPRI* of L.

LEMMA 5.14
If $A =< \mu, \nu >$ is an *IFW2AI* of L, then $A =< \mu, \nu >$ is an *IFW2APRI* of L.

PROOF. Let $A =< \mu, \nu >$ be an *IFW2AI* of L. Let $a, b, c \in L, a \wedge b \wedge c \neq 0$. Since $A =< \mu, \nu >$ is an *IFW2AI*, we get

$$\mu(a \wedge b \wedge c) \leq \mu(a \wedge b) \vee \mu(b \wedge c) \vee \mu(c \wedge a)$$
$$\nu(a \wedge b \wedge c) \geq \nu(a \wedge b) \wedge \nu(b \wedge c) \wedge \nu(c \wedge a)$$

Since $A \subseteq \sqrt{_wA}$, we get the result. □

EXAMPLE 5.15
Consider the lattice shown in Figure 8. The only *WPI* of L containing the ideal $I = (f]$ is $J = (n]$. We have $\sqrt{_wI} = (n]$. We note that I is a *2APRI* of L. Hence, by Lemma 5.3, $A =< \chi_I, \chi_{I^c} >$ is an *IF2APRI* of L. We note that I is not a weakly 2-absorbing ideal of L, as $n \wedge o \wedge p = c \in I$, but $n \wedge o \notin I$, $n \wedge p \notin I$ and $o \wedge p \notin I$. We have

$$\chi_I(n \wedge o \wedge p) = 1 \nleq \chi_I(n \wedge o) \vee \chi_I(o \wedge p) \vee \chi_I(p \wedge n) = 0$$
$$\chi_{I^c}(n \wedge o \wedge p) = 0 \ngeq \chi_{I^c}(n \wedge o) \wedge \chi_{I^c}(o \wedge p) \wedge \chi_{I^c}(p \wedge n) = 1.$$

This $A =< \chi_I, \chi_{I^c} >$ is not an *IFW2AI* of L.

LEMMA 5.16
Let $A =< \mu, \nu >$ be an *IFI* of L. If $\sqrt{_wA}$ is an *IFWPI*, then A is an *IFW2APRI*.

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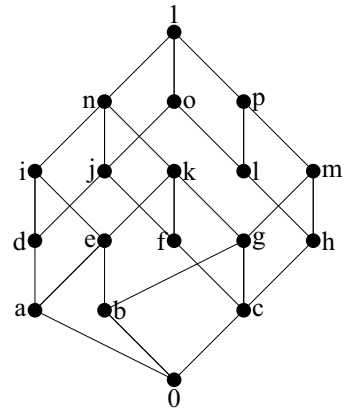


Figure 8. A representative from the family of 18-element lattices.

PROOF. Let A be an *IFI* of L. Suppose that $\sqrt{_wA}$ is an *IFWPI*. If A is not an *IFW2APRI*, then there exist $a, b, c \in L, a \wedge b \wedge c \neq 0$ such that

$$\mu(a \wedge b \wedge c) \not\leq \mu(a \wedge b) \vee \sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(c \wedge a)$$
$$\nu(a \wedge b \wedge c) \not\geq \nu(a \wedge b) \wedge \sqrt{_w\nu}(b \wedge c) \wedge \sqrt{_w\nu}(c \wedge a).$$

This implies that

$$\mu(a \wedge b) \vee \sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(c \wedge a) < \mu(a \wedge b \wedge c)$$
$$\nu(a \wedge b) \wedge \sqrt{_w\nu}(b \wedge c) \wedge \sqrt{_w\nu}(c \wedge a) > \nu(a \wedge b \wedge c).$$

Since $\sqrt{_wA}$ is an *IFWPI*, we have

$$\sqrt{_w\mu}(a \wedge b \wedge c) = \sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(a) = \sqrt{_w\mu}(a \wedge c) \vee \sqrt{_w\mu}(b).$$
$$\sqrt{_w\nu}(a \wedge b \wedge c) = \sqrt{_w\nu}(b \wedge c) \wedge \sqrt{_w\nu}(a) = \sqrt{_w\nu}(a \wedge c) \wedge \sqrt{_w\nu}(b).$$

Hence,

$$\begin{aligned}\sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(a \wedge c) &= \sqrt{_w\mu}(b \wedge c) \vee \sqrt{_w\mu}(a) \vee \sqrt{_w\mu}(c)\\ &= \sqrt{_w\mu}(a \wedge b \wedge c) \vee \sqrt{_w\mu}(c)\\ \sqrt{_w\nu}(b \wedge c) \wedge \sqrt{_w\nu}(a \wedge c) &= \sqrt{_w\nu}(b \wedge c) \wedge \sqrt{_w\nu}(a) \wedge \sqrt{_w\nu}(c)\\ &= \sqrt{_w\nu}(a \wedge b \wedge c) \wedge \sqrt{_w\nu}(c).\end{aligned}$$

Thus,

$$\mu(a \wedge b) \vee \sqrt{_w\mu}(a \wedge b \wedge c) \vee \sqrt{_w\mu}(c) < \mu(a \wedge b \wedge c)$$
$$\nu(a \wedge b) \wedge \sqrt{_w\nu}(a \wedge b \wedge c) \wedge \sqrt{_w\nu}(c) > \nu(a \wedge b \wedge c).$$

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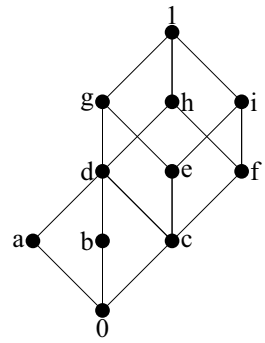


Figure 9. A representative from the family of 11-element lattices.

This implies that

$$\sqrt{{}_w\mu}(a \wedge b \wedge c) < \mu(a \wedge b \wedge c)$$
$$\sqrt{{}_w\nu}(a \wedge b \wedge c) > \nu(a \wedge b \wedge c),$$

which is not possible. Hence, A is $IFW2APRI$. □

EXAMPLE 5.17
Consider the lattice shown in Figure 9. The only WPIs of L containing the ideal $I = (a]$ are $(g]$ and $(h]$. Hence, $\sqrt{{}_wI} = (g] \cap (h] = (d]$. For any $x, y, z \in I, x \wedge y \wedge z \neq 0$, $x \wedge y \wedge z \in I$ implies that either $x \wedge y \in I$ or $y \wedge z \in \sqrt{{}_wI}$ or $x \wedge z \in \sqrt{{}_wI}$. Hence, I is a $W2APRI$ and so by lemma 5.3, $A =< \chi_I, \chi_{I^{\complement}} >$ is an $IFW2APRI$. We note that $e \wedge f = c \in \sqrt{{}_wI}$ but $e \notin \sqrt{{}_wI}$ and $f \notin \sqrt{{}_wI}$. Thus, $\sqrt{{}_wI}$ is not a WPI of L. Hence, by Theorem 3.1, $\sqrt{{}_wA}$ is not an $IFWPI$ of L.

LEMMA 5.18
Let $A =< \mu, \nu >$ be an IFI of L. Then, $\sqrt{{}_wA} = \sqrt{{}_w\sqrt{{}_wA}}$.

THEOREM 5.19
Let $A =< \mu, \nu >$ be an IFI of L. Then $\sqrt{{}_wA}$ is $IFWPI$ if and only if $\sqrt{{}_wA}$ is $IFWPRI$.

PROOF. It follows from Lemma 5.2, that if $\sqrt{{}_wA}$ is $IFWPI$, then $\sqrt{{}_wA}$ is $IFWPRI$. The converse follows from the definition of a fuzzy weakly $WPRI$ and by lemma 5.7. □

THEOREM 5.20
Let $A =< \mu, \nu >$ be an IFI of L. Then $\sqrt{{}_wA}$ is $IFW2AI$ if and only if $\sqrt{{}_wA}$ is $IFW2APRI$.

6 Intuitionistic fuzzy weakly ideals in a direct product of a lattices

In this section, we consider intuitionistic fuzzy ideals as a direct product of lattices. It is known that if $L_1, L_2, ..., L_k$ are lattices, then their Cartesian product $L = L_1 \times L_2 \times ... \times L_k$ is a lattice under componentwise operations of meet and join and if $a = (a_1, ..., a_k)$, $b = (b_1, ..., b_k)$ then $a \leq b$ iff $a_i \leq b_i$ for $i = 1, ..., k$.

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DEFINITION 6.1
Let $L = L_1 \times L_2 \times ... \times L_k$ be a direct product of lattices $L_1, L_2, ..., L_k$. A mapping $A : L \to [0,1] \times [0,1]$ is called an *IFS* of L.

THEOREM 6.2
Let $L = L_1 \times L_2 \times ... \times L_k$ be a direct product of lattices $L_1, L_2, ..., L_k$. If $A_i =< \mu_i, \nu_i >$, $1 \leq k$ are *IFI*s of L_i respectively, then for $A : L \to [0,1] \times [0,1]$, $\mu, \nu : L \to [0,1]$ defined by

$$\mu(a_1, ..., a_k) = \mu_1(a_1) \wedge ... \wedge \mu_k(a_k) \text{ and}$$
$$\nu(a_1, ..., a_k) = \nu_1(a_1) \vee ... \vee \nu_k(a_k)$$

is an *IFI* of L.

PROOF. The proof follows from the definition of the lattice operations in a direct product of lattices and that of A. □

Notation: We call the *IFS* A in Theorem 6.1 as product of the fuzzy sets μ_i and ν_i, $1 \leq k$, $\mu = \mu_1 \times ... \times \mu_k$ and $\nu = \nu_1 \times ... \times \nu_k$.

THEOREM 6.3
Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. If $A : L \to [0,1] \times [0,1]$ is an *IFI*s of L, then there exist *IFI*s A_1, A_2 of L_1 and L_2 respectively, such that $\mu = \mu_1 \times \mu_2$ and $\nu = \nu_1 \times \nu_2$. Moreover, if A is *IFPI*, then so are A_1 and A_2.

PROOF. Define $\mu_i, \nu_i : L_i \to [0,1]$ by $\mu_1(x) = \mu(x,0)$,$\nu_1(x) = \nu(x,0)$, $\mu_2(y) = \mu(0,y)$,$\nu_2(y) = \nu(0,y)$. Let $x, y \in L_1$. We have

$$\mu[(x,0) \wedge (y,0)] = \mu(x \wedge y, 0) = \mu_1(x \wedge y),$$
$$\nu[(x,0) \wedge (y,0)] = \nu(x \wedge y, 0) = \nu_1(x \wedge y) \text{ and}$$
$$\mu[(x,0) \vee (y,0)] = \mu(x \vee y, 0) = \mu_1(x \vee y),$$
$$\nu[(x,0) \vee (y,0)] = \nu(x \vee y, 0) = \nu_1(x \vee y).$$

Hence,

$$\mu_1(x \wedge y) \wedge \mu_1(x \vee y) = \mu[(x,0) \wedge (y,0)] \wedge \mu[(x,0) \vee (y,0)],$$
$$\nu_1(x \wedge y) \vee \nu_1(x \vee y) = \nu[(x,0) \wedge (y,0)] \vee \nu[(x,0) \vee (y,0)].$$

As A is an *IFI*, we get

$$\begin{aligned}
\mu_1(x \wedge y) \wedge \mu_1(x \vee y) &= \mu[(x,0) \wedge (y,0)] \wedge \mu[(x,0) \vee (y,0)] \\
&\geq \mu(x,0) \wedge \mu(y,0) \\
&= \mu_1(x) \wedge \mu_2(y), \\
\nu_1(x \wedge y) \vee \nu_1(x \vee y) &= \nu[(x,0) \wedge (y,0)] \vee \nu[(x,0) \vee (y,0)] \\
&\leq \nu(x,0) \vee \nu(y,0) \\
&= \nu_1(x) \vee \nu_1(y).
\end{aligned}$$

Also,

$$\mu_1(x \vee y) = \mu[(x,0) \vee (y,0)] = \mu(x,0) \wedge \mu(y,0) = \mu_1(x) \wedge \mu_1(y).$$
$$\nu_1(x \wedge y) = \nu[(x,0) \wedge (y,0)] = \nu(x,0) \vee \nu(y,0) = \nu_1(x) \vee \nu_1(y).$$

Thus, A_1 is an *IFI* of L_1. Similarly, we can show that A_2 is an *IFI* of L_2. The second part follows from the definition of an *IFPI*. We have

$$\mu(x,y) = \mu(x,0) \vee \mu(0,y) = \mu(x,0) \wedge \mu(0,y) = \mu_1(x) \wedge \mu_2(y),$$
$$\nu(x,y) = \nu(x,0) \wedge \nu(0,y) = \nu(x,0) \vee \nu(0,y) = \nu_1(x) \vee \nu_2(y).$$

□

LEMMA 6.4
Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. If A is an *IFWPI*s of L, then the *IFI* A_1 and A_2 in Theorem 6.2 are *IFWPI*.

PROOF. Let $x, y \in L_1, x \wedge y \neq 0$. We have

$$\begin{aligned}
\mu_1(x \wedge y) &= \mu(x \wedge y, 0) \\
&= \mu[(x,0) \wedge (y,0)] \\
&\leq \mu[(x,0)] \vee \mu[(y,0)] \\
&= \mu_1(x) \vee \mu_1(y) \\
\nu_1(x \wedge y) &= \nu(x \wedge y, 0) \\
&= \nu[(x,0) \wedge (y,0)] \\
&\geq \nu[(x,0)] \wedge \nu[(y,0)] \\
&= \nu_1(x) \wedge \nu_1(y)
\end{aligned}$$

Thus, $< \mu_1, \nu_1 >$ is an *IFWPI* of L_1. Similarly, we can show that $< \mu_2, \nu_2 >$ is an *IFWPI* of L_2. □

EXAMPLE 6.5
Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let A_1, A_2 be *IFPI*s of L_1 and L_2, respectively. Then $\mu = \mu_1 \times \mu_2$ and $\nu = \nu_1 \times \nu_2$ need not be an *IFWPI*s of L. Consider the lattices L_1 and L_2 as shown in Figure 10. The ideal $I = (f]$ is a *WPI* of L_1 and $J = (x]$ that of L_2. However, the ideal $I \times J = ((f,x)]$ is not a *WPI* of L as $(b,x) \wedge (a,x) = (0,x) \in I \times J$ but $(b,x) \notin I \times J$ and $(a,x) \notin I \times J$.

REMARK
From Example 6.1, we conclude that in general,

$$\sqrt{\mu \times \nu} \neq \sqrt{\mu} \times \sqrt{\nu}.$$

THEOREM 6.6
Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let A_1, A_2 be *IFI*s of L_1 and L_2, respectively. Suppose that $A_1(0_1) = A_2(0_2) = (1,0)$, where 0_1 is the least element of L_1 and 0_2 that of L_2. If $A = A_1 \times A_2$ is an *IFW2AI* of L, then A_1 is an *IFW2AI* of L_1 and A_2 that of L_2.

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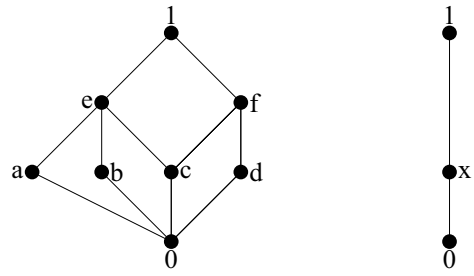


Figure 10. Two lattices with WPI.

PROOF. Let $a, b, c \in L_1, a \wedge b \wedge c \neq 0$. Since A is an *IFW2AI* of L, we have

$$\mu(a \wedge b \wedge c, 0_2) \leq \mu(a \wedge b, 0_2) \vee \mu(b \wedge c, 0_2) \vee \mu(a \wedge c, 0_2) \text{ and}$$
$$\nu(a \wedge b \wedge c, 0_2) \leq \nu(a \wedge b, 0_2) \wedge \nu(b \wedge c, 0_2) \wedge \nu(a \wedge c, 0_2).$$

By the definition of A, we can write

$$\begin{aligned}\mu_1(a \wedge b \wedge c) \wedge \mu_2(0_2) &\leq [\mu_1(a \wedge b) \wedge \mu_2(0_2)] \\ &\quad \vee [\mu_1(b \wedge c) \wedge \mu_2(0_2)] \vee [\mu_1(a \wedge c) \wedge \mu_2(0_2)] \text{ and} \\ \nu_1(a \wedge b \wedge c) \vee \nu_2(0_2) &\geq [\nu_1(a \wedge b) \wedge \nu_2(0_2)] \\ &\quad \wedge [\nu_1(b \wedge c) \wedge \nu_2(0_2)] \wedge [\mu_1(a \wedge c) \wedge \nu_2(0_2)].\end{aligned}$$

By using $A_2(0_2) = (1, 0)$, we get

$$\mu_1(a \wedge b \wedge c) \leq \mu_1(a \wedge b) \vee \mu_1(b \wedge c) \vee \mu_1(a \wedge c) \text{ and}$$
$$\nu_1(a \wedge b \wedge c) \geq \nu_1(a \wedge b) \wedge \nu_1(b \wedge c) \wedge \nu_1(a \wedge c).$$

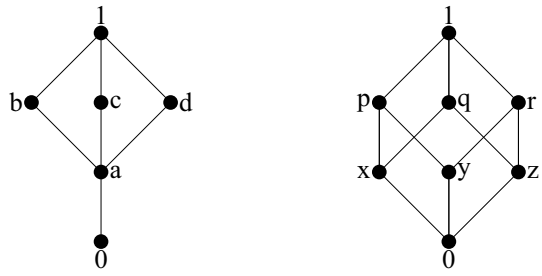
Thus, A_1 is an *IFW2AI* of L_1. Similarly,we can prove that A_2 is an *IFW2AI* of L_2. □

THEOREM 6.7
Let $L = L_1 \times L_2 \times ... \times L_k$ be a direct product of lattices $L_1, L_2, ..., L_k$. Let A_i, $1 \leq k$ be *IFI*s of L_i, respectively. Suppose that for each $i = 1, 2, ..., k$, $A_i(0_i) = (1, 0)$, where 0_i is the least element of L_i. If $A = A_1 \times A_2 \times ... \times A_k$ is an *IFW2AI* of L, then A_i, is an *IFW2AI* of L_i, $i = 1, ..., k$.

The following example shows that the converse of Theorem 6.5 need not hold.

EXAMPLE 6.8
Consider the lattices L_1, L_2 and $L = L_1 \times L_2$ as shown in the Figure 11. We note that $I = (a]$ and $J = (x]$ are *W2AI*s. However, the ideal $K = I \times J$ is not a *W2AI* of L, as $(b, 1) \wedge (c, y) \wedge (1, x) = (a, 0) \in K$ but $(b, 1) \wedge (c, y) = (a, y) \notin K$, $(c, y) \wedge (1, x) = (c, 0) \notin K$ and $(b, 1) \wedge (1, x) = (b, x) \notin K$. By Lemma 4.2, $< \chi_I, \chi_{I^{\complement}} >$ and $< \chi_J, \chi_{J^{\complement}} >$ are *IFW2AI* and $\chi_K = \chi_I \times \chi_J$ are *IF2AI* and $\chi_K = \chi_I \times \chi_J$ is not *IFW2AI*.

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L_1 L_2

Figure 11. Two lattices with W2AI.

THEOREM 6.9
Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let A_1, A_2 be *IFI*s of L_1 and L_2, respectively. Suppose that (i) $A_1(0_1) = A_2(0_2) = (1, 0)$, where 0_1 is the least element of L_1 and 0_2 that of L_2 and (ii) $A_1(1_1) = A_2(1_2) = (0, 1)$, where 1_1 is the greatest element of L_1 and 1_2 that of L_2. If $A = A_1 \times A_2$ is an *IFW2AI* of L, then A_1 is an *IFWPI* of L_1 and A_2 that of L_2.

PROOF. Suppose that A_1 is not an *IFWPI* of L_1. Then there exist $a, b \in L_1, a \wedge b \neq 0$ such that

$$\mu(a \wedge b) \not\leq \mu(a) \vee \mu(b) \text{ and } \nu(a \wedge b) \not\geq \nu(a) \wedge \nu(b).$$

Consider the elements $x = (a, 1), y = (1, 0), z = (b, 1)$ from L. Then $x \wedge y \wedge z \neq 0$. We note the following: $\mu(x \wedge y \wedge z) = \mu(a \wedge b, 0) = \mu_1(a \wedge b) \wedge \mu_2(0) = \mu_1(a \wedge b)$. $\mu(x \wedge y) = \mu(a, 0) = \mu_1(a) \wedge \mu_2(0) = \mu_1(a)$. $\mu(y \wedge z) = \mu(b, 0) = \mu_1(b) \wedge \mu_2(0) = \mu_1(b)$. $\mu(x \wedge z) = \mu(a \wedge b, 1) = \mu_1(a \wedge b) \wedge \mu_2(1) = \mu_2(1) = 0$. $\nu(x \wedge y \wedge z) = \nu(a \wedge b, 0) = \nu_1(a \wedge b) \vee \nu_2(0) = \nu_1(a \wedge b)$. $\nu(x \wedge y) = \nu(a, 0) = \nu_1(a) \vee \nu_2(0) = \nu_1(a)$. $\nu(y \wedge z) = \nu(b, 0) = \nu_1(b) \vee \nu_2(0) = \nu_1(b)$. $\nu(x \wedge z) = \nu(a \wedge b, 1) = \nu_1(a \wedge b) \vee \nu_2(1) = \nu_2(1) = 1$. Since A is an *IFW2AI*, we have

$$\mu(x \wedge y \wedge z) \leq \mu(x \wedge y) \vee \mu(y \wedge z) \vee \mu(x \wedge z),$$

$$\nu(x \wedge y \wedge z) \geq \nu(x \wedge y) \wedge \nu(y \wedge z) \wedge \nu(x \wedge z),$$

i.e.

$$\mu_1(a \wedge b) \leq \mu_1(a) \vee \mu_1(b) \vee 0 = \mu_1(a) \vee \mu_1(b),$$

$$\nu_1(a \wedge b) \geq \nu_1(a) \wedge \nu_1(b) \wedge 0 = \nu_1(a) \wedge \nu_1(b),$$

a contradiction. Hence, A_1 is an *IFWPI*. Similarly, we can show that A_2 is an *IFWPI*. □

THEOREM 6.10
Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2. Let A_1, A_2 be *IFWPI*s of L_1 and L_2, respectively. If $A = A_1 \times A_2$, then A is an *IFW2AI* of L.

PROOF. Let $x_1 = (a, x), x_2 = (b, y), x_3 = (c, z) \in L$ such that $x_1 \wedge x_2 \wedge x_3 \neq 0$. To show that A is an *IFW2AI*, we need to show that

$$\mu[(a, x) \wedge (b, y) \wedge (c, z)] \leq \mu[(a, x) \wedge (b, y)] \vee \mu[(b, y) \wedge (c, z)] \vee \mu[(a, x) \wedge (c, z)],$$

$$\nu[(a, x) \wedge (b, y) \wedge (c, z)] \geq \nu[(a, x) \wedge (b, y)] \wedge \nu[(b, y) \wedge (c, z)] \wedge \nu[(a, x) \wedge (c, z)],$$

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i.e. show that

$$\mu(a \wedge b \wedge c, x \wedge y \wedge z) \leq \mu(a \wedge b, x \wedge y) \vee \mu(b \wedge c, y \wedge z) \vee \mu(a \wedge c, x \wedge z),$$
$$\nu(a \wedge b \wedge c, x \wedge y \wedge z) \geq \nu(a \wedge b, x \wedge y) \wedge \nu(b \wedge c, y \wedge z) \wedge \nu(a \wedge c, x \wedge z).$$

We have

$$\mu(a \wedge b \wedge c, x \wedge y \wedge z) = \mu(a \wedge b \wedge c) \wedge (x \wedge y \wedge z),$$
$$\nu(a \wedge b \wedge c, x \wedge y \wedge z) = \nu(a \wedge b \wedge c) \vee (x \wedge y \wedge z).$$

As A_1, A_2 be *IFWPI*s, we can write

$$\mu_1(a \wedge b \wedge c) = \mu_1(a) \vee \mu_1(b) \vee \mu_1(c),$$
$$\nu_1(a \wedge b \wedge c) = \nu_1(a) \wedge \nu_1(b) \wedge \nu_1(c)$$

and

$$\mu_2(x \wedge y \wedge z) = \mu_2(x) \vee \mu_2(y) \vee \mu_2(z),$$
$$\nu_2(x \wedge y \wedge z) = \nu_2(x) \wedge \nu_2(y) \wedge \nu_2(z).$$

Also, we have

$$\begin{aligned}\mu(a \wedge b, x \wedge y) \vee \mu(b \wedge c, y \wedge z) \vee \mu(a \wedge c, x \wedge z) &= [\mu_1(a \wedge b) \wedge \mu_2(x \wedge y)] \\ &\vee [\mu_1(b \wedge c) \wedge \mu_2(y \wedge z)] \\ &\vee [\mu_1(a \wedge c) \wedge \mu_2(x \wedge z)], \\ \nu(a \wedge b, x \wedge y) \wedge \nu(b \wedge c, y \wedge z) \wedge \nu(a \wedge c, x \wedge z) &= [\nu_1(a \wedge b) \vee \nu_2(x \wedge y)] \\ &\wedge [\nu_1(b \wedge c) \vee \nu_2(y \wedge z)] \\ &\wedge [\nu_1(a \wedge c) \vee \mu_2(x \wedge z)].\end{aligned}$$

Since A_1, A_2 be *IFWPI*s, we can write

$$\begin{aligned}&\{[\mu_1(a) \vee \mu_1(b)] \wedge [\mu_2(x) \vee \mu_2(y)]\} \\ &\vee \{[\mu_1(b) \vee \mu_1(c)] \wedge [\mu_2(y) \vee \mu_2(z)]\} \\ &\vee \{[\mu_1(a) \vee \mu_1(c)] \wedge [\mu_2(x) \vee \mu_2(z)]\}, \text{ and} \\ &\{[\nu_1(a) \wedge \nu_1(b)] \wedge [\nu_2(x) \wedge \nu_2(y)]\} \\ &\wedge \{[\nu_1(b) \wedge \nu_1(c)] \wedge [\nu_2(y) \wedge \nu_2(z)]\} \\ &\wedge \{[\nu_1(a) \wedge \nu_1(c)] \wedge [\nu_2(x) \wedge \nu_2(z)]\}.\end{aligned}$$

By applying distributivity,

$$[\mu_1(a) \vee \mu_1(b) \vee \mu_1(c)] \wedge [\mu_2(x) \vee \mu_2(y) \vee \mu_2(z)],$$
$$[\nu_1(a) \wedge \nu_1(b) \wedge \nu_1(c)] \wedge [\nu_2(x) \wedge \nu_2(y) \wedge \nu_2(z)].$$

Thus, A is *IFW2AI*. □

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DEFINITION 6.11
[28] A lattice L is called an integral lattice, if for nonzero $x, y \in L$, $x \wedge y \neq 0$.

THEOREM 6.12
Let $L = L_1 \times L_2$ be a direct product of lattices L_1, L_2 with 0. Suppose that L_1 is an integral lattice and $A_1 =< \mu_1, \nu_1 >$ is a nonconstant *IFI* of L_1 such that $A_1(0) = (1, 0)$. The following statements are equivalent:

(1) $A_1 \times B$ is an *IF*2*AI* of L, where $B =< \chi_{L_2}, \chi_{L_2^{\complement}} >$.
(2) $A_1 \times B$ is an *IFW*2*AI* of L.
(3) $A_1 \times B$ is an *IF*2*AI* of L.
(4) A_1 is an *IF*2*AI* of L.
(5) A_1 is an *IFW*2*AI*of L.

PROOF. (1) $\Rightarrow$ (2): Obvious.
(2) $\Rightarrow$ (3): Let $x = (a_1, b_1), y = (a_2, b_2),$ and $z = (a_3, b_3)$ be elements of L.
Case 1. Suppose that $x \wedge y \wedge z \neq 0$. Then clearly,

$$(A_1 \times B)(x \wedge y \wedge z) \leq (A_1 \times B)(x \wedge y) \vee (A_1 \times B)(y \wedge z) \vee (A_1 \times B)(x \wedge z).$$

Case 2. Suppose that $x \wedge y \wedge z = 0$. Then $a_1 \wedge a_2 \wedge a_3 = 0$. Since L_1 is an integral lattice, at least one of a_i must be 0. Without loss generality, we may assume that $a_1 = 0$. We have to show that

$$(A_1 \times B)(x \wedge y \wedge z) \leq (A_1 \times B)(x \wedge y) \vee (A_1 \times B)(y \wedge z) \vee (A_1 \times B)(x \wedge z).$$

The L.H.S. of the equation can be written as

$$\begin{aligned}(A_1 \times B)(x \wedge y \wedge z) &= (A_1 \times B)(a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3)\\ &= A_1(a_1 \wedge a_2 \wedge a_3) \wedge B(b_1 \wedge b_2 \wedge b_3)\\ &= 1.\end{aligned}$$

Since we have assumed that $a_1 = 0$, we have

$$(A_1 \times B)(x \wedge y) = (A_1 \times B)(0, b_1 \wedge b_2) = A_1(0) \wedge B(b_1 \wedge b_2) = 1.$$

Hence, the R.H.S. of the equation is equal to 1. We get the result.
(3) $\Rightarrow$ (4): Let $a_1, a_2, a_3 \in L_1$.Let $b \in L_2$. We have

$$\begin{aligned}&A_1(a_1 \wedge a_2 \wedge a_3) = A_1(a_1 \wedge a_2 \wedge a_3) \wedge 1\\ &= A_1(a_1 \wedge a_2 \wedge a_3) \wedge B(b)\\ &= (A_1 \times B)((a_1, b) \wedge (a_2, b) \wedge (a_3, b))\\ &\leq (A_1 \times B)((a_1, b) \wedge (a_2, b)) \vee (A_1 \times B)((a_2, b) \wedge (a_3, b))\\ &\vee (A_1 \times B)((a_1, b) \wedge (a_3, b))\\ &= [A_1(a_1 \wedge a_2) \wedge B(b)] \vee [A_1(a_2 \wedge a_3) \wedge B(b)] \vee [A_1(a_1 \wedge a_3) \wedge B(b)]\\ &= A_1(a_1 \wedge a_2) \vee A_1(a_2 \wedge a_3) \vee A_1(a_1 \wedge a_3) \text{ as } B(b) = 1.\end{aligned}$$

Thus, A is *IF*2*AI*.
(4) $\Rightarrow$ (5): Obvious

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(5) $\Rightarrow$ (1): To show that $A_1 \times B$ is *IF2AI*. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in L$. Case 1. Suppose that $a_1 \wedge a_2 \wedge a_3 \neq 0$. We have

$$\begin{aligned}
&(A_1 \times B)((a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3)) \\
&= (A_1 \times B)(a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3) \\
&= A_1(a_1 \wedge a_2 \wedge a_3) \wedge B(b_1 \wedge b_2 \wedge b_3) \\
&= A_1(a_1 \wedge a_2 \wedge a_3) \text{ as } B(b_1 \wedge b_2 \wedge b_3) = 1 \\
&\leq A_1(a_1 \wedge a_2) \vee A_1(a_2 \wedge a_3) \vee A_1(a_1 \wedge a_3) \text{ as A is IF2AI} \\
&\leq [A_1(a_1 \wedge a_2) \wedge B(b_1 \wedge b_2)] \vee [A_1(a_2 \wedge a_3) \wedge B(b_2 \wedge b_3)] \\
&\vee [A_1(a_1 \wedge a_3) \wedge B(b_1 \wedge b_3)] \\
&= (A_1 \times B)((a_1, b_1) \wedge (a_2, b_2)) \vee (A_1 \times B)((a_2, b_2) \wedge (a_3, b_3)) \\
&\vee (A_1 \times B)((a_1, b_1) \wedge (a_3, b_3)). \text{ as}
\end{aligned}$$

Case 2. Suppose that $a_1 \wedge a_2 \wedge a_3 = 0$.Since L is an integral lattice, at least one $a_i = 0$.Without loss of generality, we assume that $a_1 = 0$. We have

$$\begin{aligned}
&(A_1 \times B)((a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3)) \\
&= (A_1 \times B)(a_1 \wedge a_2 \wedge a_3, b_1 \wedge b_2 \wedge b_3) \\
&= (A_1 \times B)(0, b_1 \wedge b_2 \wedge b_3) \\
&= A_1(0) \wedge B(b_1 \wedge b_2 \wedge b_3) \\
&= 1.
\end{aligned}$$

We have

$$\begin{aligned}
(A_1 \times B)((a_1, b_1) \wedge (a_2, b_2)) &= (A_1 \times B)((a_1 \wedge a_2, b_1 \wedge b_2)) \\
&= (A_1 \times B)((0, b_1 \wedge b_2)) \\
&= 1
\end{aligned}$$

Hence,

$$(A_1 \times B)((a_1, b_1) \wedge (a_2, b_2)) \vee (A_1 \times B)((a_2, b_2) \wedge (a_3, b_3)) = 1.$$

Thus,

$$\begin{aligned}
&(A_1 \times B)((a_1, b_1) \wedge (a_2, b_2) \wedge (a_3, b_3)) \\
&\leq (A_1 \times B)((a_1, b_1) \wedge (a_2, b_2)) \vee (A_1 \times B)((a_2, b_2) \wedge (a_3, b_3)).
\end{aligned}$$

Thus, $A_1 \times B$ is *IF2AI*. □

7 Conclusion

In this work, we introduced and examined several important concepts related to lattices, with a focus on intuitionistic fuzzy weakly prime ideals, prime intuitionistic fuzzy ideals, intuitionistic fuzzy 2-absorbing ideals, 2-absorbing intuitionistic fuzzy ideals, intuitionistic fuzzy weakly primary

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ideals, and primary intuitionistic fuzzy ideals. We provided a comprehensive characterization of intuitionistic fuzzy 2-absorbing weakly primary ideals within the lattice framework, exploring the relationships between these various notions and their connections with traditional lattice ideals. Furthermore, we extended the analysis to include these concepts in the context of lattices constructed through the Cartesian product of individual lattices. This in-depth investigation enhances our understanding of the complex relationships and behaviors of these fuzzy ideals in more intricate lattice structures.

Our research addressed several key questions:

1) The concept of bipolar-valued $2APRIFWI(L)$ will be investigated.
2) The notion of hesitant $2APRIFWI(L)$ will be introduced.
3) The idea of rough $2APRIFWI(L)$ will be explored.
4) The theory of neutrosophic vague $2APRIFWI(L)$ will be elucidated.
5) The concept of complex $2APRIFWI(L)$ will be defined.

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