ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

\[ f(x, y, y') = 0 \Rightarrow y = f(x) \]

\[ y' = f(x, y) \quad (1) \]

where the derivative \( y' \) appears only on the left side of (1).

Many, but not all, first-order DEs can be written in standard form by algebraically solving for \( y' \) and then setting \( f(x, y) \) equal to the right side of the resulting equation.

The right side of (1) can always be written as a quotient of two other functions \( M(x, y) \) and \(-N(x, y)\). Then (1) becomes

\[ \frac{dy}{dx} = \frac{M(x, y)}{-N(x, y)} \]

which is equivalent to the differential form

\[ M(x, y)dx + N(x, y)dy = 0 \quad (2) \]

Eq: \( y' = \frac{x^2 + y^2}{x - y} \) standard form.

\[ \frac{dy}{dx} = \frac{x^2 + y^2}{x - y} \]

\[ \Rightarrow (x^2 + y^2)dx + (y - x)dy = 0 \]

Notice that the coefficient of \( dy \) is the negative sign of \( x - y \).
1) **SEPARABLE DIFFERENTIAL EQUATIONS**

An equation of the form

$$A(x)dx + B(y)dy = 0$$

is called an equation with variables separable or simply a separable equation.

- $A(x) \rightarrow$ a function only of $x$
- $B(y) \rightarrow$ a function only of $y$.

The solutions of a separable DE are obtained by performing integration of each term.

$$\int A(x)dx + \int B(y)dy = C$$

**Example:**

1) Solve the following separable DE:

$$x^2(y+1)dx + y^2(x-1)dy = 0$$

in the case of multiplication

$$\int \left( \frac{x^2}{x-1} \right) dx + \int \left( \frac{y^2}{y+1} \right) dy = 0$$

$$\int \left( x+1 + \frac{1}{x-1} \right) dx + \int \left( y-1 + \frac{1}{y+1} \right) dy = 0$$

$$x^2 + x + \ln|x-1| + \frac{y^2 - y + \ln|y+1|}{2} = C \quad \text{general solution}$$

2) \( y' + \frac{x\sin{x}}{y\cos{y}} = 0 \)

$$\frac{dy}{dx} + \frac{x\sin{x}}{y\cos{y}} = 0 \implies \frac{dy}{dx} = -\frac{x\sin{x}}{y\cos{y}}$$

$$\int x\sin{x}dx + \int y\cos{y}dy = 0$$

We must use Partial Integration Rule
\[ \int x \sin x \, dx + \int ye^{-y} \, dy = 0 \]

\[ \begin{align*}
  x &= u \\
  -\sin x \, dx &= dv \\
  dx &= dv \\
  -\cos x &= v \\
  y &= u \\
  \cos y \, dy &= dv \\
  dy &= dv \\
  \sin y &= v
\end{align*} \]

\[ \sum udv = u \cdot v - \int v \, du \]  
Partial Integration Rule

\[-x \cos x + \int \cos x \, dx + y \sin y - \int \sin y \, dy = c \]

\[-x \cos x + \int \sin x \, dx + y \sin y + \cos y + c = 0 \]  
General solution

3) \[ y^2 + e^{x+y} = 0 \]

\[ \begin{align*}
  \frac{dy}{dx} + e^x \cdot e^y &= 0 \\
  dy + e^x \cdot e^y \, dx &= 0 \\
  \int e^{-y} \, dy + \int e^x \, dx &= 0
\end{align*} \]

\[ -e^{-y} + e^x = c \]  
We can arrange the eq.

\[ e^{x+y} - 1 = ce^{-y} \]  
General solution

4) \[ (x^2 y + xy) \, dx + (x+1) \, dy = 0 \]

Find the particular solution at the point \((0,2)\)

\[ y(x^2 + x) \, dx + (x+1) \, dy = 0 \]

\[ xy \, dx + dy = 0 \]

\[ \int x \, dx + \int \frac{dy}{y} = 0 \]

\[ x^2 + \ln y = c \]  
General solution.

Find the particular solution at the point \(y(0) = 2\) ??

\[ \ln 2 = c \Rightarrow \frac{x^2}{2} + \ln y = \ln 2 \]  
Particular solution.
DEs that can be transformed into separable DEs

Consider the equation

\[
\frac{dy}{dx} = f(ax + by + c)
\]

where \(a, b, c\) are constants.

This equation can be transformed into a separable DE by making the transformation

\[ax + by + c = u\].

If we take the derivatives, we get

\[a + bx' = u'\]

\[\Rightarrow \quad y' = \frac{u' - a}{b}\]

If we substitute this equation in the original DE,

\[y' = f(u)\]

\[\Rightarrow \quad \frac{u' - a}{b} = f(u)\]

\[\Rightarrow \quad u' = b f(u) + a\]

\[\Rightarrow \quad \frac{du}{a + bf(u)} = dx\]

Now we get a separable DE and we can integrate,

\[\Rightarrow \quad \int \frac{du}{a + bf(u)} = \int dx + c\]

\[\Rightarrow \quad x = \varphi(u) + c\]

Eq: \(y' = (x + y + 1)^2\) solve this DE.

This equation is of the form \(y' = f(ax + by + c)\)

\[x + y + 1 = u\]

\[1 + y' = u'\]

\[y' = u' - 1\]

\[u' - 1 = u^2\]
\[ u' - 1 = u^2, \quad u^1 = u^2 + 1 \]

\[ \frac{du}{dx} = u^2 + 1 \quad \text{Separable O.D.E.} \]

\[ \int \frac{du}{u^2 + 1} = \int dx \]

\[ \arctan u = x + c \]

\[ u = x + y + 1 \]

\[ \arctan (x + y + 1) = x + c \]

\[ x + y + 1 = \tanh (x + c) \quad \text{general solution.} \]

\[ \text{E.g.: } y' = 2\sqrt{2x+y+1} \quad \text{Solve this O.D.E.} \]

\[ 2x + y + 1 = u \]

\[ 2 + y' = u' \]

\[ y' = u' - 2 \]

\[ u' - 2 = 2\sqrt{u} \]

\[ u' = 2(1 + \sqrt{u}) \]

\[ \frac{du}{dx} = 2(1 + \sqrt{u}) \quad \text{Separable O.D.E.} \]

\[ \int \frac{du}{2(1 + \sqrt{u})} = \int dx + c \]

\[ u = t^2 \]

\[ du = 2t \, dt - 1 \]

\[ \int \frac{t \, dt}{2(1 + t)} = x + c \]

\[ \int (1 - \frac{1}{1+t}) \, dt = x + c \]

\[ t = \ln |1 + t| = x + c \]

\[ \sqrt{u} - \ln |1 + \sqrt{u}| = x + c \]

\[ \sqrt{2x+y+1} - \ln |1 + \sqrt{2x+y+1}| = x + c \quad \text{general solution.} \]
2) HOMOGENEOUS EQUATIONS

A DE in the form \( y' = f(x, y) \) is homogeneous if
\[
 f(tx, ty) = t^n f(x, y)
\]
for every real number \( t \).

A function \( f \) is called homogeneous of degree \( n \) if
\[
 f(tx, ty) = t^n f(x, y)
\]
for example: the function \( f(x, y) = x^2 + y^2 \) is homogeneous of degree 2, since
\[
f(tx, ty) = t^2 x^2 + t^2 y^2 = t^2 (x^2 + y^2) = t^2 f(x, y)
\]

If \( M \) and \( N \) in \( M(x, y)\,dx + N(x, y)\,dy = 0 \) are both homogeneous functions of the same degree, then the DE is a homogeneous DE.

Also! The equation \( M(x, y)\,dx + N(x, y)\,dy = 0 \) is said to be homogeneous if, when written in the derivative form \( (dy/dx) = f(x, y) \) \( \), there exists a function \( g \) such that \( f(x, y) \) can be expressed in the form \( g(y/x) \).

That is, a homogeneous function can always be written in terms of \( (\frac{x}{y}) \) or \( (\frac{y}{x}) \).

E.g: \( (y + \sqrt{x^2+y^2})\,dx - x\,dy = 0 \) is this equation homogeneous?

\[
M(x, y) = y + \sqrt{x^2+y^2} \Rightarrow M(tx, ty) = ty + \sqrt{t^2 x^2 + t^2 y^2}
\]
\[
M(tx, ty) = t M(tx, ty) \Rightarrow M \text{ is homo. of degree 1}
\]
\[
N(x, y) = -x \Rightarrow N(tx, ty) = (tx) = t N(x, y) \Rightarrow N \text{ is h.o. of degree 1}
\]

Thus we conclude that \( (y + \sqrt{x^2+y^2})\,dx - x\,dy = 0 \) is hom.
Theorem: If $M(x,y)dx + N(x,y)dy = 0$ is a homogeneous DE, then the change of variables $y = ux$ transforms the DE into a separable equation in the variables $u$ and $x$.

Proof: Since $M(x,y)dx + N(x,y)dy = 0$ is homogeneous, it may be written in the form $\frac{dy}{dx} = g(y/x)$.

Let $y = ux$. Then $\frac{dy}{dx} = u + x\frac{du}{dx}$ and the DE becomes $u + x\frac{du}{dx} = g(u)$ or $[u - g(u)]dx + xdu = 0$. This equation is separable. Separating the variables, we obtain

$$\frac{du}{u - g(u)} + \frac{dx}{x} = 0$$

Result: To solve a homogeneous DE of the form $M(x,y)dx + N(x,y)dy = 0$,

1) Let $y = ux$
2) Transform the homogeneous equation into a separable equation.
3) Perform integrating.

Example: $(x.e^{y/x} + y)dx - xdy = 0$

$M(tx,ty) = txe^{y/x} + ty = tM(x,y) \Rightarrow M$ is homo. of degree 1.
$N(tx,ty) = -t \Rightarrow N$ is homo. of degree 1.

So given DE is homo. of degree 1.

$y = ux \Rightarrow dy = udx + xdu$

If we arrange the DE in terms of $y/x$

$(e^{y/x} + y/x)dx - dy = 0$
\[(e^u + u)dx - (xdu + ud\!x) = 0\]
\[(e^u + u - y)dx - xdu = 0\]

\[-\int \frac{dx}{x} - \int e^{-u}du = 0\]

\[\ln |x| + e^{-u} = c\]

Put \[u = \frac{y}{x}\]
\[\ln |x| + e^{-\frac{y}{x}} = c\]  General solution.

Example: \[(2x + 3y)dx + (y - x)dy = 0\]  Solve the DE.

\[M(tx, ty) = 2tx + 3ty = tM \Rightarrow M \text{ is hom. of degree 1}\]
\[N(tx, ty) = ty - tx = tN \Rightarrow N \text{ is hom. of degree 1}\]
So, the given DE is hom. of degree 1.

Let \[y = ux \Rightarrow dy = udx + xdu\]

\[(2x + 3ux)dx + (ux - x)(udx + xdu) = 0\]
\[x(2 + 3u)dx + x(u - 1)(udx + xdu) = 0\]
\[(2 + 3u)dx + u^2dx + uxdx = 0\]
\[(2 + 3u + u^2 - u)dx + (u - 1)xdu = 0\]
\[\int \frac{dx}{x} + \int \frac{u - 1}{u^2 + 2u + 2}du = 0\]

\[\ln |x| + \frac{1}{2} \int \frac{(2u + 2)}{u^2 + 2u + 2}du - \int \frac{du}{u^2 + 2u + 2} = 0\]

\[\ln |x| + \frac{1}{2} \ln |u^2 + 2u + 2| = 2 \arctan (u + 1) + C\]

\[\ln |x| + \frac{1}{2} \ln \left| \frac{y^2 + 2y + 2}{x^2} \right| = 2 \arctan \left( \frac{y + 1}{x} \right) + C\]  \text{General solution.}
Example: \( y' = \frac{y^2 + 2xy}{x^2} \)

\[
\frac{dy}{dx} = \frac{y^2 + 2xy}{x^2}
\]

\[
\frac{(y^2 + 2xy)\,dx - x^2\,dy}{M} = 0
\]

\[
M(tx, ty) = t^2y^2 + 2t^2xy = t^2M \Rightarrow M \text{ is hom. of degree } 2.
\]

\[
N(tx, ty) = t^2x^2 = t^2N \Rightarrow N \text{ is hom. of degree } 2.
\]

So the given DE is hom. of degree 2.

Let \( y = ux \Rightarrow dy = u\,dx + x\,du \)

\[
\left[ u^2x^2 + 2x(u x) \right]\,dx - x^2 \left( u\,dx + x\,du \right) = 0
\]

\[
x^2(u^2 + 2u)\,dx - x^2(u\,dx + x\,du) = 0
\]

\[
(u^2 + 2u - u)\,dx - x\,du = 0
\]

\[
(u^2 + u)\,dx - x\,du = 0
\]

\[
\int \frac{dx}{x} - \int \frac{du}{u(u+1)} = 0
\]

\[
\int \frac{dx}{x} - \left[ \int \frac{du}{u} - \int \frac{du}{u+1} \right] = 0
\]

\[
\ln |x| - \ln |u| + \ln |u+1| = \ln c
\]

\[
\ln |u| - \ln |u+1| = \ln |x| + \ln c
\]

\[
\frac{u}{u+1} = ce^{-x} \Rightarrow \frac{y}{x+y} = cx \quad \text{general solution}
\]
3) DEs that can be transformed into separable or homogeneous equations:

Consider the equation:

\[(a_1 x + b_1 y + c_1) \, dx + (a_2 x + b_2 y + c_2) \, dy = 0\] (*)

where \(a_1, b_1, c_1, a_2, b_2, c_2\) are constants.

**Case 1:** If \(\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\), then the transformation

\[
\begin{cases}
    x = x_1 + h \\
    y = y_1 + k
\end{cases}
\]

where \((h, k)\) is the solution of the system

\[
\begin{align*}
    a_1 x + b_1 y + c_1 &= 0 \\
    a_2 x + b_2 y + c_2 &= 0
\end{align*}
\] (**)

reduces equation (*) to the homogeneous equation,

\[(a_1 x_1 + b_1 y_1) \, dx_1 + (a_2 x_1 + b_2 y_1) \, dy_1 = 0\]

in the variables \(x_1\) and \(y_1\).

Observe that the inequality \(\frac{a_1}{a_2} \neq \frac{b_1}{b_2}\) corresponds to the existence of a unique solution of the system (**).
Case 2: If \( \frac{a_2}{a_1} = \frac{b_2}{b_1} = k \), then the transformation \( z = a_1 x + b_1 y \) reduces the equation (4) to a separable equation in the variables \( x \) and \( t \).

Observe that the equality \( \frac{a_2}{a_1} = \frac{b_2}{b_1} = k \) corresponds to the existence of infinitely many solutions of the system (4).

\[
\begin{vmatrix}
  a_1 & b_1 \\
  a_2 & b_2 \\
\end{vmatrix} = 0
\]

Example: \( (x-y+1)dx + (x+y-1)dy = 0 \)

\[
\begin{vmatrix}
  1 & 1 \\
  -1 & 1 \\
\end{vmatrix} = 1+1 = 2 \neq 0 \quad \text{or} \quad \frac{a_2}{a_1} = \frac{1}{1} = \frac{b_2}{b_1} = -1
\]

Therefore, this is case 1, we make the transformation

\[
x = x_1 + u \\
y = y_1 + v
\]

where \((u, v)\) is the solution of the system:

\[
\begin{align*}
x - y + 1 &= 0 \\
x + y - 1 &= 0
\end{align*}
\]

The solution of the system \((0, 0) = (u, v)\) and so the transformation is

\[
\begin{align*}
dx &= dx_1 \\
dy &= dy_1
\end{align*}
\]
If we substitute these transformations in the original DE, 
\[
\begin{align*}
(x-y+1)dx + (x+y-1)dy &= 0 \\
(x_1-(y_1+1)+1)dx_1 + (x_1+y_1+1-1)dy_1 &= 0 \\
(x_1-y_1-1+1)dx_1 + (x_1+y_1)dy_1 &= 0 \\
(x_1-y_1)dx_1 + (x_1+y_1)dy_1 &= 0
\end{align*}
\]
the DE is hom. now.

These transformations reduce the original DE to a hom. equation of the first degree b/c
\[
\begin{align*}
&x_1 \to tx_1, \quad y_1 \to ty_1 \\
x_1-y_1 = tx_1-ty_1 = t(x_1-y_1) \\
x_1+y_1 = tx_1+ty_1 = t(x_1+y_1)
\end{align*}
\]

To solve the hom. equation let \( y_1 = ux_1 \)
\[
dy_1 = udx_1 + x_1 du \\
(x_1-ux_1)dx_1 + (x_1+ux_1)(udx_1 + x_1 du) = 0
\]
\[
x_1(1-u)dx_1 + x_1udx_1 + x_1^2 du + u2x_1 dx_1 + u,x_1^2 du = 0
\]
\[
(x_1-ux_1+ux_1+u^2x_1)dx_1 + (x_1^2+ux_1^2)du = 0
\]
\[
x_1(1+u^2)dx_1 + x_1^2(1+u)du = 0
\]

separable DE:
\[
\begin{align*}
\int \frac{dx_1}{x_1} + \int \frac{1+u}{1+u^2} du &= 0 \\
\ln |x_1| + \int \frac{du}{1+u^2} + \frac{1}{2} \int \frac{2u}{u^2+1} du &= C \\
\ln |x_1| + \arctan u + \frac{1}{2} \ln |u^2+1| &= C \\
\ln \left| \frac{x_1 \sqrt{u^2+1}}{c} \right| &= -\arctan u
\end{align*}
\]
\[
\frac{x \cdot \sqrt{(y/x)^2 + 1}}{c} = e^{-\arctan(y/x)} \quad \begin{cases} x = x_1 \\ y = y_1 + 1 \end{cases}
\]

\[
x \cdot \sqrt{(y-1/x)^2 + 1} = c \cdot e^{-\arctan(y-1/x)}
\]

**Example:** \(\frac{a_1}{b_1} + (1/y + 1)dx + (a_2 - b_2)(1/y-1)dy = 0\)

\[
\begin{vmatrix} 4 & 4 \\ 1 & 1 \end{vmatrix} = 4 - y = 0 \quad \text{or} \quad a_1 = \frac{1}{a_2} = \frac{1}{y}
\]

This is case 2. So the transformation \(z = x + y\) reduces our DE to a separable equation.

\(y = z - x\) if we substitute all these in the original DE,

\((4z + 1)dx + (z - 1)(dz - dx) = 0\)

\((4z + 1)dx - (z - 1)dz + (1 - z)dx = 0\)

\((4z + 1 + 1 - z)dx + (z - 1)dz = 0\)

\((3z + 2)dx + (z - 1)dz = 0\) a separable DE.

\[
\int dx + \int \left(\frac{z - 1}{3z + 2}\right) dz = 0
\]

\[
z + \frac{1}{3} \ln |3z + 2| = C
\]

\[
x + \frac{1}{3} (x+y) + \frac{5}{9} \ln |3(x+y) + 2| = C
\]
Example: \((x+y-2)\,dx + (y-x)\,dy = 0\)

\[
\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1+1 = 2 \neq 0
\]

That is, \(\frac{a_2}{a_1} = -1 \neq \frac{b_2}{b_1}\)

(Case 1) \(x = x_1 + h\) \((h, k)\) is the solution of the system:

\[
\begin{align*}
x + y &= 2 \\
-x + y &= 0
\end{align*}
\]

\[
2y = 2 \implies y = 1, \quad x = 1 + h, \quad h = 1, \quad k = 1
\]

\[
\begin{align*}
x &= x_1 + 1 \\
y &= y_1 + 1 \quad \text{let } y_1 = u x_1
\end{align*}
\]

\[
(x_1 + 1 + y_1 + 1 - 2)\,dx_1 + (y_1 + 1 - x_1 - 1)\,dy_1 = 0
\]

\[
(x_1 + y_1)dx_1 + (y_1 - x_1)\,dy_1 = 0 \quad \text{now we obtain a hom. DE of degree 1. So let } y_1 = u x_1
\]

\[
ch_y = u\,dx_1 + x_1 du
\]

\[
(x_1 + u x_1)\,dx_1 + (u x_1 - x_1)(u\,dx_1 + x_1 du) = 0
\]

\[
(x_1 + u x_1 + u^2 x_1 - y x_1)\,dx_1 + x_1^2 (u - 1)\,du = 0
\]

\[
\int \frac{dx_1}{x_1} + \int \frac{u - 1}{1 + u^2} du = 0
\]

\[
\ln x_1 + \frac{1}{2} \ln \frac{2u}{1 + u^2} du + \int \frac{du}{1 + u^2} = c \quad \begin{cases} u = \frac{y_1}{x_1} \\ u = \frac{y - 1}{x - 1} \end{cases}
\]

\[
\ln x_1 + \frac{1}{2} \ln |1 + u^2| - \arctan u = c
\]

\[
\ln(x-1) + \frac{1}{2} \ln \left(\frac{y-1}{x-1}\right) - \arctan \left(\frac{y-1}{x-1}\right) = c \quad \text{general solution.}
\]