Bol. Soc. Mat. Mex. (2022)28:38
https://doi.org/10.1007/s40590-022-00430-x

Sociedad Matemática Mexicana

ORIGINAL ARTICLE



Mersenne numbers which are products of two Pell numbers

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Received: 9 July 2021 / Accepted: 1 April 2022
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Abstract
Let $(P_n)_{n\geq 0}$ be the Pell sequence defined by $P_0 = 0$, $P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$ and $M_k = 2^k - 1$ be the k-th Mersenne number for $k \geq 1$. We show that the Diophantine equation $P_n P_m = M_k$ with $m \leq n$ has only the unique positive integer solution $(n, m, k) = (1, 1, 1)$.

Keywords Pell numbers · Mersenne numbers · Diophantine equations · Linear forms in logarithms

Mathematics Subject Classification 11B39 · 11J86 · 11D61

1 Introduction

Let $(F_n)_{n\geq 0}$ and $(P_n)_{n\geq 0}$ be the Fibonacci and Pell sequences given by $F_0 = P_0 = 0$, $F_1 = P_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ and $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$, respectively. Over years, not only Fibonacci, Lucas or Pell numbers themselves but also their products and sums are also investigated by a number of authors from various point of view, for example see [1, 9, 22, 25, 26]. Especially in the last decade, products of two Fibonacci, Lucas or Pell numbers which are having a particular form such as repdigits or x-coordinates of Pell equations has been investigated in [11, 12, 14, 16, 19, 20, 23, 27]. The sums of two or more Fibonacci, Lucas or Pell numbers which are of the form 2^k are also investigated in [5–8]. Recently [17],

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Published online: 27 April 2022

the authors investigated the Y-coordinates of Pell equations which are of the from $2^k - 1$. In [4] the authors investigated all k-Fibonacci numbers which are Mersenne numbers while in [21] the authors determined all the Mersenne numbers which are Padovan and Perrin numbers. More recently, in [15], authors find all repdigits in base b as products of two Fibonacci numbers.

In this paper, we search the products two Pell numbers which are Mersenne numbers, that is the numbers of the form $M_k = 2^k - 1$, and we get the following result.

Theorem 1 *The equation*

$$P_n P_m = M_k, \quad 1 \le m \le n, \quad k \ge 0 \tag{1}$$

has only the positive integer solution $(n, m, k) = (1, 1, 1)$.

In other words, by the above theorem, we find all products $P_n P_m$ which are base 2 repdigits. Note also that a similar problem for the products of two Lucas numbers $L_n L_m$ which are Mersenne numbers has already been solved in [16].

The proof of the above theorem depends on the combination of two results one of them is due to Matveev [24] which gives us a general lower bound for linear forms in logarithms and the other one is due to Dujella and Pethő [13] which is a version of reduction method given in Baker-Davenport lemma [2]. When we apply these two results we frequently utilize the software Maple for all calculations and computations.

2 Preliminaries

The Binet's formula for Pell numbers are given by

$$P_n = \frac{\gamma^n + \delta^n}{2\sqrt{2}} \qquad n \ge 0,$$

where

$$\gamma := 1 + \sqrt{2} \quad \text{and} \quad \delta := 1 - \sqrt{2}$$

are the roots of the equation $x^2 - 2x - 1 = 0$.

Using the Binet's Formula, one can see that the inequalities

$$\gamma^{n-2} \le P_n \le \gamma^{n-1} \tag{2}$$

hold for all positive integers n.

Let η be an algebraic number of degree d with minimal polynomial

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}),$$

where the a_i's are relatively prime integers with $a_0 > 0$ and the $\eta^{(i)}$'s are conjugates of η. The logarithmic height of η is defined by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, for a rational number p/q with $\gcd(p,q) = 1$ and $q > 0$, $h(p/q) = \log \max\{|p|, q\}$. The following properties are frequently used in calculations of the logarithmic height $h(\eta)$:

- $h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2$.
- $h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma)$.
- $h(\eta^s) = |s|h(\eta)$.

Theorem 2 (*Matveevs Theorem*) [24] *or* [10, *Theorem* 9.4] *Assume that* $\gamma_1, \ldots, \gamma_t$ *are positive real algebraic numbers in a real algebraic number field* $\mathbb{K}$ *of degree* $d_{\mathbb{K}}$, $b_1, \ldots, b_t$ *are rational integers, and*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1,$$

is not zero. Then,

$$|\Lambda| > \exp\left(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log B) A_1 \cdots A_t\right),$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{d_{\mathbb{K}} h(\gamma_i), |\log \gamma_i|, 0.16\}, \quad \text{for all} \quad i = 1, \ldots, t.$$

Now, we cite the following lemma from [3, Lemma 4] which is a variation of a result due to Dujella and Pethő [13, Lemma 5(a)] and we use it to reduce some upper bounds on the variables. For a real number θ, we put $||\theta|| = \min\{|\theta - n| : n \in \mathbb{Z}\}$, the distance from θ to the nearest integer.

Lemma 1 *Let* M *be a positive integer, let* p/q *be a convergent of the continued fraction of the irrational* γ *such that* $q > 6M$, *and let* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *If* $\epsilon := ||\mu q|| - M||\gamma q|| > 0$, *then there is no solution to the inequality*

$$0 < u\gamma - v + \mu < AB^{-w},$$

in positive integers u,v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

3 Proof of theorem 1

Assume that the equation (1) holds. For $1 \leq m \leq n \leq 200$, a brute force search shows that the equation (1) has only solution $(n, m, k) = (1, 1, 1)$ in this range. Therefore, we may take $n > 200$.

If $n = m$, then the equation $P_n^2 = 2^k - 1$ implies that P_n is odd and hence $P_n \equiv 1 (\mathrm{mod}\ 8)$. Thus, from the congruence $1 \equiv 2^k - 1 (\mathrm{mod}\ 8)$, we see that $k = 1$ and $P_n = P_m = 1$. Therefore, assume that $n > m$. If $P_m = 1$ then the equation $P_n = 2^k - 1$ has no positive integer solution when $n > 1$ [5]. It is also clear that $P_m \neq 2$. Therefore, from now on, we take $m > 2$.

From (1), using Binet's formula, we write

$$2^{k-1} < 2^k - 1 = P_n P_m \leq \gamma^{n+m-2}. \tag{3}$$

Therefore, from the inequality

$$k - 1 < (\log \gamma / \log 2)(n + m - 2) < (3/2)(2n - 2),$$

we get that $k < 3n - 2$ which we will use it later repeatedly.

Now, we rewrite equation (1) as

$$\frac{\gamma^n - \delta^n}{2\sqrt{2}} \cdot \frac{\gamma^m - \delta^m}{2\sqrt{2}} = 2^k - 1,$$
$$\gamma^{n+m} - 2^{k+3} = \gamma^n \delta^m + \delta^n \gamma^m - \delta^{n+m} - 8.$$

Dividing both sides of the above equation by γ^{n+m} and taking absolute values of both sides of it we get that

$$\left| 1 - \frac{2^{k+3}}{\gamma^{n+m}} \right| < \frac{|\delta|^m}{\gamma^m} + \frac{|\delta|^n}{\gamma^n} + \frac{|\delta|^{n+m}}{\gamma^{n+m}} + \frac{8}{\gamma^{n+m}}$$
$$< \frac{1}{\gamma^m} + \frac{1}{\gamma^n} + \frac{1}{\gamma^{2m}} + \frac{8}{\gamma^{2m}}.$$

Therefore, we have that

$$\Lambda_1 := \left| 1 - \gamma^{-(n+m)} 2^{k+3} \right| < \frac{11}{\gamma^m}. \tag{4}$$

Now, we may apply Theorem 2 to the left side of the inequality (4) with $\eta_1 := \gamma$, $\eta_2 := 2$ and $b_1 := -(n + m)$, $b_2 := k + 3$. Since η_1 and η_2 are belongs to real quadratic number field $\mathbb{K} = \mathbb{Q}(\sqrt{2})$, we take $d_{\mathbb{K}} = 2$, the degree of the number field $\mathbb{K}$. We show that Λ_1 is nonzero. If $\Lambda_1 = 0$ then we get that $\gamma^{n+m} = 2^{k+3}$, which is a contradiction since the left hand side, γ^{n+m}, is a unit in the ring of algebraic integers of $\mathbb{K}$ whereas the right hand side, 2^{k+3}, is not. So $\Lambda_1 \neq 0$. Since $h(\eta_1) = (1/2) \log \gamma$, $h(\eta_2) = \log 2$ we take $A_1 = \log \gamma$, $A_2 = 2 \log 2$. Furthermore, from the inequality $k < 3n - 2$, we take $B := 3n + 1 > \max\{|-(n + m)|, |k + 3|\}$.

Now, we apply Theorem 2 to the left hand side of inequality (4) and we get that

$$\log(\Lambda_1) > -1.4 \cdot 30^5 \cdot 2^{4.5} \cdot 2^2(1+\log 2)(1+\log(3n+1))\log\gamma \cdot 2\log 2.$$

With a quick calculation, we see that

$$\log(\Lambda_1) > -6.38 \cdot 10^9(1+\log(3n+1)). \tag{5}$$

On the other hand, from (4), we know that

$$\log(\Lambda_1) < \log 11 - m\log\gamma. \tag{6}$$

From (5) and (6), after the necessary calculations, we get that

$$m < 7.24 \cdot 10^9 \cdot (1+\log(3n+1)). \tag{7}$$

Now, we rewrite (1) as

$$\frac{\gamma^n - \delta^n}{2\sqrt{2}} = \frac{2^k}{P_m} - \frac{1}{P_m},$$
$$\frac{\gamma^n}{2\sqrt{2}} - \frac{2^k}{P_m} = \frac{\delta^n}{2\sqrt{2}} - \frac{1}{P_m},$$

Dividing the both sides of the above equation by $\gamma^n/2\sqrt{2}$ and taking the absolute values of both sides of it, we get that

$$\left|1 - \frac{2^k}{P_m}\cdot\frac{2\sqrt{2}}{\gamma^n}\right| < \frac{|\delta|^n}{\gamma^n} + \frac{2\sqrt{2}}{\gamma^n P_m} < \frac{1}{\gamma^n} + \frac{1}{\gamma^n}. \tag{8}$$

Thus we have that

$$\Lambda_2 := \left|1 - \gamma^{-n}2^{k+1}\sqrt{2}P_m^{-1}\right| < \frac{2}{\gamma^n}. \tag{9}$$

Let $\eta_1 := \gamma$, $\eta_2 := 2$, $\eta_3 := \sqrt{2}/P_m$ and $b_1 := -n$, $b_2 := k+1$, $b_3 := 1$. Since η_1, η_2 and η_3 are belongs to the real quadratic number field $\mathbb{K} = \mathbb{Q}(\sqrt{2})$, we take $d_{\mathbb{K}} = 2$, the degree of the number field $\mathbb{K}$. Since $h(\eta_1) = (1/2)\log\gamma$, $h(\eta_2) = \log 2$ and

$$h(\eta_3) \le h(\sqrt{2}) + h(P_m) \le (1/2)\log 2 + (1/2)\log P_m \le (1/2)\log 2 + (m/2)\log\gamma,$$

we take

$$A_1 = \log\gamma, \quad A_2 = 2\log 2 \quad \text{and} \quad A_3 = \log 2 + m\log\gamma.$$

Since $k < 3n-2$, we take $B := 3n-1 > \max\{|-n|, k+1, 1\}$. Furthermore Λ_2 is nonzero. If $\Lambda_2 = 0$ then we have that $\gamma^n = 2^{k+1}\sqrt{2}P_m^{-1}$, which means that $\gamma^{2n} \in \mathbb{Q}$, a contradiction for $n > 0$. Therefore, $\Lambda_2 \neq 0$.

Thus, we apply Theorem 2 to Λ_2 given in (9) and we get that

$$\log(\Lambda_2) > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1+\log 2)(1+\log(3n-1)) \log\gamma \cdot 2\log 2 \cdot (\log 2 + m\log\gamma). \tag{10}$$

On the other hand, from (9), we also know that

$$\log(\Lambda_2) < \log 2 - n\log\gamma. \tag{11}$$

Combining (10) and (11) and simplifying it we get that

$$n < 1.35 \cdot 10^{12} \cdot (1+\log 3n)(\log 2 + m\log\gamma). \tag{12}$$

The equation (12) together with (7) implies that

$$n < 3.2 \cdot 10^{25}. \tag{13}$$

Now, we use the theory of continued fractions to get a better bound on m and in fact we will show that $m < 78$. On the contrary, assume that $m \geq 78$. Let

$$\Gamma_1 := (k+3)\log 2 - (n+m)\log\gamma. \tag{14}$$

Then $\Lambda_1 := |\exp(\Gamma_1) - 1| < 11/\gamma^m$ and clearly $11/\gamma^m < 1/2$ for $m \geq 78$. So

$$|\Gamma_1| = |(k+3)\log 2 - (n+m)\log\gamma| < \frac{22}{\gamma^m},$$

and hence

$$\left|\frac{\log 2}{\log\gamma} - \frac{n+m}{k+3}\right| < \frac{22}{\gamma^m(k+3)\log\gamma}.$$

Since

$$\begin{aligned}\gamma^m \log\gamma \geq \gamma^{78}\log\gamma &> 6.3 \cdot 10^{29} > 4.3 \cdot 10^{27} \\ &> 44(3 \cdot 3.2 \cdot 10^{25} + 1) \\ &> 44(3n+1) > 44(k+3),\end{aligned}$$

we have that

$$\left|\frac{\log 2}{\log\gamma} - \frac{n+m}{k+3}\right| < \frac{22}{\gamma^m(k+3)\log\gamma} < \frac{1}{2(k+3)^2}$$

which means that $\dfrac{n+m}{k+3}$ is a convergent of continued fractions of $\dfrac{\log 2}{\log\gamma}$, say $\dfrac{p_i}{q_i}$. Since p_i and q_i are relatively prime integers we get that $q_i \leq k+3 < 3n+1 < 9.7 \cdot 10^{25}$. Let $[a_0, a_1, a_2, a_3, a_4, \ldots] = [0, 1, 3, 1, 2, \ldots]$ be the continued fraction expansion of $\log 2/\log\gamma$. A quick search with Maple reveals that $i < 50$ and $\max\{a_i\} = a_{27} = 100$ for $i = 1, 2, \ldots 50$. Therefore, from the well-known property of continued fractions, see for example [18, Theorem 1.1.(iv)], we write

$$\frac{22}{\gamma^m(k+3)\log\gamma} > \left|\frac{\log 2}{\log\gamma} - \frac{n+m}{k+3}\right| > \frac{1}{(a_i+2)(k+3)^2} \geq \frac{1}{102(k+3)^2}.$$

Therefore, the last inequality implies that

$$k+3 > \frac{\gamma^{2m}\log\gamma}{22\cdot 102} > \frac{\gamma^{78}\log\gamma}{22\cdot 102} > 2.8\cdot 10^{26}$$

which is a contradiction because of the fact that $k+3<3n+1<9.7\cdot 10^{25}$. Therefore, we conclude that $m<78$. Using this new upper bound of m, from (12), we can get a new upper bound of n as $n<3.6\cdot 10^{15}$.

Let

$$\Gamma_2 := (k+1)\log 2 - n\log\gamma + \log\frac{\sqrt{2}}{P_m} \tag{15}$$

so that $\Lambda_2 := |\exp(\Gamma_2)-1|<2/\gamma^n<1/2$. Then $|\Gamma_2|<4/\gamma^n$. Therefore, from (15),

$$0<\left|(k+1)\frac{\log 2}{\log\gamma} - n + \frac{\log(\sqrt{2}/P_m)}{\log\gamma}\right| < \frac{4}{\gamma^n\log\gamma}. \tag{16}$$

Now, we take

$$M = 1.08\cdot 10^{16} > 3n-1 > k+1 \qquad \text{and} \qquad \tau := \frac{\log 2}{\log\gamma}.$$

Then in the continued fraction expansion of τ, we take q_{75}, the denominator of the $75th$ convergent of τ, which exceeds $6M$. Now, we use Maple to calculate

$$\epsilon_m := ||\mu_m q_{75}|| - M||\tau q_{75}||$$

for each $m\in\{3,\ldots,78\}$ where

$$\mu_m := \frac{\log(\sqrt{2}/P_m)}{\log\gamma}, \quad q_{75} = 350606324429750630494407125482754 16414$$

and we get that

$$0.16\cdot 10^{-22} < \epsilon_m, \qquad \text{for all} \quad m\in\{3,\ldots,78\}.$$

Let $A := \dfrac{4}{\log\gamma}$, $B := \gamma$ and $\omega := n$. Thus from Lemma (1), we find that

$$n \leq \frac{\log(Aq_{75}/0.16\cdot 10^{-22})}{\log B} < 163,$$

which is a contradiction since $n>200$. This completes the proof.

Acknowledgements This work has been supported by Yildiz Technical University Scientific Research Projects Coordination Unit under project number 4356.

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