The trace formula for Sturm–Liouville operator with operator coefficient

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We will obtain a formula for the trace of the Sturm–Liouville operator with operator coefficient. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1367866]

I. INTRODUCTION

Let H be a separable Hilbert space. Let us consider the operators L_0 and L in the space $H_1 = L_2(H;[0,\pi])$ which are formed by the following differential expression:

$$l_0(y) = -y''(x), \quad l(y) = -y''(x) + Q(x)y(x),$$

with the same boundary condition $y(0) = y(\pi) = 0$, respectively. Suppose that the operator function $Q(x)$ in the expression $l(y)$ satisfies the following conditions:

(1) For $\forall x \in [0,\pi]$, $Q(x)$: $H \to H$ is a self-adjoint kernel operator. Moreover, $Q(x)$ has continuous derivative of second order with respect to the norm in space $\sigma_1(H)$ in the interval $[0, \pi]$ and for $x \in [0,\pi]$, $Q^{(i)}(x)$: $H \to H$ are self-adjoint operators $(i = 1,2)$.

(2) $\mathrm{Sup}_{[0,\pi]} \|Q(x)\|_H < \frac{3}{2}$.

(3) There is an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ of the space H such that

$$\sum_{n=1}^{\infty} \|Q(x)\varphi_n\| < \infty.$$

Here $\sigma_1(H)$ is the space of kernel operators from H to H as in Ref. 1. Moreover $\|.\|_H$ and $\|.\|$ denote the norms in H and H_1, respectively, and $\mathrm{tr}\, A = \mathrm{trace}\, A$ denotes the sum of eigenvalues of a kernel operator A.

In Ref. 2, it is proved that the spectrum of the operator L is a subset of the union of the intervals $[m^2 - \frac{3}{2}, m^2 + \frac{3}{2}]$ $(m = 1,2,...)$ and that spectrum is a disjoint eigenvalue whose points, not belonging to the set $\{m^2\}_{m=1}^{\infty}$, have finite multiplicity. Again in Ref. 2, it is proved that m^2 $(m = 1,2,...)$ is a eigenvalue of L which has finite or infinite multiplicity and that

$$\lim_{n \to \infty} \lambda_{mn} = m^2$$

such that $\{\lambda_{mn}\}_{n=1}^{\infty}$ are eigenvalues which belong to the interval $[m^2 - \frac{3}{2}, m^2 + \frac{3}{2}]$. Moreover, the trace formula in the form

$$\sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} (\lambda_{mn} - m^2) - \frac{1}{\pi} \int_0^{\pi} \mathrm{tr}\, Q(x)\, dx \right] = -\frac{\mathrm{tr}\, Q(0) + \mathrm{tr}\, Q(\pi)}{4} + \frac{1}{2\pi} \int_0^{\pi} \mathrm{tr}\, Q(x)\, dx$$

has been found for the self-adjoint operator L in Ref. 2.

In this work, we will find a formula for the sum of series

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$$\sum_{m=1}^{\infty}\left\{\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^4)-\frac{2m^2}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx-\frac{1}{2\pi}[\operatorname{tr}Q'(0)-\operatorname{tr}Q'(\pi)]\right.$$
$$\left.-\frac{1}{2\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx-\frac{1}{2\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)\,dx\right)^2\right\}.$$

The trace formulas for the scalar differential operators have been found by Gelfand and Levitan,[3] Dikiy,[4] Halberg and Kramer[5] and many other works. The list of the works on this subject is given in Levitan and Sargsyan[6] and Fulton and Pruess.[7] Note that we have only a few works on the trace of differential operators with operator coefficient. In Ref. 8, the trace of the Sturm–Liouville operator with unbounded operator coefficient has been investigated.

Let R_λ^0 and R_λ be the resolvents of the operators L_0 and L, respectively. The spectrum of the operator L_0 is the set $\{m^2\}_{m=1}^{\infty}$ and every point of this set is the eigenvalue of L_0 which has infinite multiplicity. The eigenvectors corresponding to eigenvalue m^2 are given by the form

$$\psi_{mn}=\sqrt{\frac{2}{\pi}}\operatorname{Sin}mx\cdot\varphi_n,\quad n=1,2,\ldots. \tag{1.1}$$

II. SOME FORMULAS ABOUT EIGENVALUES AND THE OPERATOR QR_λ^0

Since the operator Q satisfies the condition (3) and the system (1,1) of eigenvectors of operator L_0 is a orthonormal basis of the space H_1, it can be seen that QR_λ^0: $H_1\to H_1$ is a kernel operator for every $\lambda\notin\{m^2\}_{m=1}^{\infty}$. Then, from the relation

$$R_\lambda-R_\lambda^0=-R_\lambda QR_\lambda^0 \tag{2.1}$$

we see that $R_\lambda-R_\lambda^0\in\sigma_1(H)$ for each λ which belongs to the resolvent set of L. In this case, from Ref. 2, since

$$\operatorname{tr}(R_\lambda-R_\lambda^0)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_{mn}-\lambda}-\frac{1}{m^2-\lambda}\right)$$

we obtain

$$\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda^2\operatorname{tr}(R_\lambda-R_\lambda^0)\,d\lambda=\sum_{m=1}^{p}\sum_{n=1}^{\infty}(m^4-\lambda_{mn}^2),$$

where $b_p=p^2+p$. From (2.1) and this last relation, we write

$$\sum_{m=1}^{p}\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^4)=\sum_{j=1}^{N}M_{pj}+M_p^{(N)}. \tag{2.2}$$

Here N is a integer,

$$M_{pj}=\frac{(-1)^{j+1}}{2\pi i}\int_{|\lambda|=b_p}\lambda^2\operatorname{tr}[R_\lambda^0(QR_\lambda^0)^j]\,d\lambda$$

and

$$M_p^{(N)}=\frac{(-1)^N}{2\pi i}\int_{|\lambda|=b_p}\lambda^2\operatorname{tr}[R_\lambda(QR_\lambda^0)^{N+1}]d\lambda. \tag{2.3}$$

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Since the operator function QR_λ^0 in the domain $\mathbb{C}\backslash\{m^2\}_{m=1}^{\infty}$ is analytic with respect to the norm in $\sigma_1(H_1)$, we can show that

$$M_{pj}=\frac{(-1)^j}{\pi i j}\int_{|\lambda|=b_p}\lambda\,\mathrm{tr}\,(QR_\lambda^0)^j\,d\lambda. \tag{2.4}$$

From relations (1.1) and (2.4), we write

$$\begin{aligned}M_{p1}&=-\frac{1}{\pi i}\int_{|\lambda|=b_p}\lambda\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(QR_\lambda^0\psi_{mn},\psi_{mn})\,d\lambda\\&=2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(Q\psi_{mn},\psi_{mn})\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-m^2}\,d\lambda\\&=2\sum_{m=1}^{p}\sum_{n=1}^{\infty}m^2(Q\psi_{mn},\psi_{mn})\\&=\frac{2}{\pi}\sum_{m=1}^{p}m^2\int_0^{\pi}\mathrm{tr}\,Q(x)\,dx-\frac{2}{\pi}\sum_{m=1}^{p}m^2\int_0^{\pi}\mathrm{tr}\,Q(x)\,\mathrm{Cos}\,2mx\,dx.\end{aligned}$$

On the other hand, since

$$\int_0^{\pi}\mathrm{tr}\,Q(x)\,\mathrm{Cos}\,2mx\,dx=\frac{\mathrm{tr}\,Q'(\pi)-\mathrm{tr}\,Q'(0)}{4m^2}-\frac{1}{4m^2}\int_0^{\pi}\mathrm{tr}\,Q''(x)\,\mathrm{Cos}\,2mx\,dx,$$

we have

$$M_{p1}=\frac{2}{\pi}\sum_{m=1}^{p}m^2\int_0^{\pi}\mathrm{tr}\,Q(x)\,dx+\frac{p}{2\pi}[\mathrm{tr}\,Q'(0)-\mathrm{tr}\,Q'(\pi)]+\frac{1}{2\pi}\sum_{m=1}^{p}\int_0^{\pi}\mathrm{tr}\,Q''(x)\,\mathrm{Cos}\,2mx\,dx. \tag{2.5}$$

Now, let us compute M_{p2}: From (2.4) we write

$$M_{p2}=\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\,\mathrm{tr}\,(QR_\lambda^0)^2\,d\lambda=\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}((QR_\lambda^0)^2\psi_{mn},\psi_{mn})\,d\lambda. \tag{2.6}$$

Moreover, we know that

$$QR_\lambda^0\psi_{mn}=\frac{Q\psi_{mn}}{m^2-\lambda}$$

and

$$\begin{aligned}(QR_\lambda^0)^2\psi_{mn}&=(m^2-\lambda)^{-1}QR_\lambda^0(Q\psi_{mn})\\&=(m^2-\lambda)^{-1}QR_\lambda^0\left\{\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}(Q\psi_{mn},\psi_{rq})\psi_{rq}\right\}\\&=(m^2-\lambda)^{-1}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}(r^2-\lambda)^{-1}(Q\psi_{mn},\psi_{rq})Q\psi_{rq}.\end{aligned}$$

If we write this last expression, in (2.6), we find

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$$M_{p2}=\frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\frac{(Q\psi_{mn},\psi_{rq})(Q\psi_{rq},\psi_{mn})}{(\lambda-m^2)(\lambda-r^2)}\right]d\lambda$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}|(Q\psi_{mn},\psi_{rq})|^2\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{(\lambda-m^2)(\lambda-r^2)}d\lambda$$

$$=\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=1}^{p}\sum_{q=1}^{\infty}|(Q\psi_{mn},\psi_{rq})|^2\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{(\lambda-m^2)(\lambda-r^2)}d\lambda$$

$$+\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}|(Q\psi_{mn},\psi_{rq})|^2\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{(\lambda-m^2)(\lambda-r^2)}d\lambda$$

$$+\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{p}\sum_{q=1}^{\infty}|(Q\psi_{mn},\psi_{rq})|^2\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{(\lambda-m^2)(\lambda-r^2)}d\lambda$$

$$+\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}|(Q\psi_{mn},\psi_{rq})|^2\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{(\lambda-m^2)(\lambda-r^2)}d\lambda$$

$$=\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=1}^{p}\sum_{q=1}^{\infty}|(Q\psi_{mn},\psi_{rq})|^2+2\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}|(Q\psi_{mn},\psi_{rq})|^2\frac{m^2}{m^2-r^2}$$

$$=\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}|(Q\psi_{mn},\psi_{rq})|^2-\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left(1+\frac{2m^2}{r^2-m^2}\right)|(Q\psi_{mn},\psi_{rq})|^2$$

$$=\sum_{m=1}^{p}\sum_{n=1}^{\infty}\|Q\psi_{mn}\|^2-\sum_{m=1}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\frac{r^2+m^2}{r^2-m^2}|(Q\psi_{mn},\psi_{rq})|^2.$$

Here if we take

$$\alpha_p=\alpha_p(n,q)=\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\frac{r^2+m^2}{r^2-m^2}|(Q\psi_{mn},\psi_{rq})|^2,$$

then we write

$$M_{p2}=\sum_{m=1}^{p}\sum_{n=1}^{\infty}\|Q\psi_{mn}\|^2-\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_p. \tag{2.7}$$

From Eq. (1.1) we have

$$\alpha_p=\alpha_{p1}-\alpha_{p2}+\alpha_{p3}, \tag{2.8}$$

where

$$\alpha_{p1}=\pi^{-2}\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\frac{r^2+m^2}{r^2-m^2}\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\operatorname{Cos}(m-r)x\,dx\right|^2,$$

$$\alpha_{p2}=2\pi^{-2}\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\frac{r^2+m^2}{r^2-m^2}\operatorname{Re}\left[\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\cdot\operatorname{Cos}(m-r)x\,dx\right.$$

$$\left.\times\int_0^{\pi}\overline{(Q(x)\varphi_n,\varphi_q)_H}\operatorname{Cos}(m+r)x\,dx\right], \tag{2.9}$$

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$$\alpha_{p3}=\pi^{-2}\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\frac{r^2+m^2}{r^2-m^2}\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\operatorname{Cos}(m+r)x\,dx\right|^2. \tag{2.10}$$

For any integers p and i, we let

$$E=\{(r,m):r,m\in N;r-m=i;m\leqslant p;r>p\}.$$

Then, for the expression α_{p1}, we can write

$$\alpha_{p1}=\pi^{-2}\sum_{i=1}^{\infty}\left(\sum_{m,r\in E}\frac{r^2+m^2}{r^2-m^2}\right)\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\operatorname{Cos}(ix)\,dx\right|^2. \tag{2.11}$$

The equalities

$$\sum_{m,r\in E}\frac{r^2+m^2}{r^2-m^2}=p+\frac{1}{2}+i^2O(p^{-1})\quad\text{for}\quad i\leqslant p \tag{2.12}$$

and

$$\sum_{m,r\in E}\frac{r^2+m^2}{r^2-m^2}=O(p)\quad\text{for}\quad i\geqslant p \tag{2.13}$$

can be proved easily. Here $O(p^{-1})$ and $O(p)$ depend on p and i and they satisfy the inequalities

$$|O(p^{-1})|<\operatorname{const}p^{-1},\quad |O(p)|<\operatorname{const}p.$$

From (2.11)–(2.13), we obtain

$$\begin{aligned}\alpha_{p1}&=\pi^{-2}\left(p+\frac{1}{2}\right)\sum_{i=1}^{\infty}\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\operatorname{Cos}ix\,dx\right|^2\\&\quad+\sum_{i=1}^{\infty}i^2O(p^{-1})\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\operatorname{Cos}ix\,dx\right|^2\\&\quad+\sum_{i=p+1}^{\infty}O(p)\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\operatorname{Cos}ix\,dx\right|^2\\&=\alpha_{p1}^{(1)}+\alpha_{p1}^{(2)}+\alpha_{p1}^{(3)}.\end{aligned}$$

Here,

$$\begin{aligned}\alpha_{p1}^{(1)}&=\pi^{-2}\left(p+\frac{1}{2}\right)\sum_{i=1}^{\infty}\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\operatorname{Cos}ix\,dx\right|^2\\&=\frac{2p+1}{4\pi}\int_0^{\pi}|(Q(x)\varphi_n,\varphi_q)_H|^2dx-\frac{2p+1}{4\pi^2}\int_0^{\pi}|(Q(x)\varphi_n,\varphi_q)_H\,dx|^2,\end{aligned}$$

$$|\alpha_{p1}^{(2)}|=\left|\sum_{i=1}^{p}i^2O(p^{-1})\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\operatorname{Cos}ix\,dx\right|^2\right|\leqslant\operatorname{const}p^{-1}\int_0^{\pi}|(Q'(x)\varphi_n,\varphi_q)_H|^2\,dx,$$

and

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$$|\alpha_{p1}^{(3)}|=\left|\sum_{i=p+1}^{\infty} O(p)\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H \operatorname{Cos} ix\,dx\right|^2\right| \leqslant \text{const} p^{-1}\int_0^{\pi}|(Q'(x)\varphi_n,\varphi_q)_H|^2\,dx.$$

From (2.8) and these last relations, we find

$$\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p1}=\frac{2p+1}{4\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx-\frac{2p+1}{4\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)\,dx\right)^2+O(p^{-1}). \tag{2.14}$$

Since $Q(x)$ satisfies the condition (1), it can be shown that

$$\left|\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{pj}\right|<\text{const}\,p^{-1}\quad (j=2,3). \tag{2.15}$$

By using (2.7), (2.8), (2.14) and (2.15), we find

$$M_{p2}=\sum_{m=1}^{p}\sum_{n=1}^{\infty}\|Q\psi_{mn}\|^2-\frac{2p+1}{4\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx+\frac{2p+1}{4\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q^2(x)\,dx\right)^2+O(p^{-1}), \tag{2.16}$$

where $O(p^{-1})$ depends on p and satisfies the inequality

$$|O(p^{-1})|<\text{const}\,p^{-1}.$$

Now, let us compute the sum $\Sigma_{m=1}^{p}\Sigma_{n=1}^{\infty}\|Q\psi_{mn}\|^2$:

$$\begin{aligned}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\|Q\psi_{mn}\|^2&=\frac{2}{\pi}\sum_{m=1}^{p}\sum_{n=1}^{\infty}\int_0^{\pi}\operatorname{Sin}^2 mx(Q^2(x)\varphi_n,\varphi_n)_H\,dx\\&=\frac{p}{\pi}\int_0^{\pi}\sum_{n=1}^{\infty}(Q^2(x)\varphi_n,\varphi_n)\,dx-\frac{1}{\pi}\sum_{m=1}^{p}\int_0^{\pi}\sum_{n=1}^{\infty}(Q^2(x)\varphi_n,\varphi_n)_H\operatorname{Cos}2mx\,dx\\&=\frac{p}{\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx-\frac{1}{\pi}\sum_{m=1}^{p}\int_0^{\pi}\operatorname{tr}Q^2(x)\operatorname{Cos}2mx\,dx.\end{aligned}$$

If we put this expression into (2.16), we have

$$\begin{aligned}M_{p2}=&\frac{2p-1}{4\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)+\frac{2p+1}{4\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)\,dx\right)^2\\&-\frac{1}{\pi}\sum_{m=1}^{p}\int_0^{\pi}\operatorname{tr}Q^2(x)\operatorname{Cos}2mx\,dx+O(p^{-1}).\end{aligned} \tag{2.17}$$

III. A FORMULA FOR THE TRACE OF THE OPERATOR *L*

In this section, we will find a formula for the sum

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^4)-\frac{2m^2}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx-C\right],$$

where

$$C=\frac{1}{2\pi}[\operatorname{tr}Q'(0)-\operatorname{tr}Q'(\pi)]+\frac{1}{2\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx+\frac{1}{2\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)\,dx\right)^2.$$

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For this, we will first show that the formulas

$$\lim_{p\to\infty} M_{pj}=0, \quad j\geqslant 3, \tag{3.1}$$

$$\lim_{p\to\infty} M_p^N=0, \quad N\geqslant 6, \tag{3.2}$$

are satisfied. It is not difficult to see that the inequalities

$$\|QR_\lambda^0\|_{\sigma_1(H_1)}<C_1 \quad (|\lambda|=b_p), \tag{3.3}$$

$$\|R_\lambda^0\|<C_1p^{-1} \quad \|R_\lambda\|<C_1p^{-1} \quad (|\lambda|=b_p), \tag{3.4}$$

are true. Here $C_1>0$ is a constant. From (2.4), (3.3), and (3.4) and since $Q(x)$ satisfies the condition (2), we have

$$\begin{aligned}
|M_{pj}| &= \frac{1}{\pi j}\left|\int_{|\lambda|=b_\rho} \lambda \operatorname{tr}(QR_\lambda^0)^j\, d\lambda\right| \\
&\leqslant \frac{b_p}{\pi j}\int_{|\lambda|=b_p} \|(QR_\lambda^0)^j\|_{\sigma_1(H_1)}|\, d\lambda| \\
&\leqslant \frac{b_p}{\pi j}\int_{|\lambda|=b_p} \|(QR_\lambda^0)\|_{\sigma_1(H_1)}\|(QR_\lambda^0)^{j-1}\||d\lambda| \\
&\leqslant \frac{C_1b_p}{\pi j}\int_{|\lambda|=b_p} \|Q\|^{j-1}\|R_\lambda^0\|^{j-1}|d\lambda| \\
&\leqslant \frac{C_1^jb_p}{\pi j}\int_{|\lambda|=b_p}\left(\frac{3}{2}\right)^{j-1} p^{1-j}|d\lambda| \\
&< C_2p^{5-j} \quad (C_2>0).
\end{aligned}$$

And so, we find

$$\lim_{p\to\infty} = M_{pj}=0 \quad (j\geqslant 6),$$

but we claim that this is true also for $j=3,4,5$. Let us try this for $j=3$. It can be seen that

$$M_{pj}=\frac{(-1)^j}{\pi ij}\sum_{m_1=1}^{\infty}\sum_{n_1=1}^{\infty}\cdots\sum_{m_j=1}^{\infty}\sum_{n_j=1}^{\infty}{}^{*}\left[\int_{|\lambda|=b_p}\lambda\prod_{l=1}^{j}(m_l^2-\lambda)^{-1}\,d\lambda\right]\cdot\prod_{l=1}^{j}(Q\psi_{m_ln_l},\psi_{m_{g(l)}n_{g(l)}}). \tag{3.5}$$

Here the symbol ''*'' denotes that there are numbers, between $m_1^4, m_2^4, \ldots, m_j^4$, less than or greater than b_p and

$$g(l)=\begin{cases} l+1, & l<j, \\ 1, & l=j. \end{cases}$$

For $j=3$, we write

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$$M_{p3}=-\frac{1}{3\pi i}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}\sum_{q=1}^{\infty}\sum_{R=1}^{\infty}\sum_{N=1}^{\infty}*\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{(m^2-\lambda)(r^2-\lambda)(R^2-\lambda)}$$
$$\cdot(Q\psi_{mn},\psi_{rq})(Q\psi_{rq},\psi_{RN})(Q\psi_{RN},\psi_{mn}). \tag{3.6}$$

Let us take

$$F(m,r,R)=\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{N=1}^{\infty}(Q\psi_{mn},\psi_{rq})(Q\psi_{rq},\psi_{rN})(Q\psi_{rN},\psi_{mn}).$$

Since $F(m,r,R)=F(r,m,R)=F(R,m,r)=F(R,r,m)=F(m,R,r)=F(r,R,m)$ and from (3.6), we obtain

$$\begin{aligned}M_{p3}&=\frac{1}{\pi i}\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\sum_{R=p+1}^{\infty}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{(\lambda-m^2)(\lambda-r^2)(\lambda-R^2)}F(m,r,R)\\&+\frac{1}{\pi i}\sum_{m=1}^{p}\sum_{\substack{r=1\\m\neq r}}^{p}\sum_{R=p+1}^{\infty}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{(\lambda-m^2)(\lambda-r^2)(\lambda-R^2)}F(m,r,R)\\&+\frac{1}{\pi i}\sum_{m=1}^{p}\sum_{R=p+1}^{\infty}\int_{|\lambda|=b_p}\frac{\lambda\,d\lambda}{(\lambda-m^2)^2(\lambda-R^2)}F(m,m,R)\\&=2\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\sum_{R=p+1}^{\infty}\frac{m^2}{(r^2-m^2)(R^2-m^2)}F(m,r,R)\\&+4\sum_{m=1}^{p}\sum_{\substack{r=1\\m\neq r}}^{p}\sum_{R=p+1}^{\infty}\frac{m^2}{(m^2-r^2)(m^2-R^2)}F(m,r,R)\\&-2\sum_{m=1}^{p}\sum_{R=p+1}^{\infty}\frac{R^2}{(R^2-m^2)^2}(F(m,m,R)).\end{aligned}\tag{3.7}$$

Let

$$F_1(m,r,R)=\pi^{-3}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{N=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H\,\mathrm{Cos}(m-r)x\,dx\cdot\int_0^{\pi}(Q(x)\varphi_q,\varphi_N)_H$$
$$\times\mathrm{Cos}(r-R)x\,dx\int_0^{\pi}(Q(x)\varphi_N,\varphi_n)_H\,\mathrm{Cos}(R-m)x\,dx, \tag{3.8}$$

$$F_2(m,r,R)=F(m,r,R)-F_1(m,r,R),$$

$$A_{pi}=\sum_{m=1}^{p}\sum_{r=p+1}^{\infty}\sum_{R=p+1}^{\infty}\frac{m^2}{(r^2-m^2)(R^2-m^2)}F_i(m,r,R),$$

$$B_{pi}=\sum_{m=1}^{p}\sum_{\substack{r=1\\m\neq r}}^{p}\sum_{R=p+1}^{\infty}\frac{m^2}{(m^2-r^2)(m^2-R^2)}F_i(m,r,R),$$

$$C_{pi}=\sum_{m=1}^{p}\sum_{R=p+1}^{\infty}\frac{R^2}{(R^2-m^2)^2}F_i(m,m,R).$$

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It can be shown that

$$A_{p1}=2\sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>R}}^{\infty}\sum_{R=p+1}^{\infty}\frac{m^2}{(r^2-m^2)(R^2-m^2)}F_1(m,r,R)+\sum_{m=1}^{p}\sum_{R=p+1}^{\infty}\frac{m^2}{(R^2-m^2)^2}F_1(m,m,R),$$

$$B_{p1}=-\sum_{m=1}^{p}\sum_{\substack{r=1\\ m<r}}^{p}\sum_{R=p+1}^{\infty}\frac{R^2}{(R^2-m^2)(R^2-r^2)}F_1(m,r,R). \tag{3.9}$$

Hence, we write M_{p3} as in the form

$$M_{p3}=4A_p^1-4B_p^1-2C_p^1+2A_{p2}+4B_{p2}-2C_{p2} \tag{3.10}$$

such that

$$A_p^1=\sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>R}}^{\infty}\sum_{R=p+1}^{\infty}\frac{m^2}{(r^2-m^2)(R^2-m^2)}F_1(m,r,R),$$

$$B_p^1=-B_{p1}, \tag{3.11}$$

$$C_p^1=\sum_{m=1}^{p}\sum_{R=p+1}^{\infty}\frac{1}{R^2-m^2}F_1(m,m,R).$$

Let $E_1=\{(m,r,R):m,r,R\in N;r-m=i;R-m=j;m\leqslant p;r,R>p\}$, where p, i and j are integers such that $p\geqslant j$, $i\geqslant j$ Then

$$\begin{aligned}A_p^1&=\sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>R,R-m\leqslant p}}^{\infty}\sum_{R=p+1}^{2p}\frac{m^2}{(r^2-m^2)(R^2-m^2)}F_1(m,r,R)\\&\quad+\sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>R,R-m>p}}^{\infty}\sum_{R=p+1}^{\infty}\frac{m^2}{(r^2-m^2)(R^2-m^2)}F_1(m,r,R)\\&=\pi^{-3}\sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\left[\left(\sum_{m,r,R\in E_1}\frac{m^2}{(r^2-m^2)(R^2-m^2)}\right)\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{N=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)\right]_H\\&\quad\times\operatorname{Cos}ix\cdot\int_0^{\pi}(Q(x)\varphi_q,\varphi_N)_H\operatorname{Cos}(i-j)x\,dx\int_0^{\pi}(Q(x)\varphi_N,\varphi_n)_H\operatorname{Cos}ix\,dx\\&\quad+\sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>R,R-m>p}}^{\infty}\sum_{R=p+1}^{\infty}\frac{m^2}{(r^2-m^2)(R^2-m^2)}F_1(m,r,R).\end{aligned} \tag{3.12}$$

If we let

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$$\beta_{ij}=\pi^{-3}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{N=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_H \operatorname{Cos} ix\,dx\int_0^{\pi}(Q(x)\varphi_q,\varphi_N)_H$$

$$\times\operatorname{Cos}(i-j)x\,dx\cdot\int_0^{\pi}(Q(x)\varphi_N,\varphi_n)_H\operatorname{Cos} ix\,dx, \tag{3.13}$$

$$A_p^{11}=\sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\left[\left(\sum_{m,r,R\in E_1}\frac{m^2}{(r^2-m^2)(R^2-m^2)}\right)\beta_{ij}\right],$$

$$A_p^{12}=\sum_{m=1}^{p}\sum_{\substack{r=p+1\\ r>R,R-m>p}}^{\infty}\sum_{R=p+1}^{\infty}\frac{m^2}{(r^2-m^2)(R^2-m^2)}F_1(m,r,R),$$

then we write

$$A_p^1=A_p^{11}+A_p^{12}. \tag{3.14}$$

On the other hand, it can be shown that

$$\sum_{m,r,R\in E_1}\frac{m^2}{(r^2-m^2)(R^2-m^2)}=\frac{1}{4i}+\frac{j}{p}O(1),$$

where $O(1)$ satisfies the condition

$$|O(1)|<\text{const}$$

and it depends on p, i and j. Moreover, if $Q(x)$ has a continuous derivative of second order at $[0,\ \pi]$ with respect to the norm in $\sigma_1(H)$, then we can show that

$$|\beta_{ij}|<\frac{\text{const}}{i^2j^2}. \tag{3.15}$$

Hence, we have

$$A_p^{11}=\sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\left[\frac{\beta_{ij}}{4i}+\frac{O(1)}{pi^2j}\right]. \tag{3.16}$$

Since

$$\left|\sum_{i=2}^{\infty}\sum_{j=1}^{p}\frac{O(1)}{pi^2j}\right|\leqslant\text{const}\left(\sum_{i=1}^{\infty}i^{-2}\right)\sum_{j=1}^{p}j^{-1}<\text{const}\ p^{-1}\ln p,$$

we find

$$A_p^{11}=\sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\frac{\beta_{ij}}{4i}+o(1). \tag{3.17}$$

Here $o(1)$ is an expression which satisfies the condition

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$$\lim_{p\to\infty} o(1)=0,$$

and it depends on p. From (3.14) and (3.17) we obtain

$$A_p^1=\sum_{i=2}^{\infty}\sum_{\substack{j=1\\ i>j}}^{p}\frac{\beta_{ij}}{4i}+A_p^{12}+o(1). \tag{3.18}$$

Now, let us find a formula for the B_p^1:

Let

$$B_p^{11}=\sum_{m=1}^{p}\sum_{\substack{r=1\\ m<r,R-m\leqslant p}}^{p}\sum_{R=p+1}^{2p}\frac{R^2}{(R^2-m^2)(R^2-r^2)}F_1(m,r,R)$$

and

$$B_p^{12}=\sum_{m=1}^{p}\sum_{\substack{r=1\\ m<r,R-m>p}}^{p}\sum_{R=p+1}^{\infty}\frac{R^2}{(R^2-m^2)(R^2-r^2)}F_1(m,r,R).$$

Then, from (3.9) and (3.11), we have

$$B_p^1=B_p^{11}+B_p^{12}. \tag{3.19}$$

By using (3.8) and (3.13), we write B_p^{11} as in the form

$$B_p^{11}=\sum_{j=2}^{p}\sum_{\substack{i=1\\ j>i}}^{p-1}\left(\sum_{m,r,R\in E_2}\frac{R^2}{(R^2-m^2)(R^2-r^2)}\right)\beta_{ij}, \tag{3.20}$$

where E_2 is a set defined by

$$E_2=\{(m,r,R):m,r,R\in N;r-m=i;R-m=j;m,r\leqslant p;R>p\}$$

such that $i<j\leqslant p$. Moreover, we can show that

$$\sum_{m,r,R\in E_2}\frac{R^2}{(R^2-m^2)(R^2-r^2)}=\frac{1}{4j}+\frac{i}{p}O(1). \tag{3.21}$$

From (3.15), (3.20) and (3.21) we obtain

$$B_p^{11}=\sum_{j=2}^{p}\sum_{\substack{i=1\\ j>i}}^{p-1}\left[\frac{\beta_{ij}}{4j}+\frac{O(1)}{pj^2i}\right]$$

and, since $\beta_{ij}=\beta_{ji}$, we write

$$B_p^{11}=\sum_{j=2}^{p}\sum_{\substack{i=1\\ i<j}}^{p-1}\frac{\beta_{ji}}{4j}+o(1).$$

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By using (3.19) and this last equation, we find

$$B_p^1=\sum_{\substack{i=2\\ i>j}}^{p}\sum_{j=1}^{p-1}\frac{\beta_{ij}}{4i}+B_p^{12}+o(1).$$

On the other hand, by using (3.15), it can be shown that

$$\sum_{i=2}^{p}\sum_{j=1}^{p-1}\frac{\beta_{ij}}{i}=\sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\frac{\beta_{ij}}{4i}+o(1).$$

Thus, we have

$$B_p^1=\sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\frac{\beta_{ij}}{4i}+B_p^{12}+o(1). \tag{3.22}$$

From (3.10), (3.18) and (3.22) we obtain

$$M_{p3}=4A_p^{12}-4B_p^{12}-2C_p^1+2A_{p2}+4B_{p2}-2C_{p2}+o(1).$$

Here, it can be seen that

$$\lim_{p\to\infty}A_p^{12}=\lim_{p\to\infty}B_p^{12}=\lim_{p\to\infty}C_p^1=\lim_{p\to\infty}A_{p2}=\lim_{p\to\infty}B_{p2}=\lim_{p\to\infty}C_{p2}=0.$$

For this reason, it follows that

$$\lim_{p\to\infty}M_{p3}=0.$$

In a similar form, it is proved that

$$\lim_{p\to\infty}M_{p4}=\lim_{p\to\infty}M_{p5}=0.$$

Now let us prove the formula (3.2): By employing (2.3), (3.3) and (3.4), we obtain

$$\begin{aligned}|M_p^{(N)}|&=\frac{1}{2\pi}\left|\int_{|\lambda|=b_p}\lambda^2\,\mathrm{tr}\,[R_\lambda(QR_\lambda^0)^{N+1}]\,d\lambda\right|\\ &\leqslant\frac{b_p^2}{2\pi}\int_{|\lambda|=b_p}\|R_\lambda(QR_\lambda^0)^{N+1}\|_{\sigma_1(H_1)}|d\lambda|\\ &\leqslant b_p^2\int_{|\lambda|=b_p}\|R_\lambda\|\|(QR_\lambda^0)^{N+1}\|_{\sigma_1(H_1)}|d\lambda|\\ &\leqslant C_1b_p^2p^{-1}\int_{|\lambda|=b_p}\|QR_\lambda^0\|^N\|QR_\lambda^0\|_{\sigma_1H_1}|d\lambda|\\ &\leqslant\mathrm{const}\,p^{5-N}.\end{aligned}$$

This shows that

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$$\lim_{p\to\infty} M_p^{(N)}=0; \quad N\geqslant 6.$$

The main result of this article is given by the following theorem.

Theorem 3.1: If the operator function $Q(x)$ satisfies the conditions (1)–(3), then

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^4)-\frac{2m^2}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx-C\right]$$
$$=\frac{C}{2}+\frac{1}{8}[\operatorname{tr}Q''(0)+\operatorname{tr}Q''(\pi)]-\frac{1}{4}[\operatorname{tr}Q^2(0)+\operatorname{tr}Q^2(\pi)],$$

where

$$C=\frac{1}{2\pi}[\operatorname{tr}Q'(0)-\operatorname{tr}Q'(\pi)]+\frac{1}{2\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx+\frac{1}{2\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)\,dx\right)^2.$$

Proof: By using the relations (2.2), (2.5), (2.17), (3.1) and (3.2), we find

$$\sum_{m=1}^{p}\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^4)=\frac{2}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx\sum_{m=1}^{p}m^2+\frac{p}{2\pi}[\operatorname{tr}Q'(0)-\operatorname{tr}Q'(\pi)]$$
$$+\frac{1}{2\pi}\sum_{m=1}^{p}\int_0^{\pi}\operatorname{tr}Q''(x)\operatorname{Cos}2mx\,dx+\frac{2p-1}{4\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx$$
$$+\frac{2p+1}{4\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)\,dx\right)^2-\frac{1}{\pi}\sum_{m=1}^{p}\int_0^{\pi}\operatorname{tr}Q^2(x)\operatorname{Cos}2mx\,dx+o(1).$$

Or, by taking

$$g(x)=\operatorname{tr}Q''(x)-2\operatorname{tr}Q^2(x),$$

we get

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^4)-\frac{2m^2}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx-C\right]$$
$$=\frac{1}{4\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)\,dx\right)^2-\frac{1}{4\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx+\frac{1}{2\pi}\sum_{m=1}^{\infty}\int_0^{\pi}g(x)\operatorname{Cos}2mx\,dx. \tag{3.23}$$

Here, we note that

$$\frac{1}{2\pi}\sum_{m=1}^{\infty}\int_0^{\pi}g(x)\operatorname{Cos}mx\,dx=\frac{1}{4\pi}\sum_{m=1}^{\infty}\left[\int_0^{\pi}g(x)\operatorname{Cos}mx\,dx+(-1)^m\int_0^{\pi}g(x)\operatorname{Cos}mx\,dx\right]$$
$$=\frac{1}{8}\left\{\sum_{m=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}g(x)\operatorname{Cos}mx\,dx\right]\operatorname{Cos}m0+\frac{1}{\pi}\int_0^{\pi}g(x)dx\operatorname{Cos}0\right\}$$
$$+\frac{1}{8}\left\{\sum_{m=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}g(x)\operatorname{Cos}mx\,dx\right]\operatorname{Cos}m\pi\right.$$
$$\left.+\frac{1}{\pi}\int_0^{\pi}g(x)\,dx\operatorname{Cos}0\pi\right\}-\frac{1}{4\pi}\int_0^{\pi}g(x)\,dx$$
$$=\frac{1}{8}[g(0)+g(\pi)]-\frac{1}{4\pi}\int_0^{\pi}g(x)\,dx.$$

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By considering this in (3.23), we obtain

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^4)-\frac{2m^2}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx-C\right]$$

$$=\frac{1}{4\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)dx\right)^2-\frac{1}{4\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx+\frac{1}{8}[\operatorname{tr}Q''(0)-2\operatorname{tr}Q^2(0)+\operatorname{tr}Q''(\pi)$$

$$-2\operatorname{tr}Q^2(\pi)]-\frac{1}{4\pi}\int_0^{\pi}\operatorname{tr}Q''(x)\,dx+\frac{1}{2\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx$$

$$=\frac{1}{4\pi^2}\operatorname{tr}\left(\int_0^{\pi}Q(x)\,dx\right)^2+\frac{1}{4\pi}\int_0^{\pi}\operatorname{tr}Q^2(x)\,dx+\frac{1}{4\pi}[\operatorname{tr}Q'(0)-\operatorname{tr}Q'(\pi)]$$

$$+\tfrac{1}{8}[\operatorname{tr}Q''(0)+\operatorname{tr}Q''(\pi)]-\tfrac{1}{4}[\operatorname{tr}Q^2(0)+\operatorname{tr}Q^2(\pi)]$$

or

$$\sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}(\lambda_{mn}^2-m^4)-\frac{2m^2}{\pi}\int_0^{\pi}\operatorname{tr}Q(x)\,dx-C\right]$$

$$=\frac{C}{2}+\frac{1}{8}[\operatorname{tr}Q''(0)+\operatorname{tr}Q''(\pi)]-\frac{1}{4}[\operatorname{tr}Q^2(0)+\operatorname{tr}Q^2(\pi)].$$

This proves the theorem.

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