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A second regularized trace formula for a higher order differential operator

Erdal Gül

Department of Mathematics, Faculty of Arts and Science, Yıldız Technical University, Istanbul, Turkey

Correspondence
Erdal Gül, Department of Mathematics, Faculty of Arts and Science, Yıldız Technical University, Istanbul, Turkey.
Email: gul@yildiz.edu.tr

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In this paper, we obtain a second regularized trace formula on $L_2([0,\pi];H)$ for a higher order self-adjoint differential operator with unbounded operator-valued coefficient, where H is a separable Hilbert space.

KEYWORDS

Hilbert space, regularized trace, resolvent, self-adjoint operator, spectrum

MSC CLASSIFICATION

47A10; 34L20; 34L05

1 | INTRODUCTION

The trace of an operator is usually defined as the total of its eigenvalues. For the differential operator, the trace concept so called regularized trace is quite different, and it can be considered as a more general form of traces for matrices or trace class operators. The regularized trace of the classical Sturm-Liouville problem was first obtained by Gelfand and Levitan[1] and then by Gelfand,[2] Levitan,[3] Dikii,[4,5] Halberg and Kramer,[6] Gorbachuk,[7] and Fulton and Prues.[8] For discrete operators, abstract formulas for higher order regularized traces were given by Fazullin[9] and the regularized trace of a two-dimensional harmonic oscillator was obtained by Fazullin and Murtazin.[10] For Dirac systems, the trace was calculated by Abdukadyrov.[11] The first trace formula, for a differential operator defined by two-term operator $l(y)=y^{(n)}+q(x)y$ with general regular conditions, was first obtained by Sevcenko.[12] Later, Sadovnicii[13,14] proved trace formulas for more general cases. And then, Lidskii and Sadovnicii[15-17] indicated a technique for computing trace formulas of general problems including ordinary differential equations on a finite interval. The subject has a very broad theory, and we recommend the list of studies in previous works[18-21] for more and further information.

The regularized trace formulas of the differential operators in which contains operator coefficient were firstly studied by Chalilova[22] and Adiguzelov.[23] A significant number of studies with bounded operator coefficients within this scope were given in previous studies,[24-26] and those with unbounded operator coefficients were given in previous works.[27-31] However, very few of them contain second or higher regularized trace formulas (see, e.g., previous literature[27,29,32]). In the present work, we give a general result for the second regularized trace formula of a higher order differential operator with unbounded operator-valued coefficient. The first and second regularized trace formulas for the special cases of this operator were obtained in Gül and Üçüncü[33] and Gül,[34] respectively. Trace formulas for differential operators with operator coefficients are used to derive Ambarzumyan's inverse theorems (see Yang and Wang[35]). More generally, problems with operator-valued coefficients arise in many disciplines, for example, in waveguide theory, when the environment is slowly range-dependent. Here, the boundary conditions are called radiation conditions and the coefficients themselves are differential operators.

Let us state some basic definitions and theorems for the development of our results.

Definition 1.1. A system $e_1, e_2, \ldots$ of elements of a Hilbert space H is said to be orthonormal if

$$(e_i, e_j) = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

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Definition 1.2. The orthonormal system $\{e_i\}$ in H is said to be complete if there exists no element $x \in H$ other than zero which is orthogonal to all the elements of the system $\{e_i\}$

Definition 1.3. The orthonormal system $\{e_i\}$ in H is said to be closed if the closed subspace M generated by this system coincides with H.

A closed orthonormal system is called an orthonormal basis for H.

Theorem 1.4 (Levitan and Sargsyan[36]). *The concepts of a complete and closed orthonormal system coincide.*

Note that the considerations above are concerned with separable Hilbert spaces. These spaces are characterized by the fact that there can be always found a countable basis for each one.

Definition 1.5. If with every ordered pair of elements $f, g \in H$ we associate some number $\Omega(f, g)$ in such way that

(i) $\Omega(\alpha_1 f_1 + \alpha_2 f_2, g) = \alpha_1 \Omega(f_1, g) + \alpha_2 \Omega(f_2, g)$,
(ii) $\Omega(f, \beta_1 g_1 + \beta_2 g_2) = \bar{\beta}_1 \Omega(f, g_1) + \bar{\beta}_2 \Omega(f, g_2)$,
(iii) $sup_{\|f\|<1, \|g\|<1} |\Omega(f, g)| < \infty$,

then we say that we have defined a bilinear functional in H.

Theorem 1.6 (Levitan and Sargsyan[36]). *Every bilinear functional Ω in H can be represented in the form $\Omega(f, g) = (Tf, g)$, where T is a bounded operator in H which is uniquely determined by the functional Ω, furthermore $\|T\| = \|\Omega\|$.*

Let T be a bounded linear operator defined on H. For the bilinear form (f, Tg) in H, it follows from previous theorem that there exists a unique bounded linear operator T^* for which

$$(Tf, g) = (f, T^* g)$$

and that $\|T^*\| = \|T\|$.

The operator T^* satisfying the equality above for all $f, g \in H$ is called the adjoint of T.

The general definition of the adjoint operator of an arbitrary linear operator T whose domain is dense in the Hilbert space H can be given in a similar way above. Such an operator is referred as a densely defined operator.

Definition 1.7. A linear operator T is said to be symmetric if

(i) its domain $D(T)$ is dense in H,
(ii) for every pair of elements $f, g \in D(T)$ one has $(Af, g) = (f, Ag)$.

Clearly, for a symmetric operator T, we have $T \subseteq T^*$ from which it follows that a symmetric operator always has a closure. If S is a symmetric extension of T, then $S \subseteq T^*$.

An operator T is called self-adjoint if it coincides with its adjoint; that is, $T^* = T$ and a self adjoint operator T is called positive if $(Tf, f) \geq 0$ for every $f \in H$.

Some basic properties for the self-adjoint operators:

(i) If a self-adjoint operator has an inverse, then this inverse is self-adjoint,
(ii) The eigenvalues of a self-adjoint operator are real,
(iii) Two eigenvectors f_1, f_2 of a self-adjoint operator corresponding distinct eigenvalues λ_1, λ_2 are orthogonal.

We supplement the basic information on the subject with the example of a self-adjoint operator below, and more detailed descriptions and results can be found in previous works.[37-39]

Let $H = L_2[0, \pi]$ and let $D(A)$ denote the set of all functions $u = u(t)$ satisfying the following conditions:

- $u'(t)$ is absolutely continuous on $[0, \pi]$,
- $u(0) = u(\pi) = 0$,
- $u''(t) \in H$.

Define the operator $A : D(A) \to H$ by $Au(t) = -u''(t)$. Easily, we check that A is self-adjoint, its eigenvalues are $\gamma_n = n^2$ $(n = 1, 2, \ldots)$ and corresponding orthonormal eigenvectors are $\psi_n(t) = \sqrt{\frac{2}{\pi}} \sin nt$.

Now let us introduce some notations to present our results:

We suppose that H is an infinite dimensional separable complex Hilbert space. We denote the inner product and the norm in H by $(.,.)$ and $\|.\|$, respectively. $\sigma_1(H)$ denotes the set of all trace class operators on H. Let $H_1 = L_2([0,\pi];H)$ represent the set of all strongly measurable functions f defined on $[0,\pi]$ with their values in H such that for every $g \in H$, the scalar function $(f(x),g)$ is measurable in the finite interval $[0,\pi]$ and

$$\int_0^\pi \|f(x)\|^2 dx < \infty.$$

The space H_1 is also a separable Hilbert space with the scalar product

$$(f,g)_{H_1} = \int_0^\pi (f(x),g(x))dx \quad (f,g \in H_1),$$

(see Kirillov[40]). Consider the higher order differential operators

$$L = L_0 + Q,\ L_0 = (-1)^m \frac{d^{2m}}{dx^{2m}} + A\,(m = 1,2,\ \dots\ )$$

with the identical boundary conditions $y^{(2i-1)}(0) = y^{(2i-1)}(\pi) = 0\ (i = 1,2,\ \dots\ ,m)$ on $H_1 = L_2([0,\pi];H)$.

Here, we suppose that $A : D(A) \to H$ is a densely defined operator on H such that

$$A = A^* \geq E, A^{-1} \in \sigma_\infty(H),$$

where E is identity operator on H, A^* is the adjoint operator of A and $\sigma_\infty(H)$ denotes the set of all compact operators on H.

Moreover, we suppose that $Q(x)$ satisfies the following conditions:

(a) $Q(x)$ is self-adjoint from H to H for every $x \in [0,\pi]$,
(b) $Q(x)$ is weakly measurable in $[0,\pi]$ and for every $f,g \in H$, the scalar function $(Q(x)f,\ g)$ is measurable on $[0,\pi]$,
(c) $\|Q(x)\|$ is bounded on $[0,\pi]$.

The goal of this work is to calculate a second regularized trace formula for the operator L in the form

$$\lim_{p\to\infty} \sum_{q=1}^{n_p} \left[{\lambda_q}^2 - {\mu_q}^2 - \frac{2}{\pi}\mu_q \int_0^\pi (Q(x)\varphi_{j_q}, \varphi_{j_q})dx \right]$$

$$= \frac{1}{2}[trAQ(0) + trAQ(\pi)] + \frac{(-1)^m}{2^{2m+1}}[trQ^{2m}(0) + trQ^{2m}(\pi)] - \frac{1}{\pi}\int_0^\pi trAQ(x)dx,$$

where $n_1 < n_2 < \dots$ and $j_1, j_2, \dots$ are the sequences of natural numbers, $\mu_1 \leq \mu_2 \leq \dots$ and $\lambda_1 \leq \lambda_2 \leq \dots$ are the eigenvalues of the operators L_0 and L, respectively, and $\varphi_1, \varphi_2, \dots, \varphi_n, \dots$ is a complete orthonormal sequence consisting of eigenvectors corresponding to the eigenvalues $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_n \leq \dots$ of the operator A.

2 | SOME RELATIONS ABOUT THE OPERATOR QR_λ^0

Let $D(L_0')$ denote the set of elements $y(x)$ of H_1 having the following conditions:

1. $y(x)$ has continuous derivative of the $2m$-th order, with respect to the norm in H, on $[0,\pi]$,
2. For every $x \in [0,\pi]$, $y(x) \in D(A)$ and $Ay(x)$ is continuous with respect to the norm in H,
3. $y^{(2i-1)}(0) = y^{(2i-1)}(\pi) = 0\ (i = 1,2,\ \dots\ ,m)$.

Now, we consider the operator $L_0'y = l_0(y)$ defined from $D(L_0')$ to H_1. Note that this operator is symmetric and $\overline{D(L_0')} = H_1$. Moreover, its eigenvalues and orthonormal eigenvectors corresponding these eigenvalues are of the form $k^{2m}+\gamma_j$ and $M_k \cos kx\varphi_j$ respectively, where $k = 0, 1, 2, \dots$, $j = 1, 2, \dots$, and $M_0 = 1/\sqrt{\pi}$ and $M_k = \sqrt{2/\pi}$, if $k = 1, 2, \dots$.

Set $L_0 = \overline{L_0'}$. It follows that L_0 is a closed symmetric operator. And the fact that its eigenvectors system is complete requires it to be self-adjoint. Since $Q : H_1 \to H_1$ is bounded and self-adjoint, the operator $L = L_0 + Q$ is also self-adjoint. So, we conclude that the operators L_0 and L have pure discrete spectrum (see Smirnov[39]).

If $\gamma_j \sim a.j^{\alpha}$ $(a > 0, \alpha < \infty)$ as $j \to \infty$ then by following Gül,[30] we find

$$\lambda_n,\ \mu_n \sim d.n^{\frac{2m\alpha}{2m+\alpha}} as n \to \infty, \tag{1}$$

where

$$d = \left(\frac{\alpha a^{\frac{1}{\alpha}}}{2b}\right)^{\frac{2m\alpha}{2m+\alpha}} and\ b = \int_0^{\frac{\pi}{2}} (\sin t)^{\frac{2}{\alpha}-1}(\cos t)^{\frac{1}{m}+1} dt.$$

Thus, we can determine a subsequence $\{\mu_{n_p}\}_{p=1}^{\infty}$ of the sequence $\{\mu_n\}$ such that

$$\mu_q - \mu_{n_p} > d_0\left(q^{\frac{2m\alpha}{2m+\alpha}} - n_p^{\frac{2m\alpha}{2m+\alpha}}\right)\ (q = n_p + 1, n_p + 2, \dots), \tag{2}$$

where d_0 is a positive constant.

Let $R_{\lambda}^0 = (L_0 - \lambda I)^{-1}$, $R_{\lambda} = (L - \lambda I)^{-1}$ be the resolvents of the operators L_0 and L, respectively. If $\alpha > \frac{2m}{2m-1}$ then from the formula (1), R_{λ} and R_{λ}^0 will be both trace class for $\lambda \neq \lambda_q,\ \mu_q\ (q = 1, 2, \dots)$. So we obtain

$$tr(R_{\lambda} - R_{\lambda}^0) = trR_{\lambda} - trR_{\lambda}^0 = \sum_{q=1}^{\infty}\left(\frac{1}{\lambda_q - \lambda} - \frac{1}{\mu_q - \lambda}\right). \tag{3}$$

Set $|\lambda| = b_p = 2^{-1}(\mu_{n_p} + \mu_{n_p+1})$. Obviously, we have $\mu_{n_p} < b_p < \mu_{n_p+1}$ and $\lambda_{n_p} < b_p < \lambda_{n_p+1}$ for the large values of p. By the uniformly convergence of the series $\sum_{q=1}^{\infty}\left(\frac{\lambda^2}{\lambda_q - \lambda}\right)$ and $\sum_{q=1}^{\infty}\left(\frac{\lambda^2}{\mu_q - \lambda}\right)$ on the circle $|\lambda|=b_p$ and the inequality (2), we obtain

$$\sum_{q=1}^{n_p}({\lambda_q}^2 - {\mu_q}^2) = -\frac{1}{2\pi i}\int_{|\lambda|=b_p} \lambda^2 tr(R_{\lambda} - R_{\lambda}^0)d\lambda, \tag{4}$$

and if $\gamma_j \sim a.j^{\alpha}\left(a > 0, \alpha > \frac{2m}{2m-1}\right)$ as $j \to \infty$, then the inequality

$$\|R_{\lambda}^0\|_{\sigma_1(H_1)} < \frac{2(1+2\delta)}{d_0\delta}{n_p}^{1-\delta}\ \left(\delta = \frac{2m\alpha}{2m+\alpha} - 1\right) \tag{5}$$

is satisfied on the circle $|\lambda|=b_p$.

Furthermore, if $\gamma_j \sim a.j^{\alpha}$ $(0 < a, \alpha < \infty)$ as $j \to \infty$ then by the conditions $(a), (b), (c)$ on $Q(x)$ and for the large values of p, we get

$$\|R_{\lambda}\|_{H_1} < \frac{4}{d_0}{n_p}^{-\delta}. \tag{6}$$

By using Equation (4) together with the famous Hilbert identity

$$R_{\lambda} = R_{\lambda}^0 - R_{\lambda}QR_{\lambda}^0\ (\lambda \in \rho(L) \cap \rho(L_0)),$$

it follows that

$$\sum_{q=1}^{n_p}({\lambda_q}^2 - {\mu_q}^2) = \sum_{j=1}^{s} M_{pj} + M_p^{(s)}, \tag{7}$$

where

$$M_{pj} = \frac{(-1)^j}{\pi i j} \int\limits_{|\lambda|=b_p} \lambda\, tr[(QR_\lambda^0)^j] d\lambda \,(\, j = 1, 2, \ \dots\ ), \tag{8}$$

$$M_p^{(s)} = \frac{(-1)^s}{2\pi i} \int\limits_{|\lambda|=b_p} \lambda^2 tr[R_\lambda (QR_\lambda^0)^{s+1}] d\lambda\,. \tag{9}$$

3 | THE SECOND REGULARIZED TRACE FORMULA

Theorem 3.1. *If the conditions (a), (b), (c) on Q(x) are satisfied and if* $\gamma_j \sim aj^\alpha \left(a > 0, \alpha > \frac{2m+2\sqrt{2m}}{2\sqrt{2m}-\sqrt{2}-1}\right)$ *as* $j \to \infty$, *then we have*

$$\lim_{p\to\infty} M_{pj} = 0 \,(\, j = 2, 3, 4, \ \dots\ ).$$

Proof. If we consider the formula (8) for $j = 2$, we get

$$M_{p2} = \frac{1}{\pi i} \sum_{j=1}^{n_p} \sum_{k=n_p+1}^{\infty} \left[\int\limits_{|\lambda|=b_p} \frac{\lambda}{(\lambda-\mu_j)(\lambda-\mu_k)} d\lambda \right] (Q\psi_j, \psi_k)_{H_1} (Q\psi_k, \psi_j)_{H_1}. \tag{10}$$

It follows that

$$|M_{p2}| \le \|Q\|_{H_1}^2 b_p \Phi_p, \tag{11}$$

where $\Phi_p < \frac{\delta+1}{\delta d_0}\, n_p^{-\frac{\delta}{\delta+1}}$. By using this inequality, we obtain

$$\lim_{p\to\infty} M_{p2} = 0 \quad \left(\alpha > \frac{2m}{2m-1}\right). \tag{12}$$

Let us prove that

$$\lim_{p\to\infty} M_{p3} = 0. \tag{13}$$

From the formula (8), it follows that

$$\begin{aligned} M_{p3} = &\sum_{j=1}^{n_p} \sum_{k=1}^{n_p} \sum_{s=n_p+1}^{\infty} [F(\,j,k,s) + F(s,k,j) + F(\,j,s,k)] \\ &+ \sum_{j=1}^{n_p} \sum_{k=n_p+1}^{\infty} \sum_{s=n_p+1}^{\infty} [F(\,j,k,s) + F(s,k,j) + F(\,j,s,k)], \end{aligned} \tag{14}$$

where

$$F(\,j,k,s) = g(\,j,k,s)(Q\psi_j, \psi_k)_{H_1} (Q\psi_k, \psi_s)_{H_1} (Q\psi_s, \psi_j)_{H_1},$$

$$g(\,j,k,s) = \frac{1}{3\pi i} \int\limits_{|\lambda|=b_p} \frac{\lambda}{(\lambda-\mu_j)(\lambda-\mu_k)(\lambda-\mu_s)} d\lambda.$$

By considering the equalities $g(\,j,k,s) = \overline{g(\,j,k,s)}$ and $Q = Q^*$, we have

$$F(s,k,j) = \overline{F(\,j,k,s)}\ ,\ F(k,j,s) = \overline{F(\,j,k,s)}\ ,F(\,j,s,k) = \overline{F(\,j,k,s)}. \tag{15}$$

From (14) and (15), we write

$$M_{p3} = T_1 + T_2 \tag{16}$$

with

$$T_1 = A + 2\bar{A}, \tag{17}$$

$$T_2 = B + 2\bar{B}, \tag{18}$$

and

$$A = \sum_{j=1}^{n_p} \sum_{k=1}^{n_p} \sum_{s=n_p+1}^{\infty} F(j,k,s),$$

$$B = \sum_{j=1}^{n_p} \sum_{k=n_p+1}^{\infty} \sum_{s=n_p+1}^{\infty} F(j,k,s).$$

It is not difficult to see that the following inequalities are satisfied:

$$|A| \leq \|Q\|^3 b_p \frac{2}{{d_0}^2} \left(\frac{\delta+1}{\delta}\right) {n_p}^{\frac{1-\delta^2}{\delta+1}}, \tag{19}$$

$$|B| \leq \|Q\|^3 b_p \frac{4}{{d_0}^2} \left(\frac{\delta+1}{\delta}\right)^2 {n_p}^{-\frac{2\delta}{\delta+1}} \quad \left(\alpha > \frac{2m}{2m-1}\right). \tag{20}$$

By (16), (17), (18), (19), and (20), we find

$$\lim_{p\to\infty} M_{p3} = 0 \left(\alpha > \frac{2m + 2\sqrt{2m}}{2\sqrt{2m} - \sqrt{2} - 1}\right). \tag{21}$$

Now, it remains to show that the equality

$$\lim_{p\to\infty} M_{pj} = 0 (\, j = 4, 5, 6, \ \dots \,)$$

holds. From the formula (8), we find

$$|M_{pj}| \leq c \int_{|\lambda|=b_p} \|R_\lambda^0\| \|R_\lambda^0\|^{j-1} |d\lambda|, \tag{22}$$

where c is a constant. In the case $Q(x) \equiv 0$, since $R_\lambda = R_\lambda^0$ from inequality (6), we have

$$\|R_\lambda^0\|_{H_1} < \frac{4}{d_1} {n_p}^{-\delta} \left(|\lambda| = b_p; \alpha > \frac{2m\alpha}{2m+\alpha} - 1\right). \tag{23}$$

By using inequalities (22),(23), and (5), we obtain

$$|M_{pj}| < c \int_{|\lambda|=b_p} {n_p}^{1-\delta} {n_p}^{-\delta(j-1)} |d\lambda| < c b_p {n_p}^{1-\delta j}.$$

If we consider the large values of p, since

$$b_p = \frac{1}{2}(\mu_{n_p+1} + \mu_{n_p}) \leq c {n_p}^{\delta+1},$$

we obtain

$$|M_{pj}| < c {n_p}^{2-\delta(j-1)}.$$

Clearly, if $j > 1 + 2\delta^{-1}$ $\left(\delta > \frac{2}{3}\ or\ \alpha > \frac{10m}{6m-5}\right)$, we find

$$\lim_{p\to\infty} M_{pj} = 0 (\, j = 4, 5, 6, \ \dots \,). \tag{24}$$

By considering the results (12) and (21) for $\alpha > \frac{2m+2\sqrt{2m}}{2\sqrt{2m}-\sqrt{2}-1} > \frac{10m}{6m-5}$, we obtain

$$\lim_{p\to\infty} M_{pj} = 0 \ (\, j = 2, 3, 4, \ \dots \,). \tag{25}$$

Thus, the proof is completed. □

The eigenvalues of the operator L_0 are of the form $k^{2m}+\gamma_j$ $(k=0,1,2,\ \dots\ , j=1,2,\ \dots\ )$, and so we write

$$\mu_q = {k_q}^{2m}+\gamma_{j_q}(q=1,2,\ \dots\ ). \tag{26}$$

We state the following additional three conditions on $Q(x)$ to obtain the second regularized trace formula:

(d) $Q(x)$ has weak derivative of order $4m$ on $[0,\pi]$ and the function $(Q^{(4m)}(x)u,v)$ is continuous for every $u,v\in H$.
(e) For every $x\in[0,\pi]$, $Q^{(i)}(x):H\to H(i=0,1,\ \dots\ ,4m)$ are self-adjoint.
(f) For every $x\in[0,\pi]$, $Q^{(4m)}(x)$, $AQ^{(2i)}(x)\in\sigma_1(H)(i=0,1,\ \dots\ ,2m)$ and the functions $\|Q^{(4m)}(x)\|_{\sigma_1(H)}$, $\|AQ^{(2i)}(x)\|_{\sigma_1(H)}$ $(i=0,1,\ \dots\ ,2m)$ are bounded and measurable on $[0,\pi]$.

Theorem 3.2. *By the assumptions (d), (e), (f) on Q(x) and the fact* $\gamma_j\sim aj^{\alpha}\left(a>0,\alpha>\frac{2m+2\sqrt{2m}}{2\sqrt{2m}-\sqrt{2}-1}\right)$ *as* $j\to\infty$, *we have*

$$\begin{aligned}&\lim_{p\to\infty}\sum_{q=1}^{n_p}\left[{\lambda_q}^2-{\mu_q}^2-\frac{2}{\pi}\mu_q\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right]\\&=\frac{1}{2}[trAQ(0)+trAQ(\pi)]+\frac{(-1)^m}{2^{2m+1}}\left[trQ^{(2m)}(0)+trQ^{(2m)}(\pi)\right]-\frac{1}{\pi}\int_0^{\pi}trAQ(x)dx,\end{aligned} \tag{27}$$

where $j_1, j_2,\ \dots$ *are the positive integers satisfying the equality (26).*

Proof. By the equality (8), we write

$$M_{p1}=-\frac{1}{\pi i}\int_{|\lambda|=b_p}\lambda\, tr(QR_{\lambda}^0)d\lambda. \tag{28}$$

Since QR_{λ}^0 is trace class for every $\lambda\in\rho(L_0)$ and $\{\psi_q(x)\}_{q=1}^{\infty}$ is an orthonormal basis of H_1, then

$$tr(QR_{\lambda}^0)=\sum_{q=1}^{\infty}(QR_{\lambda}^0\psi_q,\psi_q)_{H_1}.$$

By considering the equality

$$R_{\lambda}^0\psi_q=(L_0-\lambda I)^{-1}\psi_q=(\mu_q-\lambda I)^{-1}\psi_q$$

and substituting $tr(QR_{\lambda}^0)$ into the equality (28), we obtain

$$\begin{aligned}M_{p1}&=-\frac{1}{\pi i}\int_{|\lambda|=b_p}\left(\lambda\sum_{q=1}^{\infty}(QR_{\lambda}^0\psi_q,\psi_q)_{H_1}\right)d\lambda\\&=-\frac{1}{\pi i}\int_{|\lambda|=b_p}\left(\lambda\sum_{q=1}^{\infty}\frac{1}{\mu_q-\lambda}(Q\psi_q,\psi_q)_{H_1}\right)d\lambda\\&=\frac{1}{\pi i}\sum_{q=1}^{\infty}(Q\psi_q,\psi_q)_{H_1}\int_{|\lambda|=b_p}\frac{\lambda}{\lambda-\mu_q}d\lambda.\end{aligned} \tag{29}$$

Since the orthonormal eigenvectors corresponding to the eigenvalues $k^{2m}+\gamma_j(k=0,1,2,\ \dots\ ;j=1,2,\ \dots\ )$, of the operator L_0 are $M_k\cos kx.\varphi_j$, respectively, then

$$\psi_q(x)=M_{k_q}\cos k_q x.\varphi_{j_q}(q=1,2,\ \dots\ ). \tag{30}$$

Moreover,

$$\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\lambda-\mu_q}d\lambda=\begin{cases}\mu_q, \text{ if } q\le n_p,\\ 0, \text{ if } q>n_p.\end{cases} \tag{31}$$

By using (29), (30), and (31), we find

$$M_{p1}=\sum_{q=1}^{n_p}M_{k_q}^2\mu_q\int_0^\pi(Q(x)\varphi_{j_q},\varphi_{j_q})dx+\sum_{q=1}^{n_p}M_{k_q}^2\mu_q\int_0^\pi\cos 2k_qx(Q(x)\varphi_{j_q},\varphi_{j_q})dx.$$

the equality (7) gives

$$\begin{aligned}&\sum_{q=1}^{n_p}\left[\lambda_q{}^2-\mu_q{}^2-M_{k_q}{}^2\mu_q\int_0^\pi(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right]=\\&\sum_{q=1}^{n_p}M_{k_q}^2\mu_q\int_0^\pi\cos 2k_qx(Q(x)\varphi_{j_q},\varphi_{j_q})dx+\sum_{j=2}^{s}M_{pj}+M_p^{(s)}.\end{aligned} \tag{32}$$

Since the series

$$\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}M_k^2\mu\int_0^\pi\cos 2kx(Q(x)\varphi_j,\varphi_j)dx$$

is absolutely convergent, we have

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}M_{k_q}^2\mu_q\int_0^\pi\cos 2k_qx(Q(x)\varphi_{j_q},\varphi_{j_q})dx=\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}M_k^2\mu\int_0^\pi\cos 2kx(Q(x)\varphi_j,\varphi_j)dx. \tag{33}$$

Since

$$M_k=\begin{cases}\frac{1}{\sqrt{\pi}}, \text{ if } k=0\\ \sqrt{\frac{2}{\pi}}, \text{ if } k=1,2,\ldots\end{cases}$$

then

$$\begin{aligned}&\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}M_k^2\mu\int_0^\pi\cos 2kx(Q(x)\varphi_j,\varphi_j)dx\\&=\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}M_k^2(k^{2m}+\gamma_j)\int_0^\pi\cos 2kx(Q(x)\varphi_j,\varphi_j)dx\\&=\frac{1}{2}\frac{(-1)^m}{2^{2m}}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}M_k^2\left[\int_0^\pi(Q^{(2m)}(x)\varphi_j,\varphi_j)\cos kx\,dx+(-1)^k\int_0^\pi(Q^{(2m)}(x)\varphi_j,\varphi_j)\cos kx\,dx\right]\\&\quad+\frac{1}{2}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty}M_k^2\left[\int_0^\pi(AQ(x)\varphi_j,\varphi_j)\cos kx\,dx+(-1)^k\int_0^\pi(AQ(x)\varphi_j,\varphi_j)\cos kx\,dx\right]\\&=\frac{1}{2}\frac{(-1)^m}{2^{2m}}\sum_{j=1}^{\infty}\left[\sum_{k=0}^{\infty}\left(M_k^2\int_0^\pi(Q^{(2m)}(x)\varphi_j,\varphi_j)\cos kx\,dx\right)\cos(k0)\right.\\&\quad\left.+\sum_{k=0}^{\infty}\left(M_k^2\int_0^\pi\left(Q^{(2m)}(x)\varphi_j,\varphi_j\right)\cos kx\,dx\right)\cos(k\pi)\right]\end{aligned} \tag{34}$$

$$+\frac{1}{2}\sum_{j=1}^{\infty}\left[\sum_{k=0}^{\infty}\left(M_k^2\int_0^{\pi}(AQ(x)\varphi_j,\varphi_j)\cos kx\,dx\right)\cos(k0)\right.$$
$$\left.+\sum_{k=0}^{\infty}\left(M_k^2\int_0^{\pi}(AQ(x)\varphi_j,\varphi_j)\cos kx\,dx\right)\cos(k\pi)\right].$$

The sums according to k on the right hand side of the last equality are the values at the point 0 and π of the Fourier series of the functions $(Q^{(2m)}(x)\varphi_j,\varphi_j)_H$ and $(AQ(x)\varphi_j,\varphi_j)_H$ having second order derivative according to the functions $\{\cos kx\}_{k=0}^{\infty}$ in the interval $[0,\pi]$, respectively. Therefore, by (33) and (34), we find

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}M_{k_q}^2\mu_q\int_0^{\pi}\cos 2k_qx(Q(x)\varphi_{j_q},\varphi_{j_q})dx$$
$$=\frac{1}{2}\sum_{j=1}^{\infty}[(AQ(0)\varphi_j,\varphi_j)+(AQ(\pi)\varphi_j,\varphi_j)]+\frac{(-1)^m}{2^{2m+1}}\sum_{j=1}^{\infty}\left[(Q^{(2m)}(0)\varphi_j,\varphi_j)+(Q^{(2m)}(\pi)\varphi_j,\varphi_j)\right]$$

or

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}M_{k_q}^2\mu_q\int_0^{\pi}\cos 2k_qx(Q(x)\varphi_{j_q},\varphi_{j_q})dx$$
$$=\frac{1}{2}[trAQ(0)+trAQ(\pi)]+\frac{(-1)^m}{2^{2m+1}}\left[trQ^{(2m)}(0)+trQ^{(2m)}(\pi)\right]. \tag{35}$$

In a similar way to the proof of the equality (24), by using (5) and (6), one can show that

$$\lim_{p\to\infty}M_p^{(s)}=0\left(1+\frac{5}{\delta-1}<s\right). \tag{36}$$

By using the equalities (32), (35), (36), and Theorem 2.1, we obtain

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\left[{\lambda_q}^2-{\mu_q}^2-M_{k_q}^2\mu_q\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right]$$
$$=\frac{1}{2}[trAQ(0)+trAQ(\pi)]+\frac{(-1)^m}{2^{2m+1}}\left[trQ^{(2m)}(0)+trQ^{(2m)}(\pi)\right]. \tag{37}$$

Let us arrange the finite sum on the left hand side of the last equality as follows:

$$\sum_{q=1}^{n_p}\left[{\lambda_q}^2-{\mu_q}^2-M_{k_q}^2\mu_q\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right]$$
$$=\sum_{q=1}^{n_p}\left[{\lambda_q}^2-{\mu_q}^2-\frac{2}{\pi}\mu_q\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right]+\frac{1}{\pi}\sum_{\substack{q\le n_p\\k_q=0}}^{n_p}\left[\gamma_{j_q}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right]. \tag{38}$$

Furthermore, we have

$$\left|\sum_{j=1}^{n}(AQ(x)\varphi_j,\varphi_j)\right|\le\sum_{j=1}^{\infty}\left|(AQ(x)\varphi_j,\varphi_j)\right|\le\|AQ(x)\|_{\sigma_1(H)}(n=1,2,\ \dots\ ). \tag{39}$$

By hypothesis, the function $\|AQ(x)\|_{\sigma_1(H)}$ is bounded and measurable in the interval $[0,\pi]$. Therefore,

$$\int_0^{\pi}\|AQ(x)\|_{\sigma_1(H)}dx<\infty. \tag{40}$$

By (39), (40), and using Lebesgue's Monotone Convergence Theorem for Series, we obtain

$$\sum_{j=1}^{\infty}\gamma_j\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)dx=\int_0^{\pi}\left[\sum_{j=1}^{\infty}(AQ(x)\varphi_j,\varphi_j)\right]dx=\int_0^{\pi}trAQ(x)dx. \tag{41}$$

Moreover, since we have $j_{q_1}\neq j_{q_2}$ for every different natural numbers q_1 and q_2 that satisfy the condition $k_{q_1}=k_{q_2}=0$, we get

$$\lim_{p\to\infty}\sum_{\substack{q\le n_p\\ k_q=0}}^{n_p}\left[\gamma_{j_q}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right]=\sum_{j=1}^{\infty}\gamma_j\int_0^{\pi}(Q(x)\varphi_j,\varphi_j)dx. \tag{42}$$

By (41) and (42), we find

$$\lim_{p\to\infty}\sum_{\substack{q\le n_p\\ k_q=0}}^{n_p}\left[\gamma_{j_q}\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right]=\int_0^{\pi}trAQ(x)dx. \tag{43}$$

Thus, by (37), (38), and (43), we obtain

$$\lim_{p\to\infty}\sum_{q=1}^{n_p}\left[{\lambda_q}^2-{\mu_q}^2-\frac{2}{\pi}\mu_q\int_0^{\pi}(Q(x)\varphi_{j_q},\varphi_{j_q})dx\right]$$

$$=\frac{1}{2}[trAQ(0)+trAQ(\pi)]+\frac{(-1)^m}{2^{2m+1}}[trQ^{2m}(0)+trQ^{2m}(\pi)]-\frac{1}{\pi}\int_0^{\pi}trAQ(x)dx.$$

This completes the proof. □

4 | CONCLUSIONS

In this work, we have calculated a new second regularized formula in terms of coefficients and boundary conditions of a self-adjoint higher order differential operator defined in a separable Hilbert space. Such formulas can be used to solve inverse problems and can be applied to index theory of some special operators. These are also important for applied mathematics, mathematical physics, quantum mechanics, and some branches of engineering.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

ORCID

Erdal Gül https://orcid.org/0000-0003-0626-0148

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