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Note on a nonlinear reaction-diffusion process

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Abstract

We revised an Adomian decomposition method proposed by Lesnic [1] applied for solving a nonlinear reaction-diffusion equation. Applying the Adomian decomposition method in a standard form, the author has solved the problem when the power n of the reaction term is small enough. But the Adomian decomposition method in the proposed form in fact, is not applicable for larger values of n. We demonstrated the correct way of solution, which is applicable for all values of n.

Key words: Reaction-diffusion process, Adomian decomposition method, Boundary value problems

1. INTRODUCTION

In [1] Lesnic proposed a standard approach to apply the Adomian decomposition method (ADM) for solving the reaction-diffusion problem. The main difficulty in the application of ADM in proposed form is that the corresponding approximate solution is a divergent or very slowly convergent series on $[0, L]$ for $n \geq 2$ (see[1]). We consider a reaction-diffusion process governed by the equation

$$u'' + u^n = 0, \quad 0 < x < L, \tag{1}$$

with boundary conditions

$$u(0) = u(L) = 0, \tag{2}$$

where $u(x)$ represents the steady-state temperature for the corresponding reaction-diffusion equation with the reaction term u^n, n is the power of the reaction term (heat source), L is the length of the sample (heat conductor). For $n = 3$, the problem (1)-(2) has a unique positive solution (as can be seen by phase-plane analysis, for example) which is given in terms of elliptic functions. For an arbitrary real number $n \neq 3$, the general reaction-diffusion process is more complicated. We analyze the ADM used in [1] and demonstrate the efficient way of solution of the problem.

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2. The Adomian decomposition method

In [1] the author used the ADM to solve the problem (1)-(2). By defining the differential operator $L_{xx} = d^2/dx^2$, Eq. (1) becomes

$$L_{xx}u = -u^n. \tag{3}$$

If we take the inverse two-fold integration operator $L_{xx}^{-1} = \int_0^x \int_0^x dxdx$ then Eq. (1) subject to (2) yields

$$u = \gamma x - L_{xx}^{-1}\left(u^n\right), \tag{4}$$

where $\gamma = u'(0)$ is to be determined by imposing $u(L) = 0$.
The decomposition method seeks the solution as a series [2,3],

$$u = u_0 + u_1 + u_2 + ..., \tag{5}$$

where

$$u_0 = bx,\ u_{n+1} = -L_{xx}^{-1}(A_n),\ n \geq 0 \tag{6}$$

and A_n are the Adomian polynomials given by

$$A_n = \frac{1}{n!}\left[\frac{d}{d\lambda_n}\left(\sum_{k=0}^{n} u_k\lambda^k\right)^p\right]_{\lambda=0}. \tag{7}$$

The first few Adomian polynomials are given by [2,3]

$$\begin{aligned} A_0 &= u_0^n,\ A_1 = nu_0^{n-1}u_1,\ A_2 = nu_0^{n-1}u_2 + \frac{n(n-1)}{2}u_0^{n-2}u_1^2, \\ A_3 &= nu_0^{n-1}u_3 + n(n-1)u_0^{n-2}u_1u_2 + \frac{n(n-1)(n-2)}{6}u_0^{n-3}u_1^3, ... \end{aligned} \tag{8}$$

and so on. Consequently, calculating a few terms in the recurrence relation (6) we obtain

$$\begin{aligned} u_0(x) &= bx,\ u_1(x) = -\frac{b^n x^{n+2}}{(n+1)(n+2)},\ u_2(x) = \frac{nb^{2n-1}x^{2n+3}}{2(n+1)^2(n+2)(2n+3)}, \\ u_3(x) &= -\frac{3n\left(n^2+n-1\right)b^{3n-2}x^{3n+4}}{3!(n+1)^3(n+2)^2(2n+3)(3n+4)}, ... \end{aligned} \tag{9}$$

Then taking $u_{approx} = u_0 + u_1 + u_2 + ... + u_p$ the value of b can be determined as a smallest positive solution of the equation $u_{approx}(L) = 0$ [1]. We shall show that

this (standard) type approach is not efficient in most cases (this type of approach is useful in case of initial value problems).
Let us analyze this difficulty in the case of $n = 2$ and $L = 1$. It follows from (8) and (9) that

$$\begin{aligned} u_0 &= bx,\ u_1 = -\frac{b^2}{12}(x)^4,\ u_2 = \frac{1}{252}b^3x^7, \\ u_3 &= -\frac{1}{6048}b^4x^{10},\ u_4 = \frac{1}{1008*12*13}b^5x^{13}, \end{aligned}$$

$$\begin{aligned} u_5 &= -\frac{185}{3302\,208}\frac{1}{15*16}b^6x^{16}, \\ u_6 &= \frac{26\,381}{79\,252\,992}\frac{1}{18*19}b^7x^{19}, \\ u_7 &= -\frac{61\,535}{4169\,926\,656}\frac{1}{21*22}b^8x^{22}, \end{aligned}$$

$$\begin{aligned} u_8 &= \frac{1389\,887\,029}{41\,740\,965\,826\,560}\frac{1}{24*25}b^9x^{25}, \\ u_9 &= -\frac{63\,088\,428\,391}{30\,523\,081\,260\,672\,000}\frac{1}{27*28}b^{10}x^{28}. \end{aligned}$$

Let us take $u = u_0 + ... + u_9$ and find b from $u(1) = 0 :\ b = 20.57$. If we replace this value into the expression for u_9 we obtain

$$u_9(1) = -\frac{63\,088\,428\,391}{30\,523\,081\,260\,672\,000}\frac{1}{27*28}(20.57)^{10} = -37080.0.$$

That is the "remainder" ($u_9(x)$) is large enough and does not approach 0. The situation remains the same as we consider the more terms in approximation. (Since b must be found from $u(1) = 0$ rewriting this relationship we get

$$\left(b - \frac{b^2}{12}\right) + \left(\frac{1}{252}b^3 - \frac{5}{336*9*10}b^4\right) + \left(\frac{1}{1008*12*13}b^5 - \frac{185}{3302\,208}\frac{1}{15*16}b^6\right) \tag{10}$$

$$+ \left(\frac{26\,381}{79\,252\,992}\frac{1}{18*19}b^7 - \frac{61\,535}{4169\,926\,656}\frac{1}{21*22}b^8\right)$$

$$+ \left(\frac{1389\,887\,029}{41\,740\,965\,826\,560}\frac{1}{24*25}b^9 - \frac{63\,088\,428\,391}{30\,523\,081\,260\,672\,000}\frac{1}{27*28}b^{10}\right) + ... = 0$$

and so at least one of the parentheses must be < 0. It is easy to see that then $b > 12$ -consider the first parentheses in (10)- and b increases for large enough values of approximation's components. That is why the coefficient a_p on $u_p = a_p x^{3p+1}$ becomes large enough as $p \to \infty$).
To overcome these difficulties we offer the next two versions of the ADM: 1) Since the solution must be symmetric with respect to the line $x = L/2$ try to represent the solution as a series of $(x - \frac{L}{2})^k$. Since the solution reaches the maximum at $L/2$ take $u'(L/2) = 0$. We take

$$u_0 = b,$$

where b is an unknown constant to be further determined. Instead of initial-boundary conditions we consider initial conditions

$$u_k(L/2) = 0, \ u_k'(L/2) = 0$$

for each component u_k, $k = 1, 2, ...$
2) To get rapidly convergent series solution we offer to take an initial approximation as $u_0 = a \sin \pi x$ where a is an unknown constant to be further determined (by taking $x = L/2$ in $u'' + u^n = 0$, where $u = u_{approx}$). This version is more suitable when n is positive integer.

3. Applications

Example 3.1. Consider the problem

$$\begin{aligned} u'' + u^2 &= 0, \\ u(0) &= u(1) = 0 \end{aligned}$$

and apply the version 1 of ADM. Now we have from

$$u_0 = b, \ u_{n+1} = -L_{xx}^{-1}(A_n), n \geq 0,$$

that

$$\begin{aligned} u_0 &= b, \\ u_1 &= -0.5b^2 (x - 0.5)^2, \\ u_2 &= \frac{1}{12} b^3 (x - 0.5)^4, \\ u_3 &= -1.388\,9 \times 10^{-2} b^4 (x - 0.5)^6, \end{aligned}$$

$$\begin{aligned} u_4 &= 1.984\,1\times 10^{-3}b^5\left(x-0.5\right)^8, \\ u_5 &= -2.755\,8\times 10^{-4}b^6\left(x-0.5\right)^{10}, \\ u_6 &= 3.674\,3\times 10^{-5}b^7\left(x-0.5\right)^{12}, \\ u_7 &= -4.794\,8\times 10^{-6}b^8\left(x-0.5\right)^{14} \end{aligned}$$

$$\begin{aligned} u_8 &= 6.140\,7\times 10^{-7}b^9\left(x-0.5\right)^{16}, \\ u_9 &= -7.757\,7\times 10^{-8}b^{10}\left(x-0.5\right)^{18} \end{aligned}$$

Let us take $u_{approx}=u_0+u_1+...+u_9$ and find b from $u_{approx}(0)=0$−some numerical method can be used for finding b (we used standard LaTex computation): $b=11.79$. Thus we get an approximate solution

$$\begin{aligned} u &= 11.79-0.5(11.79)^2\left(x-0.5\right)^2+\frac{1}{12}(11.79)^3\left(x-0.5\right)^4-1.388\,9\times 10^{-2}(11.79)^4\left(x-0.5\right)^6 \\ &+1.984\,1\times 10^{-3}(11.79)^5\left(x-0.5\right)^8-2.755\,8\times 10^{-4}(11.79)^6\left(x-0.5\right)^{10} \\ &+3.674\,3\times 10^{-5}(11.79)^7\left(x-0.5\right)^{12}-4.794\,8\times 10^{-6}(11.79)^8\left(x-0.5\right)^{14} \\ &+6.140\,7\times 10^{-7}(11.79)^9\left(x-0.5\right)^{16}-7.757\,7\times 10^{-8}(11.79)^{10}\left(x-0.5\right)^{18}. \end{aligned}$$

Now let us apply the version 2 of ADM: We take $u_0=a\sin\pi x$ and $u_{n+1}=-L_{xx}^{-1}(A_n), n\geq 0$:

$$\begin{aligned} u_0 &= a\sin(\pi x), \\ u_1 &= -\frac{1}{4}a^2x^2+\frac{1}{4}a^2x-\frac{1}{8\pi^2}a^2\cos^2\pi x+\frac{1}{8\pi^2}a^2\sin^2\pi x+\frac{1}{8\pi^2}a^2-a\sin\pi x, \\ u_2 &= -\frac{1}{2\pi^2}a^3x^2\sin\pi x-\frac{2}{\pi^3}a^3x\cos\pi x+\frac{1}{2\pi^2}a^3x\sin\pi x-\frac{1}{24\pi^4}a^3\cos^2\pi x\sin\pi x, \\ &+\frac{a^3\cos\pi x}{\pi^3}+\frac{a^3\sin^3\pi x}{72\pi^4}+\frac{27}{8\pi^4}a^3\sin\pi x+\frac{a^2x^2}{2}+\frac{a^2\cos^2\pi x}{4\pi^2}-\frac{a^2\sin^2\pi x}{4\pi^2}- \\ &\frac{a^2x}{2}-\frac{a^2\left(\pi+4a\right)}{4\pi^3}, \end{aligned}$$

$u_3=\frac{1}{160}a^4x^5-\frac{1}{192}a^4x^4-\frac{1}{4}a^2x^2-\frac{1}{480}a^4x^6-\frac{3}{\pi^3}a^3\cos\pi x-\frac{81}{8\pi^4}a^3\sin\pi x-\frac{2}{\pi^5}a^4\sin\pi x$
$-\frac{3}{32\pi^2}a^4x^3+\frac{3}{64\pi^2}a^4x^4-\frac{435}{256\pi^4}a^4x^2-\frac{1}{8\pi^2}a^2\cos^2\pi x-\frac{451}{288\pi^6}a^4\cos^2\pi x$
$+\frac{25}{18\,432\pi^6}a^4\cos^4\pi x+\frac{1}{8\pi^2}a^2\sin^2\pi x-\frac{1}{24\pi^4}a^3\sin^3\pi x+\frac{451}{288\pi^6}a^4\sin^2\pi x$
$+\frac{25}{18\,432\pi^6}a^4\sin^4\pi x+\frac{1}{8\pi^4}a^3\cos^2\pi x\sin\pi x+\frac{9}{64\pi^4}a^4x^2\cos^2\pi x-\frac{9}{64\pi^4}a^4x^2\sin^2\pi x$
$+\frac{6}{\pi^3}a^3x\cos\pi x-\frac{3}{2\pi^2}a^3x\sin\pi x-\frac{25}{3072\pi^6}a^4\cos^2\pi x\sin^2\pi x+\frac{25}{32\pi^5}a^4\cos\pi x\sin\pi x$

$-\frac{9}{64\pi^4}a^4x\cos^2\pi x+\frac{3}{2\pi^2}a^3x^2\sin\pi x+\frac{9}{64\pi^4}a^4x\sin^2\pi x$
$-\frac{25}{16\pi^5}a^4x\cos\pi x\sin\pi x+\frac{1}{3840\pi^4}a^2\left(180\pi^2a^2+4\pi^4a^2+960\pi^4+6525a^2\right)x$
$+\frac{9613}{6144\pi^6}a^4+\frac{3}{\pi^3}a^3+\frac{1}{8\pi^2}a^2.$
(We used very easy LaTex tools for finding integrals). Now we calculate the number a from the relationship

$$u''+u^2=0,$$

where $u\approx u_0+u_1+u_2+u_3$ and $x=0.5$. We obtain $a=11.569$. The Table 3.1 shows the errors in the solutions and comparison with the errors in [1]. As it is seen from table the error dramatically increases as x goes to 1 and the approximate solution in fact, has no relationship with the solution in case of standard ADM application [1].

Table 3.1. Errors in approximate solution for example 3.1

x	$u_{appr}=\sum_{k=0}^{9}\mu_k$ $(u_0=b)$	$u_{appr}=\sum_{k=0}^{3}\mu_k$ $(u_0=a\sin\pi x)$	Error in $u''+u^2=0$ $(u_0=b)$	Error in $u''+u^2=0$ $(u_0=a\sin x)$	Error in $u''+u^2=0$ in [1] $u_{appr}=\sum_{k=0}^{9}\mu_k$
0.0	0	0.0	0.0	$2.566\,5\times10^{-3}$	0.0
0.1	3.300 9	3.297 6	0.114 5	$2.594\,2\times10^{-2}$	$4.903\,6\times10^{-5}$
0.2	6.471 2	6.469 1	6.061×10^{-3}	$9.843\,5\times10^{-2}$	$1.977\,2\times10^{-4}$
0.3	9.212 3	9.210 9	$5.585\,2\times10^{-3}$	0.136 82	$1.117\,1\times10^{-3}$
0.4	11.108	11.107	$4.505\,7\times10^{-3}$	$3.547\,4\times10^{-2}$	$9.882\,2\times10^{-2}$
0.5	11.79	11.788	0.00	$5.400\,3\times10^{-2}$	5.736 6
0.6	11.108	11.107	$4.505\,7\times10^{-3}$	$3.547\,4\times10^{-2}$	198.22
0.7	9.212 3	9.210 9	$5.585\,2\times10^{-3}$	0.136 82	4292.8
0.9	3.300 9	3.297 6	0.114 5	$2.594\,2\times10^{-2}$	7946 90

Example 3.2. Consider the problem

$$\begin{aligned} u''+u^3 &= 0,\\ u(0) &= u(1)=0. \end{aligned}$$

We take $u_0=a\sin\pi x$ and $u_{n+1}=-L_{xx}^{-1}(A_n), n\geq 0$:

$$\begin{aligned}
u_0 &= a\sin(\pi x),\\
u_1 &= -\frac{1}{12\pi}\left(\frac{1}{3\pi}a^3\sin3\pi x-\frac{9}{\pi}a^3\sin\pi x+12\pi a\sin\pi x\right),\\
u_2 &= \frac{1}{1440\pi^3}\left(\frac{2460}{\pi}a^5\sin\pi x-\frac{290}{3\pi}a^5\sin3\pi x+\frac{6a^5\sin5\pi x}{5\pi}-3240\pi a^3\sin\pi x+120\pi a^3\sin3\pi x\right),\\
u_3 &= -\frac{1}{756\,000\pi^5}(63\,000\pi^3a^3\sin3\pi x-\frac{3925\,250}{\pi}a^7\sin\pi x-1701\,000\pi^3a^3\sin\pi x\\
&\quad+\frac{470\,470}{3\pi}a^7\sin3\pi x-\frac{12\,614}{5\pi}a^7\sin5\pi x+\frac{130}{7\pi}a^7\sin7\pi x+6457\,500\pi a^5\sin\pi x\\
&\quad-253\,750\pi a^5\sin3\pi x+3150\pi a^5\sin5\pi x).
\end{aligned}$$

To find a we take $u = u_0 + u_1 + u_2 + u_3$ in $u'' + u^3 = 0$ and replace $x = 0.5$. Thus we obtain $a = 3.6033$. If we use the ADM as in [1] we obtain

$$\begin{aligned} u_0 &= bx, \\ u_1 &= -\frac{1}{20}b^3x^5, \\ u_2 &= \frac{1}{480}b^5x^9, \\ u_3 &= -\frac{11}{124\,800}b^7x^{13}, \\ u_4 &= \frac{211}{56\,576\,000}b^9x^{17}, \\ u_5 &= -\frac{1607}{10\,183\,680\,000}b^{11}x^{21}. \end{aligned}$$

The Table 3.2 demonstrates the errors in the solution and the comparison with the errors in [1].

Table 3.2. Errors in approximate solution for Example 3.2

x	Error in $u'' + u^3 = 0$ $(u = u_0 + ... + u_3)$	Error in $u'' + u^2 = 0$ in [1] $(u = u_0 + ... + u_5)$
0.0	0.0	0.0
0.1	$2.763\,2 \times 10^{-3}$	$2.851\,5 \times 10^{-6}$
0.2	$1.724\,0 \times 10^{-2}$	$2.274\,5 \times 10^{-5}$
0.3	0.136 73	$7.578\,0 \times 10^{-5}$
0.4	0.121 86	$1.748\,4 \times 10^{-4}$
0.5	0.0	$5.673\,3 \times 10^{-4}$
0.6	0.121 86	$1.621\,6 \times 10^{-2}$
0.7	0.136 73	0.490 05
0.8	$1.724\,0 \times 10^{-2}$	9.079 4
0.9	$2.763\,2 \times 10^{-3}$	112.57
1.0	0.0	1004. 7

Note that our approach is efficient not only for larger values of $n \geq 2$, but also for values of $n < 2$. The following example demonstrates the advantage of the method used here for $n = 1/2$:

Example 3.3. Consider the problem

$$\begin{aligned} u'' + u^{1/2} &= 0, \\ u(0) &= u(1) = 0. \end{aligned}$$

We take $u_0 = b$ and $u_{n+1} = -L_{xx}^{-1}(A_n), n \geq 0$:

$$
\begin{aligned}
u_0 &= b, \quad A_0 = b^{1/2} \\
u_1 &= -\frac{b^{1/2}}{2}(x-0.5)^2, \; A_1 = \frac{1}{2}(b)^{-0.5}\left(-\frac{b^{1/2}}{2}(x-0.5)^2\right) \\
u_2 &= \frac{1}{4*12}(x-0.5)^4, \\
A_2 &= \frac{1}{2}(b)^{-0.5}\left(\frac{1}{4*12}(x-0.5)^4\right) - \frac{1}{2}\frac{1}{2}\frac{1}{2}(b)^{-(3/2)}\left(-\frac{b^{1/2}}{2}(x-0.5)^2\right)^2 \\
u_3 &= \frac{2.0833\times 10^{-2}}{\sqrt{b}*30}(x-0.5)^6, ...
\end{aligned}
$$

Let us take $u_{approx} = u_0 + u_1 + u_2 + u_3$ and calculate b from $u_{approx}(1) = 0$. We obtain $b = 0.335\,53$. If we proceed as in [1] we get

$$
\begin{aligned}
u_0 &= bx, \quad A_0 = b^{1/2}x^{1/2}, \\
u_1 &= -\frac{b^{1/2}}{\frac{3}{2}*\frac{5}{2}}x^{5/2}, \quad A_1 = \frac{1}{2}(bx)^{-0.5}\left(-\frac{b^{1/2}}{\frac{3}{2}*\frac{5}{2}}x^{5/2}\right) \\
u_2 &= \frac{2}{15}\frac{x^4}{12}, \\
A_2 &= \frac{1}{2}(bx)^{-0.5}\left(\frac{2}{15}\frac{x^4}{12}\right) - \frac{1}{2}\frac{1}{2}\frac{1}{2}(bx)^{-(3/2)}\left(-\frac{b^{1/2}}{\frac{3}{2}*\frac{5}{2}}x^{5/2}\right)^2, \\
u_3 &= \frac{3.3333\times 10^{-3}}{\sqrt{b}*4.5*5.5}x^{5.5},
\end{aligned}
$$

and taking $u_{approx} = bx - \frac{b^{1/2}}{\frac{3}{2}*\frac{5}{2}}x^{5/2} + \frac{2}{15}\frac{x^4}{12} + \frac{3.3333\times 10^{-3}}{\sqrt{b}*4.5*5.5}x^{5.5}$, $u_{approx}(1) = 0$ gives $b = 0.459\,33$. The Table 3.3 shows the absolute errors in the solutions and comparison with the errors in [1].

Table 3.3. Errors in approximate solution for Example 3.3.

x	Errors in $u'' + u^{1/2} = 0$ $(u = u_0 + ... + u_3)$	Errors in $u'' + u^{1/2} = 0$ in [1] $(u = u_0 + ... + u_3)$
0.0	0.0	0.0
0.1	6.5132×10^{-5}	3.5223×10^{-6}
0.2	1.1557×10^{-5}	5.2383×10^{-6}
0.3	1.4322×10^{-6}	8.159×10^{-6}
0.4	5.0011×10^{-7}	1.5937×10^{-5}
0.5	4.8554×10^{-7}	3.569×10^{-5}
0.6	5.0011×10^{-7}	7.9746×10^{-5}
0.7	1.4322×10^{-6}	1.6777×10^{-4}
0.8	1.1557×10^{-5}	3.2948×10^{-4}
0.9	6.5132×10^{-5}	6.0829×10^{-4}
0.99	2.2931×10^{-4}	1.0101×10^{-3}
1.0	2.6033×10^{-4}	1.0662×10^{-3}

4. Conclusions

The one of the main strengths of ADM is in fact its fast convergence, but the form of initial approximation and initial-boundary conditions are very important for the efficiency of the ADM. We revised an Adomian decomposition method proposed by Lesnic [1] applied for solving a nonlinear reaction-diffusion equation. The method is of remarkable simplicity, while the obtained results are of utter accuracy on the whole solution domain. The idea used here can be applied to various other nonlinear problems without any difficulty.

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