Computational and Applied Mathematics (2025) 44:264
https://doi.org/10.1007/s40314-025-03226-7

$\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes

Roghayeh Mohammadi Hesari[1] · Mustafa Sari[2] · Ismail Aydogdu[2]

Received: 7 November 2024 / Revised: 18 March 2025 / Accepted: 16 April 2025
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Abstract
In this study, we introduce $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes as a generalization of skew cyclic codes over $\mathbb{F}_4$. We characterize algebraic structures of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes by determining their generating polynomials and minimal spanning sets. We also exhibit generator matrices for these codes. Furthermore, we examine the duals of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes concerning a certain inner product and identify their generating polynomials. Finally, as an application of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes, we present some examples of near-MDS codes over $\mathbb{F}_4$ and optimal binary linear codes which are Gray images of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes.

Keywords Mixed alphabet · Skew codes · Cyclic codes · Optimal codes

Mathematics Subject Classification 94B05 · 94B60 · 81P70

1 Introduction

Cyclic codes are a key topic in the field of coding theory, which is crucial in the development of error-correcting codes for reliable communication and data storage. These codes possess unique algebraic properties that make them particularly amenable to efficient encoding and decoding algorithms. By exploiting the cyclic nature of these codes, it becomes possible to simplify the implementation of error correction techniques while achieving high-performance levels.

Skew cyclic codes are a fascinating extension of traditional cyclic codes, where the concept of cyclicity is generalized to include a new kind of symmetry. In skew cyclic codes, shifts are replaced by linear transformations defined by polynomials, introducing a novel perspective on the algebraic structures underlying error correction codes.

The concept of skew cyclic codes was originally introduced by Boucher and colleagues in Boucher et al. (2007). These codes represent a generalization of cyclic codes, forming a

✉ Ismail Aydogdu
iaydogdu@yildiz.edu.tr

Roghayeh Mohammadi Hesari
r.mohammadi1363@yahoo.com

Mustafa Sari
musari@yildiz.edu.tr

[1] Department of Mathematics, Payam Noor Qeydar University, Zanjan, Iran

[2] Department of Mathematics, Yildiz Technical University, Esenler, Istanbul 34220, Türkiye

Published online: 06 May 2025

distinct class of linear codes through the utilization of skew polynomial rings with an automorphism θ operating over a finite field $\mathbb{F}_q$ containing q elements. Within this framework, the polynomial ring is defined as, $\mathbb{F}_q[x;\theta] = \left\{a_0 + a_1x + \cdots + a_kx^k : a_i \in \mathbb{F}_q, 0 \le i \le k\right\}$, where the operation of addition adhere to the standard polynomial addition but with the multiplication rule $xa = \theta(a)x\left(a \in \mathbb{F}_q\right)$. A linear code $\mathcal{C}$ of length n over $\mathbb{F}_q^n$ is categorized as a skew cyclic code concerning the automorphism θ if it satisfies $(\theta(c_{n-1}), \theta(c_0), \ldots, \theta(c_{n-2})) \in \mathcal{C}$ for all $(c_0, c_1, \ldots, c_{n-1}) \in \mathcal{C}$. One of the prominent distinctions between classical cyclic codes and skew cyclic codes lies in the nature of their divisors; in general, the modulus $x^n - 1$ possesses a greater number of right divisors in $\mathbb{F}_q[x;\theta]$ than in $\mathbb{F}_q[x]$. Consequently, a polynomial may exhibit more roots than its degree would suggest within the skew polynomial ring. As a result, the family of skew-cyclic codes of a specified length is substantially larger than that of cyclic codes. The intrinsic complexity of factorizations in skew polynomial rings, relative to the commutative case, has spurred significant interest among researchers in exploring skew cyclic codes. Notably, the investigation of codes within skew polynomial rings has led to the discovery of numerous record-breaking codes (Abualrub et al. 2010; Boucher et al. 2007). There have been various studies defining skew cyclic codes over different algebraic structures, as evidenced by the works referenced in Aydogdu et al. (2022); Boucher et al. (2008); Gursoy and Aydogdu (2022); Patel et al. (2024); Prakash et al. (2023); Jitman et al. (2012); Melakhessou et al. (2020); Yao et al. (2015). In this paper, our focus is directed towards examining skew cyclic codes specifically within the realm of the ring $\mathbb{F}_2\mathbb{F}_4$. Linear and cyclic codes over mixed alphabets have been studied by many researchers. For instance, Aydogdu et. al generalized the concept of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes (introduced by Borges et. al. in Borges et al. (2010)) as $\mathbb{Z}_2\mathbb{Z}_{2^s}$ and $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ additive codes for positive integers r and s with a prime p in Aydogdu and Siap (2013) and Aydogdu and Siap (2015) respectively. Furthermore, cyclic codes over $\mathbb{F}_2\mathbb{F}_4$ were introduced in Abualrub et al. (2020). Moreover, skew cyclic codes over mixed alphabet codes such as $\mathbb{Z}_2\mathbb{Z}_4$ (Gursoy and Aydogdu 2022) and $\mathbb{F}_4R$(Benbelkacem et al. 2022) were introduced in the literature.

In this paper, we are interested in studying skew cyclic codes over the ring $\mathbb{F}_2 \times \mathbb{F}_4$. We introduce $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes by determining their generator polynomials in Sect. 3. Section 4 is related to generator matrices of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes. In Sect. 5, we study the duals of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes and give illustrative examples of near-MDS codes over $\mathbb{F}_4$ and optimal parameter binary linear codes derived from $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes.

2 Preliminaries

Let us consider the finite fields $\mathbb{GF}(2) = \mathbb{F}_2 = \{0, 1\}$ and $\mathbb{GF}(4) = \mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ where $\omega^2 = \omega + 1$ with $\omega^2 + \omega + 1 = 0$. Define the set,

$$\mathbb{F}_2 \times \mathbb{F}_4 = \mathbb{F}_2\mathbb{F}_4 = \{(a, b) : \ a \in \mathbb{F}_2 \text{ and } b \in \mathbb{F}_4\}.$$

$\mathbb{F}_2\mathbb{F}_4$ is an $\mathbb{F}_4$-module with the module multiplication $*$ defined as

$$d * (a, b) = (da \mod \omega, db)$$

where $d \in \mathbb{F}_4$. A nonempty subset C of $\mathbb{F}_2^r \times \mathbb{F}_4^s$ is called $\mathbb{F}_2\mathbb{F}_4$-linear code of length $r + s$ if it is an $\mathbb{F}_4$-submodule of $\mathbb{F}_2^r \times \mathbb{F}_4^s$. Moreover, $\mathbb{F}_2\mathbb{F}_4$-linear codes can be seen as linear codes over $\mathbb{F}_2$ using a specific Gray map.

Definition 1 Let $(a_0, a_1, \ldots, a_{r-1}, b_0, b_1, \ldots, b_{s-1}) \in \mathbb{F}_2^r \times \mathbb{F}_4^s$ with $b_i = p_i + \omega q_i$. We define the Gray map as follows:

$$\begin{aligned} \Psi : \mathbb{F}_2^r \times \mathbb{F}_4^s &\to \mathbb{F}_2^n \\ \Psi\left(a_0, a_1, \ldots, a_{r-1}, p_0 + \omega q_0, p_1 + \omega q_1, \ldots, p_{s-1} + \omega q_{s-1}\right) & \\ = \left(a_0, a_1, \ldots, a_{r-1}, q_0, q_1, \ldots, q_{s-1}, p_0 + q_0, p_1 + q_1, \ldots, p_{s-1} + q_{s-1}\right) & \end{aligned}$$

where $n = r + 2s$.

The Gray map Ψ is an isometry which transforms the Lee distance in $\mathbb{F}_2^r \times \mathbb{F}_4^s$ to the Hamming distance in $\mathbb{F}_2^n$. The Hamming and the Lee distance between two codewords is the Hamming weight and the Lee weight of their differences, respectively. The Hamming weight of a codeword is defined as the number of its non-zero entries, and the Lee weights of the elements of $\mathbb{F}_4$ are defined as $wt_L(0) = 0, wt_L(1) = 1, wt_L(\omega) = 2, wt_L(1 + \omega) = 1$. It is worth mentioning that for an $\mathbb{F}_2\mathbb{F}_4$-linear code C, we have $\Psi(C)$ as a binary linear code, i.e., Ψ is a linear map. The definition of the Lee weight of a codeword in $\mathbb{F}_4$ can be extended to the Lee weight of a codeword $v = (v_1, v_2) \in \mathbb{F}_2^r \times \mathbb{F}_4^s$ as follows:

$$wt(v) = wt_H\left(v_1\right) + wt_L\left(v_2\right)$$

where $wt_H\left(v_1\right)$ is the Hamming weight of v_1 and $wt_L\left(v_2\right)$ is the Lee weight of v_2. Further, the minimum distance of the $\mathbb{F}_2\mathbb{F}_4$-linear code C, denoted by $d(C)$, is naturally defined as

$$d(C) = \min\left\{d\left(c_1, c_2\right) : c_1, c_2 \in C \text{ such that } c_1 \neq c_2\right\}$$

where $d\left(c_1, c_2\right) = wt\left(c_1 - c_2\right)$. If C is an $\mathbb{F}_2\mathbb{F}_4$-linear code of length $r + s$, then the Gray image $\Psi(C)$ is a binary linear code of length $n = r + 2s$.

Moreover, we can give the the following definition of cyclic codes over $\mathbb{F}_2^r \times \mathbb{F}_4^s$.

Definition 2 A subset C of $\mathbb{F}_2^r \times \mathbb{F}_4^s$ is called an $\mathbb{F}_2\mathbb{F}_4$-cyclic code, if for any codeword $\mathbf{c} = (a_0, a_1, \ldots, a_{r-1},\ b_0, b_1, \ldots, b_{s-1}) \in \mathbb{F}_2^r \times \mathbb{F}_4^s$, its cyclic shift

$$\left(a_{r-1}, a_0, \ldots, a_{r-2},\ b_{s-1}, b_0, \ldots, b_{s-2}\right)$$

is also in C.

3 $\mathbb{F}_2\mathbb{F}_4$-Skew cyclic codes

Consider the automorphism θ of $\mathbb{F}_4$ defined by

$$\begin{aligned} \theta : \mathbb{F}_4 &\to \mathbb{F}_4 \\ x &\mapsto x^2. \end{aligned}$$

For a vector $\mathbf{c} = (a_0, a_1, \ldots, a_{r-1},\ b_0, b_1, \ldots, b_{s-1}) \in \mathbb{F}_2^r \times \mathbb{F}_4^s$,

$$\Theta(\mathbf{c}) = \left(a_{r-1}, a_0, \ldots, a_{r-2},\ \theta\left(b_{s-1}\right), \theta\left(b_0\right), \ldots, \theta\left(b_{s-2}\right)\right)$$

is called θ-skew cyclic shift of $\mathbf{c}$.

Definition 3 An $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r + s$ is a linear code C in $\mathbb{F}_2^r \times \mathbb{F}_4^s$ such that $\Theta(\mathbf{c}) \in C$ for all $\mathbf{c} \in C$.

The set $\mathbb{F}_4[x;\theta] = \left\{b_0 + b_1x + \cdots + b_{s-1}x^{s-1} : b_i \in \mathbb{F}_4,\ s \in \mathbb{Z}^+\right\}$ is called the skew polynomial ring with usual addition of polynomials and multiplication defined under the rule $xb = \theta(b)x$. The set $\mathcal{R}_{r,s} = \dfrac{\mathbb{F}_2[x]}{\langle x^r-1\rangle} \times \dfrac{\mathbb{F}_4[x;\theta]}{\langle x^s-1\rangle}$ is a ring with the following two operations for $(a_1(x), b_1(x)), (a_2(x), b_2(x)) \in \dfrac{\mathbb{F}_2[x]}{\langle x^r-1\rangle} \times \dfrac{\mathbb{F}_4[x;\theta]}{\langle x^s-1\rangle}$:

$$\begin{aligned}(a_1(x), b_1(x)) + (a_2(x), b_2(x)) &= (a_1(x) + a_2(x), b_1(x) + b_2(x)),\\ (a_1(x), b_1(x))(a_2(x), b_2(x)) &= (a_1(x)a_2(x), b_1(x)b_2(x)).\end{aligned}$$

Define the map $\phi : \mathbb{F}_4[x] \to \mathbb{F}_2[x]$ by $\sum a_i x^i \mapsto \sum L(a_i)x^i$, where $L : \mathbb{F}_4 \to \mathbb{F}_2$ is the map defined by $L(0) = L(\omega^2) = 0$, $L(1) = L(\omega) = 1$. For $f(x) \in \mathbb{F}_4[x]$ and $(a(x), b(x)) \in \mathcal{R}_{r,s}$, define the scalar multiplication as

$$f(x)(a(x), b(x)) = (\phi(f(x))a(x), f(x)b(x)).$$

Then, $\mathcal{R}_{r,s}$ becomes a left $\mathbb{F}_4[x]$-module with above addition and scalar multiplication. Furthermore, we can identify one-one correspondence between the elements in $\mathbb{F}_2^r \times \mathbb{F}_4^s$ and the elements in $\mathcal{R}_{r,s}$ by defining the following map:

$$\begin{aligned}\varphi : \mathbb{F}_2^r \times \mathbb{F}_4^s &\to \mathcal{R}_{r,s}\\ (a_0, a_1, \ldots, a_{r-1},\ b_0, b_1, \ldots, b_{s-1}) &\mapsto \left(a_0 + a_1x + \cdots + a_{r-1}x^{r-1},\ b_0 + b_1x + \cdots + b_{s-1}x^{s-1}\right)\\ &= (a(x), b(x))\end{aligned}$$

where $(a_0, a_1, \ldots, a_{r-1},\ b_0, b_1, \ldots, b_{s-1}) \in \mathbb{F}_2^r \times \mathbb{F}_4^s$. Therefore, we give the following theorem.

Theorem 1 *An $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r+s$ can be viewed as a left $\mathbb{F}_4[x]$-submodule of $\mathcal{R}_{r,s}$ and vice versa.*

Proof It is enough to show that there is a one-to-one correspondence between $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes of length $r+s$ and left $\mathbb{F}_4[x]$-submodules of $\mathcal{R}_{r,s}$.

Let C be an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r+s$. It is clear that $\varphi(C)$ is closed under addition. Since $f(x)(a(x), b(x)) \in \varphi(C)$ if $x(a(x), b(x)) \in \varphi(C)$ for all $f(x) \in \mathbb{F}_4[x]$ and $(a(x), b(x)) \in \varphi(C)$, we only prove $x(a(x), b(x)) \in \varphi(C)$ for all $(a(x), b(x)) \in \varphi(C)$. Let $\mathbf{c} = (a_0, a_1, \ldots, a_{r-1},\ b_0, b_1, \ldots, b_{s-1}) \in C$ and $\varphi(\mathbf{c}) = (a(x), b(x)) \in \varphi(C)$. Since $x^r = 1$ in $\frac{\mathbb{F}_2[x]}{\langle x^r-1\rangle}$ and $x^s = 1$, $xb = \theta(b)x$ in $\frac{\mathbb{F}_4[x;\theta]}{\langle x^s-1\rangle}$, we can write

$$\begin{aligned}x(a(x), b(x)) &= (xa(x), xb(x))\\ &= \Big(a_{r-1} + a_0x + \cdots + a_{r-2}x^{r-1},\\ &\quad \theta\left(b_{s-1}\right) + \theta(b_0)x + \cdots + \theta\left(b_{s-2}\right)x^{s-1}\Big).\end{aligned}$$

Since C is an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r+s$, we have

$$\Theta(\mathbf{c}) = \left(a_{r-1}, a_0, \ldots, a_{r-2},\ \theta\left(b_{s-1}\right), \theta(b_0), \ldots, \theta\left(b_{s-2}\right)\right) \in C$$

and so

$$\begin{aligned}\varphi(\Theta(\mathbf{c})) &= \Big(a_{r-1} + a_0x + \cdots + a_{r-2}x^{r-1},\\ &\quad \theta\left(b_{s-1}\right) + \theta(b_0)x + \cdots + \theta\left(b_{s-2}\right)x^{s-1}\Big)\\ &= x(a(x), b(x)) \in \varphi(C).\end{aligned}$$

The converse is clear. □

From now on, unless otherwise stated, by an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r+s$, we refer to a left $\mathbb{F}_4[x]$-submodule of $\mathcal{R}_{r,s}$. We also denote $g(x)$ is a right divisor of $h(x)$ by $g(x)|_r h(x)$ through the paper. The following provides the general form of left $\mathbb{F}_4[x]$-submodules of $\mathcal{R}_{r,s}$.

Theorem 2 *Every left $\mathbb{F}_4[x]$-submodule of $\mathcal{R}_{r,s}$ is generated by $(k(x),0)$ and $(l(x),g(x))$ for some polynomials $k(x), l(x) \in \mathbb{F}_2[x]$ and $g(x) \in \mathbb{F}_4[x;\theta]$, where $k(x) \mid \left(x^r-1\right)$ in $\mathbb{F}_2[x]$ and $g(x) \mid_r \left(x^s-1\right)$ in $\mathbb{F}_4[x;\theta]$.*

Proof Let C be a left $\mathbb{F}_4[x]$-submodule of $\mathcal{R}_{r,s}$ and σ be an $\mathbb{F}_4[x]$-module homomorphism defined by

$$\begin{aligned}\sigma : C \subseteq \mathcal{R}_{r,s} &\to \frac{\mathbb{F}_4[x;\theta]}{\langle x^s-1\rangle},\\ (a(x),b(x)) &\mapsto b(x).\end{aligned}$$

Since $\mathrm{Im}(\sigma)$ is a left ideal of $\frac{\mathbb{F}_4[x;\theta]}{\langle x^s-1\rangle}$ then $\mathrm{Im}(\sigma) = \langle g(x)\rangle$ for some $g(x) \mid_r \left(x^s-1\right)$ in $\mathbb{F}_4[x;\theta]$. Furthermore, consider

$$Ker(\sigma) = \left\{(a(x),0) \in \mathcal{R}_{r,s} : a(x) \in \frac{\mathbb{F}_2[x]}{\langle x^r-1\rangle}\right\}$$

and the set

$$J = \left\{a(x) \in \frac{\mathbb{F}_2[x]}{\langle x^r-1\rangle} : (a(x),0) \in Ker(\sigma)\right\}.$$

Since J is an ideal of $\frac{\mathbb{F}_2[x]}{\langle x^r-1\rangle}$ then $J = \langle k(x)\rangle$ for some $k(x) \mid \left(x^r-1\right)$ in $\mathbb{F}_2[x]$. Let $(a(x),0) \in Ker(\sigma)$ and $a(x) \in J = \langle k(x)\rangle$. So $a(x) = a_1(x)k(x)$ for some $a_1(x) \in \frac{\mathbb{F}_2[x]}{\langle x^r-1\rangle}$, which implies that

$$(a(x),0) = a_1(x)(k(x),0).$$

Therefore, $Ker(\sigma)$ is a left $\mathbb{F}_4[x]$-submodule of $C \subseteq \mathcal{R}_{r,s}$ generated by $(k(x),0)$. By the first isomorphism theorem, we have

$$\frac{C}{Ker(\sigma)} \cong \mathrm{Im}(\sigma) = \langle g(x)\rangle.$$

Since $\sigma((l(x),g(x))) = g(x)$ for some $(l(x),g(x)) \in C$ then

$$\sigma(\langle(l(x),g(x))\rangle) = \langle g(x)\rangle.$$

Consequently, we have $C = \langle(k(x),0),(l(x),g(x))\rangle$. □

Lemma 1 *Let C be a left $\mathbb{F}_4[x]$-submodule of $\mathcal{R}_{r,s}$ generated by $(k(x),0)$ and $(l(x),g(x))$ as given in Theorem 2. Then, we have*

1. *We may assume that* $\deg(l(x)) < \deg(k(x))$.
2. $k(x) \mid \phi\left(\frac{x^s-1}{g(x)}\right) l(x)$ *in* $\mathbb{F}_2[x]$.

Proof 1. If $\deg(k(x)) \leq \deg(l(x))$, then, by division algorithm for polynomials, $l(x) = k(x)l_1(x) + k_1(x)$, where $\deg(k_1(x)) \leq \deg(k(x))$ and

$$\begin{aligned}(l(x),g(x)) - l_1(x)(k(x),0) &= (l(x)-k(x)l_1(x),g(x))\\ &= (k_1(x),g(x)) \in C,\end{aligned}$$

which implies that $C = \langle(k(x),0),(k_1(x),g(x))\rangle$. Hence, we may assume that $\deg(l(x)) < \deg(k(x))$.

2. See that $\frac{x^s-1}{g(x)}(l(x),g(x)) = \left(\phi\left(\frac{x^s-1}{g(x)}\right)l(x),0\right)$. Since $\sigma\left(\left(\phi\left(\frac{x^s-1}{g(x)}\right)l(x),0\right)\right) = 0$, $\left(\phi\left(\frac{x^s-1}{g(x)}\right)l(x),0\right) \in Ker(\sigma)$ and $\phi\left(\frac{x^s-1}{g(x)}\right)l(x) \in J = \langle k(x)\rangle$, then $k(x) \mid \phi\left(\frac{x^s-1}{g(x)}\right)l(x)$ in $\mathbb{F}_2[x]$. □

Lemma 2 *Let C be a left $\mathbb{F}_4[x]$-submodule of $\mathcal{R}_{r,s}$ generated by $(k(x),0)$ and $(l(x),g(x))$ as given in Theorem 2 and $\deg(l(x)) < \deg(k(x))$. Then, the generators of C are unique.*

Proof Since $\sigma(C) = \langle g(x)\rangle$ is a left ideal of $\frac{\mathbb{F}_4[x;\theta]}{\langle x^s-1\rangle}$ and $J = \langle k(x)\rangle$ is an ideal of $\frac{\mathbb{F}_2[x]}{\langle x^r-1\rangle}$, the polynomials $g(x)$ and $k(x)$ are unique. Now, suppose that

$$C = \langle (k(x),0),(l(x),g(x))\rangle = \langle (k(x),0),(l_1(x),g(x))\rangle.$$

From Lemma 1, we may assume that $\deg(l_1(x)) < \deg(k(x))$. So, we have

$$(l(x),g(x)) - (l_1(x),g(x)) = (l(x)-l_1(x),0) \in Ker(\sigma)$$

and $k(x) \mid (l(x)-l_1(x))$, which contradicts with the assumption $\deg(l(x)), \deg(l_1(x)) < \deg(k(x))$. □

We summarize the algebraic structure of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes in the following theorem.

Theorem 3 *An $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code C of length $r+s$ can be identified as:*

1. $C = 0, \mathcal{R}_{r,s}$.
2. $C = \langle (k(x),0)\rangle$, *where* $k(x) \in \mathbb{F}_2[x]$ *and* $0 \le \deg(k(x)) \le r-1$.
3. $C = \langle (l(x),g(x))\rangle$, *where* $l(x) \in \mathbb{F}_2[x]$ *and* $g(x) \in \mathbb{F}_4[x;\theta]$ *with* $g(x) \mid_r (x^s-1)$ *in* $\mathbb{F}_4[x;\theta]$.
4. $C = \langle (k(x),0),(l(x),g(x))\rangle$, *where* $k(x), l(x) \in \mathbb{F}_2[x]$, $k(x) \mid (x^r-1)$ *in* $\mathbb{F}_2[x]$, $\deg(l(x)) < \deg(k(x))$, $g(x) \in \mathbb{F}_4[x;\theta]$ *with* $g(x) \mid_r (x^s-1)$ *in* $\mathbb{F}_4[x;\theta]$ *and* $k(x) \mid \phi\left(\frac{x^s-1}{g(x)}\right) l(x)$ *in* $\mathbb{F}_2[x]$.

Theorem 4 *Let C be an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r+s$ generated by*

$$C = \langle (k(x),0),(l(x),g(x))\rangle$$

with the polynomials $k(x), l(x)$ and, $g(x)$ are as in Theorem 2. We define the sets

$$S_1 = \bigcup_{i=0}^{r-\deg(k(x))-1} \left\{x^i (k(x),0)\right\},$$

$$S_2 = \bigcup_{i=0}^{s-\deg(g(x))-1} \left\{x^i (l(x),g(x))\right\}.$$

Then, $|C| = 2^{r-\deg(k(x))} \cdot 4^{s-\deg(g(x))}$ and $S = S_1 \cup S_2$ forms a minimal spanning set for C.

Proof For any $\mathbf{c}(x) \in C$, there exist polynomials $\lambda_1(x), \lambda_2(x) \in \mathbb{F}_4[x]$ such that

$$\mathbf{c}(x) = \lambda_1(x)(k(x),0) + \lambda_2(x)(l(x),g(x)).$$

$\lambda_1(x)(k(x),0) \in \langle S_1\rangle$ if $\deg(\lambda_1(x)) < r - \deg(k(x))$. In the case $\deg(\lambda_1(x)) \ge r - \deg(k(x))$, by division algorithm for polynomials, $\lambda_1(x) = k_1(x)\left(\frac{x^r-1}{k(x)}\right) + r_1(x)$ where $r_1(x) = 0$ or $\deg(r_1(x)) < r - \deg(k(x))$. Then, we have

$$\begin{aligned}\lambda_1(x)(k(x),0) &= \left(k_1(x)\left(\frac{x^r-1}{k(x)}\right) + r_1(x)\right)(k(x),0)\\ &= r_1(x)(k(x),0) \in \langle S_1\rangle. \end{aligned} \tag{1}$$

$\lambda_2(x)(l(x),g(x)) \in \langle S_2\rangle$ if $\deg(\lambda_1(x)) < s - \deg(g(x))$. In the case $\deg(\lambda_2(x)) \ge s - \deg(g(x))$, by the division algorithm, $\lambda_2(x) = g_1(x)\left(\frac{x^s-1}{g(x)}\right) + r_2(x)$ where $r_2(x) = 0$ or $\deg(r_2(x)) < s - \deg(g(x))$. Then, we have

$$\begin{aligned}\lambda_2(x)(l(x),g(x)) &= \left(g_1(x)\left(\frac{x^s-1}{g(x)}\right) + r_2(x)\right)(l(x),g(x))\\ &= g_1(x)\left(\frac{x^s-1}{g(x)}\right)(l(x),g(x)) + r_2(x)(l(x),g(x))\\ &= g_1(x)\left(\frac{x^s-1}{g(x)}\right)(l(x),0) + r_2(x)(l(x),g(x)).\end{aligned}$$

See that $r_2(x)(l(x), g(x)) \in \langle S_2 \rangle$ and $\phi\left(g_1(x)\left(\frac{x^s-1}{g(x)}\right)\right) l(x) \in J = \langle k(x) \rangle$. Then, $\phi\left(g_1(x)\left(\frac{x^s-1}{g(x)}\right)\right) l(x) = k_2(x)k(x)$ for some $k_2(x) \in \mathbb{F}_2[x]$ where $\deg(r_1(x)) < r - \deg(k(x))$, and so

$$g_1(x)\left(\frac{x^s-1}{g(x)}\right)(l(x), 0) = \left(\phi\left(g_1(x)\left(\frac{x^s-1}{g(x)}\right)\right) l(x), 0\right) = k_2(x)(k(x), 0) \in \langle S_2 \rangle \quad (2)$$

Combining Eq. (1) and Eq. (2), we have $\lambda_2(x)(l(x), g(x)) \in \langle S_1, S_2 \rangle$ which means that $S_1 \cup S_2$ is a minimal spanning set for C.

It follows from $\frac{C}{Ker(\sigma)} \cong \mathrm{Im}(\sigma) = \langle g(x) \rangle$ that $|C| = |Ker(\sigma)| \cdot |\sigma(C)| = 2^{r-\deg(k(x))} \cdot 4^{s-\deg(g(x))}$. □

Example 1 Consider the skew-cyclic code $C = \langle (k(x), 0), (l(x), g(x)) \rangle$ in $\mathcal{R}_{7,4} = \frac{\mathbb{F}_2[x]}{\langle x^7-1 \rangle} \times \frac{\mathbb{F}_4[x;\theta]}{\langle x^4-1 \rangle}$ where

$$k(x) = (1+x)(1+x+x^3) = 1+x^2+x^3+x^4,\ l(x) = 1+x+x^3,$$
$$g(x) = \omega^2 + \omega^2 x + x^2.$$

Therefore, C has a generator matrix of the form

$$\begin{bmatrix} 1&0&1&1&1&0&0 & 0 & 0 & 0&0 \\ 0&1&0&1&1&1&0 & 0 & 0 & 0&0 \\ 0&0&1&0&1&1&1 & 0 & 0 & 0&0 \\ 1&1&0&1&0&0&0 & 1+\omega & 1+\omega & 1&0 \\ 0&1&1&0&1&0&0 & 0 & \omega & \omega&1 \end{bmatrix}$$

and moreover, the binary image of C is a $[15, 5, 4]$ binary linear code.

4 Generator matrices of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes

In this section, we determine the standard form of the generator matrices of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes. Let C be an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r+s$ and G be a generator matrix of C. The rows of G are formed by a set of basis elements (minimal spanning set) of the linear subspace C. The code C is constituted by the all linear combinations of the rows of G. The standard form of the matrix G is a special form of the matrix which is obtained by elementary row operations.

Definition 4 Let C be an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code and r (resp. s) be the set of $\mathbb{F}_2$ (resp. $\mathbb{F}_4$) coordinate positions. Denote C_r (resp. C_s) the punctured code of C by deleting the coordinates r (resp. s).

Proposition 1 *Suppose $C = \langle (k(x), 0), (l(x), g(x)) \rangle$ is an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r+s$. Then C is permutation equivalent to an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code with the following standard form generator matrix*

$$G_S = \left[\begin{array}{ccc|ccc} I_{r-\deg(k(x))} & A_1 & A_2 & 0 & 0 & 0 \\ 0 & B_\eta & B_1 & H_1 & I_\eta & 0 \\ 0 & 0 & 0 & M_1 & M_2 & I_{s-\deg(g(x))-\eta} \end{array}\right] \quad (\star)$$

where $\eta = \deg(k(x)) - \deg(\gcd(k(x), l(x))$.

Proof The code C is generated by the matrix whose rows are the elements of the set $S_1 \cup S_2$ (by Theorem 4). Therefore, the dimensions of the matrices generated by the θ-shifts of $(k(x), 0)$ and $(l(x), g(x))$ are $r - \deg(k(x))$ and $s - \deg(g(x))$, respectively. We have that C is permutation equivalent to the following matrix.

$$\left[\begin{array}{cc|cc} I_{r-\deg(k(x))} & A & 0 & 0 \\ 0 & B & H & I_{s-\deg(g(x))} \end{array}\right]$$

It is obvious that C_r is a skew cyclic code generated by $\gcd(k(x), l(x))$. Hence, the rank of submatrix B is $\eta = \deg(k(x)) - \deg(\gcd(k(x), l(x)))$. Also, the generating matrix of C_r is permutation equivalent to the matrix

$$\begin{bmatrix} I_{r-\deg(k(x))} & A_1 & A_2 \\ 0 & B_\eta & B_1 \\ 0 & 0 & 0 \end{bmatrix},$$

where B_η is a square matrix of full rank. By applying the necessary row and column operations, we can get the standard form generator matrix in ($\star$). □

Example 2 Let $C = \langle (x^2 - 1, 0), (x - 1, (x^4 - 1)(x - \omega)) \rangle$ be an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code in $\mathcal{R}_{4,8} = \frac{\mathbb{F}_2[x]}{\langle x^4-1\rangle} \times \frac{\mathbb{F}_4[x;\theta]}{\langle x^8-1\rangle}$. Then, C has a generator matrix of the form

$$G = \left[\begin{array}{cccc|cccccccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \omega & 1 & 0 & 0 & \omega & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & \omega^2 & 1 & 0 & 0 & \omega^2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \omega & 1 & 0 & 0 & \omega & 1 \end{array}\right].$$

By using elementary row operations, we can put this matrix in the following standard form

$$G_S = \left[\begin{array}{cccc|cccccccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \omega & 1 & 0 & 0 & \omega & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & \omega & 1 & 0 & \omega & \omega & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right].$$

5 Duality of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes

In this section, we first determine the dual of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes of length $r + s$. Then, we provide some examples of optimal $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes.

5.1 Dual of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes of length $r + s$

This subsection is devoted to discussing the structural properties of the dual codes of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes.

Assume that

$$x = (x_0, x_1, \ldots, x_{r-2}, x_{r-1}, x'_0, x'_1, \ldots, x'_{s-2}, x'_{s-1})$$

and

$$y = (y_0, y_1, \ldots, y_{r-2}, y_{r-1}, y'_0, y'_1, \ldots, y'_{s-2}, y'_{s-1})$$

are elements of $\mathbb{F}_2^r \times \mathbb{F}_4^s$. The inner product is defined as

$$x \cdot y = \left(\omega \sum_{i=0}^{r-1} x_i y_i + \sum_{j=0}^{s-1} x'_j y'_j \right) \in \mathbb{F}_4.$$

Define the dual code of the $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code C as

$$C^{\perp} = \{x \in \mathbb{F}_2^r \times \mathbb{F}_4^s : \ x \cdot y = 0, \ \text{ for all } y \in C\}.$$

We have the following result whose proof is evident.

Proposition 2 *The dual of an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r+s$ is an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r+s$.*

Definition 5 (Boucher and Ulmer 2011, Definition 3) Let $f(x) = a_0 + a_1 x + \cdots + a_t x^t \in \mathbb{F}_4[x;\theta]$, where $a_t \neq 0$. Then the polynomial $f^*(x) = a_t + \theta(a_{t-1})x + \cdots + \theta^t(a_0)x^t$ is called the *reciprocal polynomial* of $f(x)$. Equivalently, $f^*(x)$ can be expressed by $f^*(x) = \sum_{i=0}^{t} \theta^i(a_{t-i})x^i$.

Lemma 3 *(Chaussade et al. 2009, Lemma 5) Let ψ be the map*

$$\psi : \mathbb{F}_4[x;\theta] \longrightarrow \mathbb{F}_4[x;\theta],$$
$$\sum_{i=0}^{n} a_i x^i \longmapsto \sum_{i=0}^{n} \theta(a_i)x^i,$$

where $a_i \in \mathbb{F}_4$. Then ψ is a ring homomorphism.

Lemma 4 *(Boucher and Ulmer 2011, Lemma 1) Let $f(x), g(x)$ be skew polynomials in $\mathbb{F}_4[x;\theta]$.*
(1) If $\deg(f) \geq \deg(g)$, then $(f(x)+g(x))^* = f^*(x) + x^{\deg f - \deg g} g^*(x)$.
(2) $(fg)^* = \psi^{\deg f}(g^*)f^*$.
(3) $(f^*)^* = \psi^n(f)$, where $\deg(f) = n$.

We denote the polynomial $\sum_{j=0}^{v-1} x^j$ by $\Gamma_v(x)$. As the unit element 1 remains fixed under automorphism θ, we have the following lemma whose proof is evident.

Lemma 5 *Let $n, n' \in \mathbb{N}$. Then $x^{nn'} - 1 = (x^n - 1)\Gamma_{n'}(x^n) = \Gamma_{n'}(x^n)(x^n - 1)$.*

Definition 6 Let $\mathbf{v}(x) = (v(x), v'(x))$ and $\mathbf{w}(x) = (w(x), w'(x))$ be any two elements in $\mathcal{R}_{r,s}$ and $m = \operatorname{lcm}(r,s)$. Define the map

$$\circ : \mathcal{R}_{r,s} \times \mathcal{R}_{r,s} \longrightarrow \mathcal{R}_m$$

such that

$$\circ(\mathbf{v}(x), \mathbf{w}(x)) = v(x)w^*(x)x^{m-1-\deg(w(x))}\Gamma_{\frac{m}{r}}(x^r) + v'(x)\psi^{m-\deg(w'(x))}(w'^*(x))x^{m-1-\deg(w'(x))}\Gamma_{\frac{m}{s}}(x^s).$$

The map $\circ$ is bilinear between left $\mathbb{F}_4[x;\theta]$-modules. We denote $\circ(\mathbf{v}(x), \mathbf{w}(x))$ by $\mathbf{v}(x) \circ \mathbf{w}(x)$.

Proposition 3 *Let $\mathbf{v}$ and $\mathbf{w}$ be elements in $\mathbb{F}_2^r \times \mathbb{F}_4^s$ with associated polynomials $\mathbf{v}(x) = (v(x), v'(x))$ and $\mathbf{w}(x) = (w(x), w'(x))$, respectively. Then, $\mathbf{w}$ is orthogonal to $\mathbf{v}$ and all its θ-shifts if and only if*

$$\mathbf{v}(x) \circ \mathbf{w}(x) = 0.$$

Proof Suppose that

$$\mathbf{v} = (v_0, v_1, \ldots, v_{r-1}, v'_0, v'_1, \ldots, v'_{s-1})$$

and

$$\mathbf{w} = (w_0, w_1, \ldots, w_{r-1}, w'_0, w'_1, \ldots, w'_{s-1}).$$

Let

$$\mathbf{v}^{(i)} = (v_{0-i}, v_{1-i}, \ldots, v_{r-1-i}, \theta^i(v'_{0-i}), \theta^i(v'_{1-i}), \ldots, \theta^i(v'_{s-1-i}))$$

be the i-th cyclic θ-shift of $\mathbf{v}$ such that $0 \le i \le m-1$. For $0 \le j_1 \le r-1$ and $0 \le j_2 \le s-1$, indices $j_1 - i$ and $j_2 - i$ are computed modulo r and s, respectively. Hence

$$\mathbf{v}^{(i)}.\mathbf{w} = 0 \quad \text{if and only if} \quad \sum_{j=0}^{r-1} v_{j-i} w_j + \sum_{v=0}^{s-1} \theta^i(v'_{v-i}) w'_v = 0.$$

Let $\mathcal{S}_i = \sum_{j=0}^{r-1} v_{j-i} w_j + \sum_{v=0}^{s-1} \theta^i(v'_{v-i}) w'_v$. We can get

$$\begin{aligned}
\mathbf{v}(x) \circ \mathbf{w}(x) &= \left[\sum_{\mu=0}^{r-1} \sum_{j=\mu}^{r-1} v_{j-\mu} w_j x^{m-1-\mu} + \sum_{\mu=1}^{r-1} \sum_{j=\mu}^{r-1} v_j w_{j-\mu} x^{m-1+\mu} \right] \Gamma_{\frac{m}{r}}(x^r) \\
&\quad + \left[\sum_{\tau=0}^{s-1} \sum_{v=\tau}^{s-1} v'_{v-\tau} \theta^{m-\tau}(w'_v) x^{m-1-\tau} + \sum_{\tau=1}^{s-1} \sum_{v=\tau}^{s-1} v'_\tau \theta^\tau(w'_{v-\tau}) x^{m-1+\tau} \right] \Gamma_{\frac{m}{s}}(x^s) \\
&= \sum_{i=0}^{m-1} \theta^{m-i}(\mathcal{S}_i) x^{m-1-i}.
\end{aligned}$$

So, $\mathbf{v}(x) \circ \mathbf{w}(x) = 0$ if and only if $\mathcal{S}_i = 0$ for $0 \le i \le m-1$.

□

Lemma 6 *Let $\mathbf{v}(x) = (v(x), v'(x))$ and $\mathbf{w}(x) = (w(x), w'(x))$ be elements in $\mathcal{R}_{r,s}$ such that $\mathbf{v}(x) \circ \mathbf{w}(x) = 0$. If $v'(x) = 0$ or $w'(x) = 0$, then*

$$v(x) w^*(x) \equiv 0 \pmod{(x^r - 1)}.$$

Respectively, if $v(x) = 0$ or $w(x) = 0$, then

$$v'(x) \psi^{m - \deg(w'(x))}(w'^*(x)) \equiv 0 \pmod{(x^s - 1)}.$$

Remark 1 Let C be an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code with the standard form generator matrix as in ($\star$). The values of $|(C^\perp)_r|$ and $|(C^\perp)_s|$ can be obtained from the projection on the first r and on the last s coordinates of following parity check matrix of C

$$H = \left[\begin{array}{ccc|ccc} A_1^t & I_\eta & 0 & 0 & B_\eta^t & B_\eta^t M_2^t \\ A_2^t & 0 & I_{\deg(k(x))-\eta} & 0 & B_1^t & B_1^t M_2^t \\ 0 & 0 & 0 & I_{\deg(g(x))} & H_1^t & M_1^t + H_1^t M_2^t \end{array}\right].$$

Lemma 7 *If $C = \langle (k(x), 0), (l(x), g(x)) \rangle$ is an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r + s$. Then*

$$\begin{aligned}
|C_r| &= 2^{r - \deg(k(x)) + \eta}, & |C_s| &= 4^{s - \deg(g(x))}, \\
|(C_r)^\perp| &= 2^{\deg(k(x)) - \eta}, & |(C_s)^\perp| &= 4^{\deg(g(x))}, \\
|(C^\perp)_r| &= 2^{\deg(k(x))}, & |(C^\perp)_s| &= 4^{\deg(g(x)) + \eta}.
\end{aligned}$$

Corollary 1 *Let $C = \langle (k(x), 0), (l(x), g(x)) \rangle$ be an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r + s$ with dual code $C^\perp = \langle (\bar{k}(x), 0), (\bar{l}(x), \bar{g}(x)) \rangle$. Then*

$$\begin{aligned}
\deg(\bar{k}(x)) &= r - \deg(\gcd(k(x), l(x))). \\
\deg(\bar{g}(x)) &= s - \deg(g(x)) - \deg(k(x)) + \deg(\gcd(k(x), l(x))).
\end{aligned}$$

Theorem 5 *If $C = \langle (k(x), 0), (l(x), g(x))\rangle$ is an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code of length $r + s$. Then $C^{\perp} = \langle (\bar{k}(x), 0), (\bar{l}(x), \bar{g}(x))\rangle$, such that*

(1) $\bar{k}(x) = \frac{x^r-1}{\gcd(k^*(x),l^*(x))}$.

(2) $\bar{l}(x) = \frac{\nu(x)(x^r-1)}{k^*(x)}$, *for some* $\nu(x) \in \mathbb{F}_2[x]$.

(3) $\overline{g}(x) = \frac{x^s-1}{\psi^{m-\deg(\frac{\mathrm{lcm}(k(x),l(x))g(x)}{l(x)})}((\frac{\mathrm{lcm}(k(x),l(x))g(x)}{l(x)})^*)}$.

Proof (1) Since $(\bar{k}(x), 0)$ is an element in $C^{\perp}$, we have

$$(\bar{k}(x), 0) \circ (k(x), 0) \equiv 0 \pmod{(x^m - 1)} \text{ and } (\bar{k}(x), 0) \circ (l(x), g(x)) \equiv 0 \pmod{(x^m - 1)}.$$

Using Lemma 6,

$$\bar{k}(x)k^*(x) \equiv 0 \pmod{(x^r - 1)},$$

and

$$\bar{k}(x)l^*(x) \equiv 0 \pmod{(x^r - 1)}.$$

Therefore

$$\bar{k}(x)\gcd(k^*(x), l^*(x)) \equiv 0 \pmod{(x^r - 1)}.$$

So, there exists $h(x) \in \mathbb{F}_2[x]$ such that

$$\bar{k}(x)\gcd(k^*(x), l^*(x)) = h(x)(x^r - 1).$$

Since $\deg(\bar{k}(x)) = r - \deg(\gcd(k(x), l(x)))$, thus $h(x) = 1$ and

$$\bar{k}(x) = \frac{x^r - 1}{\gcd(k^*(x), l^*(x))}.$$

(2) Assume that $c'(x) = (\bar{k}(x), 0) + (\bar{l}(x), \bar{g}(x))$. Hence, $c'(x) \in C^{\perp}$. Since the map $\circ$ is bilinear between left $\mathbb{F}_4[x; \theta]$-module, it follows that

$$c'(x) \circ (k(x), 0) = (\bar{k}(x), 0) \circ (k(x), 0) + (\bar{l}(x), \overline{g}(x)) \circ (k(x), 0) = 0.$$

Therefore, $\bar{l}(x)k^*(x) \equiv 0 \pmod{(x^r - 1)}$. We have,

$$\bar{l}(x)k^*(x) = \nu(x)(x^r - 1),$$

for some skew polynomial $\nu(x) \in \mathbb{F}_2[x]$. Then

$$\bar{l}(x) = \frac{\nu(x)(x^r - 1)}{k^*(x)}.$$

(3) Let

$$\begin{aligned} c(x) &= \frac{\mathrm{lcm}(k(x), l(x))}{l(x)}.(l(x), g(x)) - \frac{\mathrm{lcm}(k(x), l(x))}{k(x)}.(k(x), 0) \\ &= (0, \frac{\mathrm{lcm}(k(x), l(x))g(x)}{l(x)}). \end{aligned}$$

Thus $c(x) \in C$, which implies that

$$(\bar{l}(x), \overline{g}(x)) \circ (0, \frac{\mathrm{lcm}(k(x), l(x))g(x)}{l(x)}) \equiv 0 \pmod{(x^m - 1)}.$$

Using Lemma 6, there exists $\gamma(x) \in \mathbb{F}_4[x; \theta]$ such that

$$\overline{g}(x)\psi^{m-\deg\left(\frac{\mathrm{lcm}(k(x),l(x))g(x)}{l(x)}\right)}\left(\left(\frac{\mathrm{lcm}(k(x), l(x))g(x)}{l(x)}\right)^*\right) = \gamma(x)(x^s - 1).$$

By Corollary 1,

$$\deg(\overline{g}(x)) = s - \deg(g(x)) - \deg(k(x)) + \deg(\gcd(k(x), l(x)))$$

$$= s - \deg(g(x)) + \deg(l(x)) - \deg(\mathrm{lcm}(k(x), l(x))).$$

So

$$\begin{aligned}\deg(x^s - 1) &= s \\ &= \deg\left(\overline{g}(x)\psi^{m-\deg\left(\frac{\mathrm{lcm}(k(x),l(x))g(x)}{l(x)}\right)}\left(\left(\frac{\mathrm{lcm}(k(x), l(x))g(x)}{l(x)}\right)^*\right)\right) \\ &= \deg(\gamma(x)(x^s - 1)).\end{aligned}$$

Hence $\gamma(x) = 1$ and

$$\overline{g}(x) = \frac{x^s - 1}{\psi^{m-\deg\left(\frac{\mathrm{lcm}(k(x),l(x))g(x)}{l(x)}\right)}\left(\left(\frac{\mathrm{lcm}(k(x),l(x))g(x)}{l(x)}\right)^*\right)}.$$

□

As a special case, when θ is the identity automorphism, by Theorem 5 we have:

Remark 2 If $C = \langle(k(x), 0), (l(x), g(x))\rangle$ is an $\mathbb{F}_2\mathbb{F}_4$-cyclic code of length $r + s$. Then $C^{\perp} = \langle(\bar{k}(x), 0), (\bar{l}(x), \bar{g}(x))\rangle$, where

(1) $\bar{k}(x) = \frac{x^r-1}{\gcd(k^*(x),l^*(x))}$,

(2) $\bar{l}(x) = \frac{\nu(x)(x^r-1)}{k^*(x)}$, for some $\nu(x) \in \mathbb{F}_2[x]$ and

(3) Since $\mathrm{lcm}(k(x), l(x)).\gcd(k(x), l(x)) = k(x)l(x)$ in $\mathbb{F}_2[x]$,

$$\overline{g}(x) = \frac{(x^s - 1)\gcd(k^*(x), l^*(x))}{k^*(x)g^*(x)}.$$

5.2 Examples

In this subsection, we provide some examples of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes of length $r + s$ and we specify their parameters. We also give some examples of optimal parameter binary linear codes which are derived from $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes. From Wei (1991), recall that the support $\chi(C)$ of a code C is the set

$$\chi(C) = \left\{i : c_i \neq 0, \exists\left(c_0, c_1, \ldots, c_{n-1}\right) \in C\right\}$$

and the ith generalized Hamming weight $d_i(C)$ of C is

$$d_i(C) = \min\{|\chi(D)| : D \text{ is a subcode of } C \text{ with rank } i\}.$$

Definition 7 (Dodunekov and Landjev 2000, Definition 2.1) A linear $[n, k]$ code C is said to be near-MDS if

1. $d_1(C) = n - k$ and
2. $d_i(C) = n - k + i$ for $i = 2, 3, \ldots, k$,

where $d_i(C)$ is the ith generalized Hamming weight of C.

Example 3 Let $C = \langle((x-1)^3, (x-1)^5(x-\omega))\rangle$ be an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code in $\mathcal{R}_{4,8} = \frac{\mathbb{F}_2[x]}{\langle x^4-1\rangle} \times \frac{\mathbb{F}_4[x;\theta]}{\langle x^8-1\rangle}$. Hence, a generator matrix of C is

$$G = \left[\begin{array}{cccc|cccccccc} 1 & 1 & 1 & 1 & \omega & \omega & 1 & 0 & \omega & \omega & 1 & 0 \\ & & & & & & & & & & & \\ 1 & 1 & 1 & 1 & 0 & \omega^2 & \omega^2 & 1 & 0 & \omega^2 & \omega^2 & 1 \end{array}\right].$$

So, $d_1(C) = n - k = 10$ and $d_2(C) = n - k + 2 = 12$. Then, C is a near-MDS code.

Example 4 Consider the following $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes in $\mathcal{R}_{4,4} = \frac{\mathbb{F}_2[x]}{\langle x^4-1\rangle} \times \frac{\mathbb{F}_4[x;\theta]}{\langle x^4-1\rangle}$.

• If $C = \langle(1 + x + x^3, x - \omega^2)\rangle$, then

$$G = \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 1 & \omega^2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \omega & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & \omega^2 & 1 \end{array}\right]$$

is a generator matrix of C. We have, $d_1(C) = n-k = 5, d_2(C) = n-k+2 = 7$ and $d_3(C) = n-k+3 = 8$. Therefore, C is a near-MDS code.

• If $C = \langle (x-1, x-1) \rangle$, C has a generator matrix of the form

$$G = \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right],$$

and $\Psi(C)$ has the following generator matrix

$$\left[\begin{array}{cccc|cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{array}\right].$$

Thus, $\Psi(C)$ is an optimal code with parameters $[12, 6, 4]_2$ based on Grassl (xxx).

• If $C = \langle (x-1, 0), (0, x-1) \rangle$, then

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right],$$

and

$$\left[\begin{array}{cccc|cccccccc}
1&1&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&1&1&0&0&1&1&0&0\\
0&0&0&0&0&1&1&0&0&1&1&0\\
0&0&0&0&0&0&1&1&0&0&1&1
\end{array}\right],$$

are generator matrices of C and $\Psi(C)$, respectively. Hence, $\Psi(C)$ is an optimal code with parameters $[12, 9, 2]_2$ based on Grassl (xxx).

Example 5 Let $C = \langle (x-1, 0), (1, x-\omega^2)\rangle$ be an $\mathbb{F}_2\mathbb{F}_4$-skew cyclic code in $\mathcal{R}_{4,2} = \frac{\mathbb{F}_2[x]}{\langle x^4-1\rangle} \times \frac{\mathbb{F}_4[x;\theta]}{\langle x^2-1\rangle}$. Then, generator matrices of C and $\Psi(C)$ are

$$\left[\begin{array}{cccc|cc}
1&1&0&0&0&0\\
0&1&1&0&0&0\\
0&0&1&1&0&0\\
1&0&0&0&\omega^2&1
\end{array}\right],$$

and

$$\left[\begin{array}{cccc|cccc}
1&1&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0\\
0&0&1&1&0&0&0&0\\
1&0&0&0&1&0&0&1\\
0&0&0&0&0&1&1&1
\end{array}\right],$$

respectively. Thus, $\Psi(C)$ is an optimal code with parameters $[8, 5, 2]_2$ based on Grassl (xxx).

6 Conclusion

In this paper, we introduced $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes as a generalization of skew cyclic codes over $\mathbb{F}_4$. We gave generator polynomials of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes and classified them. We also gave minimal spanning sets and generator matrices for these codes. Furthermore, we defined a certain inner product over $\mathbb{F}_2^r \times \mathbb{F}_4^s$ and investigated the duals of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes with respect to this inner product. We determined the generator polynomials of the duals of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes. Finally, we presented several examples of linear codes over $\mathbb{F}_4$ that are near-MDS and optimal binary linear codes which are Gray images of $\mathbb{F}_2\mathbb{F}_4$-skew cyclic codes.

Funding Open access funding provided by the Scientific and Technological Research Council of Türkiye (TÜBİTAK).

Data availability No data were generated or analyzed during this study.

Declarations

Conflict of interest The authors declare that they have no conflict interest regarding the publication of this paper.

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