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Editör: Doç.Dr. Gülistan KAYA GÖK
Doç.Dr. Meral SÜER



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Güncel Konular ve Akademik Değerlendirmeler

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"Bu kitapta yer alan bölümlerde kullanılan kaynakların, görüşlerin, bulguların, sonuçların, tablo, şekil, resim ve her türlü içeriğin sorumluluğu yazar veya yazarlarına ait olup ulusal ve uluslararası telif haklarına konu olabilecek mali ve hukuki sorumluluk da yazarlara aittir."

A NEW QUANTUM SPACE AND ITS HOPF ALGEBRA STRUCTURE

Sultan ABACI ÇELİK[1]

1. INTRODUCTION

Quantum groups and noncommutative geometry have emerged as important mathematical frameworks for extending classical concepts of symmetry, space, and differential geometry beyond the commutative setting. In classical geometry, a space can be studied through its algebra of coordinate functions, while its symmetries are described by groups acting on these functions. The passage to the quantum setting replaces, in general, the commutative algebra of functions by a noncommutative algebra. Within this algebraic viewpoint, the resulting quantum symmetries are naturally described by Hopf algebras and related quantum groups. This perspective has established a close relationship between noncommutative coordinate algebras, quantum groups, and generalized geometric structures (Drinfeld, 1987; Majid, 1995; Manin, 1988, 2018).

A fundamental idea in this development is to regard a noncommutative algebra as the algebra of coordinate functions on a quantum space. Rather than starting with a classical group and deforming its multiplication or coordinate functions, Manin proposed an approach in which quantum spaces themselves provide the basic objects and quantum groups arise as their symmetry structures. In particular, quadratic algebras can be interpreted as quantum analogues of linear spaces, with the

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classical commutation relations replaced by deformation-dependent quadratic relations (Manin, 1988, 2018). This viewpoint is especially useful for constructing explicit examples of quantum spaces and for determining the quantum groups that preserve their defining relations.

The algebraic structure required to formulate quantum symmetries is provided by Hopf algebras. A Hopf algebra combines an associative algebra structure with a compatible coalgebra structure and an antipode, thereby providing an algebraic analogue of multiplication, identity, and inverse operations in classical groups. This framework is particularly suitable for quantum groups because the coordinate algebras of quantum groups are typically noncommutative and, in general, noncocommutative. The systematic development of Hopf algebras, matrix quantum groups, quantum enveloping algebras, and related constructions has therefore become a central part of quantum group theory (Drinfeld, 1987; Majid, 1995).

The relationship between quantum spaces and their symmetry quantum groups has also been investigated from the perspective of differential structures. Once the coordinate algebra of a space becomes noncommutative, the construction of differential forms cannot generally be obtained by simply imposing the classical commutation rules. Instead, the algebra of differential forms must be deformed consistently with the relations of the underlying coordinate algebra. Woronowicz developed a general theory of differential calculus on quantum groups and introduced the framework of bicovariant differential calculi, in which differential forms and exterior derivatives are compatible with the Hopf algebraic structure (Woronowicz, 1987, 1989).

The relationship between the present construction and the existing three-dimensional quantum-space literature is therefore

twofold. On the one hand, standard quantum Euclidean spaces such as $\mathbb{R}_q^3$ provide important reference models for three-dimensional noncommutative geometry and differential calculus (Cerchiai et al., 2000; Wachter, 2007; Celik, 1998; Celik & Celik, 2000; Celik et al., 2009). On the other hand, the work of Celik and collaborators demonstrates that alternative deformation principles, including supersymmetric, graded, contracted, and nonstandard constructions, can lead to genuinely new Hopf-algebraic quantum spaces (Celik, 2016, 2020, 2021; Celik & Temli, 2019). The proposed $\mathbb{C}_q^3$ is situated within this broader landscape while focusing on an ordinary three-coordinate, ungraded quantum space with a new defining deformation.

In this work, we introduce a new three-dimensional quantum space and investigate its Hopf algebraic and covariant structures together with its symmetry group. The corresponding quantum algebra $O(M_q(3,\mathbb{C}))$ is then investigated through its coactions on both the coordinate algebra and the q-deformed exterior algebra. The entries of the matrix $T = (t_{ij})$ are required to satisfy relations determined by the condition that the coactions preserve the defining relations of the quantum space and its exterior algebra. In this way, the quantum group is obtained as a symmetry object of the proposed quantum space, in accordance with the general philosophy of quantum groups as symmetries of noncommutative spaces (Manin, 1988, 2018).

The remainder of this chapter is organized as follows. Section 2 provides a brief review of the basic notions and properties of algebras, coalgebras, bialgebras, and Hopf algebras that are used in the subsequent construction. Section 3 contains the main results of the chapter. In Section 3.1, we introduce the new three-dimensional quantum space, discuss its quadratic defining relations and classical limit, and construct the associated q-deformed exterior algebra. Section 3.2 is devoted to the

construction and verification of the coproduct, counit, and antipode, establishing the Hopf algebra structure of the proposed quantum space. In Section 4, we introduce a new quantum bialgebra $O(M_q(3,\mathbb{C}))$, determine its coactions on the quantum coordinate and exterior algebras.

2. A BRIEF OVERVIEW OF THE DEFINITION OF A HOPF ALGEBRA

We first recall some basic definitions and properties of Hopf algebras that will be used throughout this work.

Definition 2.1. Let K be a field. An associative unital algebra over K is a K-vector space A equipped with two linear maps $m: A \otimes A \to A$, $\eta: K \to A$, called the multiplication and unit maps, respectively, satisfying

$$m \circ (m \otimes id) = m \circ (id \otimes m),$$

$$m \circ (\eta \otimes id) = m \circ (id \otimes \eta) = id_A.$$

For $a, b \in A$, we write $m(a \otimes b) = a \cdot b$ and $\eta(1) = 1_A$.

Definition 2.2. A coalgebra over K is a K-vector space A equipped with two linear maps $\Delta: A \to A \otimes A$, $\varepsilon: A \to K$, called the coproduct and counit, respectively, satisfying

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta,$$

$$(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id.$$

Definition 2.3. A bialgebra over K is a vector space A that is simultaneously an associative unital algebra and a coalgebra, such that the coproduct and counit are algebra homomorphisms.

Equivalently, the maps $\Delta: A \to A \otimes A$ and $\varepsilon: A \to K$ are morphisms of algebras.

Definition 2.4. A Hopf algebra over K is a bialgebra A together with a linear map $\kappa: A \to A$, called the coinverse or antipode, satisfying

$$m \circ (\kappa \otimes id) \circ \Delta = m \circ (id \otimes \kappa) \circ \Delta = \eta \circ \varepsilon.$$

Thus, a Hopf algebra may be viewed as a bialgebra equipped with an antipode. We denote a Hopf algebra by $(A, m, \eta, \Delta, \varepsilon, \kappa)$.

Theorem 2.5. Let A be a Hopf algebra with antipode κ. Then κ is an algebra antihomomorphism and a coalgebra anti-homomorphism. In particular, for all $a, b \in A$,

$$\kappa(a \cdot b) = \kappa(b) \cdot \kappa(a), \quad \kappa(1_A) = 1_A,$$

$$\Delta\big(\kappa(a)\big) = (\kappa \otimes \kappa) \circ \tau \circ \Delta(a),$$

where $\tau: A \otimes A \to A \otimes A$ is defined by $\tau(a \otimes b) = b \otimes a$.

Moreover, $\varepsilon \circ \kappa = \varepsilon$. If the antipode is bijective, then

$$\kappa^{-1}(ab) = \kappa^{-1}(b)\kappa^{-1}(a).$$

In the theory of quantum groups, the algebra A is generally noncommutative, while its coproduct need not be cocommutative. This noncommutative and noncocommutative structure provides the algebraic framework for describing quantum deformations of classical groups and spaces.

3. A NEW QUANTUM SPACE AND ITS HOPF ALGEBRA STRUCTURE

In this section, we introduce a new three-dimensional quantum space and determine its Hopf algebra structure.

3.1. A New 3d Quantum Space

A quantum space is described algebraically by a noncommutative coordinate algebra generated by quantum coordinates subject to quadratic relations. This viewpoint is central to Manin's approach to quantum groups and noncommutative geometry.

Definition 3.1. Let q be nonzero complex parameter. The three-dimensional quantum space is defined as the quadratic algebra

$$O(\mathbb{C}_q^3) = \mathbb{C}\langle x, y, z\rangle / I_q,$$

where $\mathbb{C}\langle x, y, z\rangle$ denotes the free associative algebra generated by x, y, and z, and I_q is the two-sided ideal generated by the quadratic relations

$$x \cdot y = y \cdot x, \quad x \cdot z = q^2 z.x, \quad y.z = z \cdot y. \tag{3.1}$$

The generators x, y, and z are interpreted as the noncommuting coordinate functions of the quantum space. The parameter q measures the deformation of the commutative coordinate algebra. In the limit $q \to 1$, the algebra $O(\mathbb{C}_q^3)$ becomes the ordinary polynomial algebra $\mathbb{C}[x, y, z]$.

Example 3.2. If we define a representation $\rho : O(\mathbb{C}_q^3) \to M_3(\mathbb{C})$ by

$$\rho(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^2 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} q^2 & 0 & 0 \\ 1-q^2 & 1 & 0 \\ 0 & 0 & q^2 \end{pmatrix},$$

$$\rho(z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1-q^2 & 0 & 0 \end{pmatrix},$$

then we can see that ρ is a representation of the algebra $O(\mathbb{C}_q^3)$, in the sense that the matrices $\rho(\cdot)$ preserve the relations given in (3.1) according to the rule

$$\rho_{ik}(f \cdot g) = \sum_{j=1}^{3} \rho_{ij}(f) \cdot \rho_{jk}(g)\,,$$

for all $f, g \in O(\mathbb{C}_q^3)$.

The notation $\mathbb{C}\langle x, y, z\rangle$ emphasizes that the generators x, y, and z are initially free and noncommuting. The ideal I_q imposes the quadratic relations among these generators. Hence, the quantum space is completely specified by its generators and defining relations. Equivalently, one may write

$$I_q = \langle x \cdot y - y \cdot x,\ x \cdot z - q^2\, z \cdot x,\ y \cdot z - z \cdot y \rangle.$$

The quotient by I_q identifies monomials that differ by the defining commutation relations. For example, $x \cdot y$ and $y \cdot z$ can be replaced by $y \cdot x$ and $z \cdot y$, respectively, while $x \cdot z$ can be replaced by $q^2\, z \cdot x$.

Remark 3.3. Unlike the classical coordinate algebra $\mathbb{C}[x, y, z]$, the algebra $O(\mathbb{C}_q^3)$ is generally noncommutative. For generic values of q, the products $x \cdot z$ and $z \cdot x$ are distinct elements of $O(\mathbb{C}_q^3)$, although they are related by the defining relations. This noncommutativity is the essential algebraic feature of the corresponding quantum space.

Definition 3.4. Let $\Lambda(\mathbb{C}_q^3)$ denote the exterior algebra of the algebra $O(\mathbb{C}_q^3)$. Its generators θ, φ, and ϕ are assumed to satisfy the following relations:

$\Theta_i^2 = 0,$ for $i = 1,2,3,$

$$\theta \cdot \varphi = -q^{-2}\varphi \cdot \theta,\ \ \theta \cdot \phi = -\phi \cdot \theta,\ \ \varphi \cdot \phi = -q^2 \phi \cdot \varphi, \quad (3.2)$$

where $\Theta_1 = \theta, \Theta_2 = \varphi, \Theta_3 = \phi$.

The above relations define a q-deformed exterior algebra associated with $O(\mathbb{C}_q^3)$. In particular, the relations $\Theta_i^2 = 0$, for $i = 1,2,3$ are analogous to the classical relations $dx \wedge dx = 0$, etc., where dx is differential of x, etc. The remaining relations describe the q-deformed anticommutation rules among the generators. Notice that, in the limit $q \to 1$, these relations reduce to the usual anticommutation relations. Thus, $\Lambda(\mathbb{C}_q^3)$ can be regarded as a q-deformation of the classical exterior algebra of $\mathbb{C}_q^3$.

Remark 3.5. The importance of the quantum space in Manin's approach is that quantum groups can be constructed as symmetry objects acting on such noncommutative coordinate algebras. Instead of beginning with a group and deforming its coordinate functions, one may first specify the algebra of a quantum space and then determine the quantum group that preserves its defining relations. This provides an algebraic realization of quantum groups as symmetry objects of noncommutative spaces (see, Section 4).

Remark 3.6. If additional conditions are imposed, particularly the invertibility of a suitable generator, the algebra can be endowed with a bialgebra or Hopf algebra structure. The coproduct, counit, and antipode are then required to be compatible with the defining relations of the quantum space. These structures allow the quantum space to be treated within the framework of quantum groups and provide the basis for studying its covariance and differential geometry.

3.2. Hopf Algebra Structure On $O(\mathbb{C}_q^3)$

We consider the three-dimensional quantum space whose coordinate algebra is generated by x, y, and z subject to the relations below. We then introduce the coproduct, counit, and

antipode and verify that these maps give the algebra a Hopf algebra structure.

Definition 3.7 Define a linear map

$$\Delta: O(\mathbb{C}_q^3) \rightarrow O(\mathbb{C}_q^3) \otimes O(\mathbb{C}_q^3)$$

on the generators by

$$\Delta(x) = x \otimes x,$$

$$\Delta(y) = y \otimes y,$$

$$\Delta(z) = x \otimes z + z \otimes y.$$

The map Δ is extended multiplicatively to the whole algebra.

Theorem 3.8. The coproduct Δ is compatible with the defining relations of $O(\mathbb{C}_q^3)$. Hence, it is a well-defined algebra homomorphism.

Proof. For $x \cdot y = y \cdot x$, one has

$$\Delta(x)\Delta(y) = x \cdot y \otimes x \cdot y = y \cdot x \otimes y \cdot x = \Delta(y)\Delta(x).$$

For $y \cdot z = z \cdot y$, one obtains $\Delta(y)\Delta(z) = \Delta(z)\Delta(y)$.

For $x \cdot z = q^2 z \cdot x$, one obtains

$$\Delta(x)\Delta(z) = q^2\Delta(z)\Delta(x).$$

Therefore, all defining relations are preserved by Δ.

Definition 3.9. Define the counit $\varepsilon: O(\mathbb{C}_q^3) \rightarrow \mathbb{C}$ on the generators by

$$\varepsilon(x) = 1, \quad \varepsilon(y) = 1, \quad \varepsilon(z) = 0,$$

and extend ε as an algebra homomorphism.

Theorem 3.10. The maps Δ and ε satisfy the coassociativity and counit conditions. Therefore, $O(\mathbb{C}_q^3)$ is a coalgebra. In particular,

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta)\,\Delta,$$

$$(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id.$$

Proof. For example, for z,

$$(\Delta \otimes id)\Delta(z) = x \otimes x \otimes z + x \otimes z \otimes y + z \otimes y \otimes y,$$

$$(id \otimes \Delta)\Delta(z) = x \otimes x \otimes z + x \otimes z \otimes y + z \otimes y \otimes y.$$

Definition 3.11. In the Hopf algebra extension in which x and y are invertible, define the antipode κ by

$$\kappa(x) = x^{-1}, \quad \kappa(y) = y^{-1}, \quad \kappa(z) = -x^{-1} \cdot z \cdot y^{-1}.$$

The antipode is extended as an antihomomorphism.

Theorem 3.12. The algebra $O(\mathbb{C}_q^3)$, equipped with the coproduct Δ, counit ε, and antipode κ defined above, forms a Hopf algebra, provided that x and y are invertible. The antipode identities are

$$m \circ (\kappa \otimes id) \circ \Delta = m \circ (id \otimes \kappa) \circ \Delta = \eta \circ \varepsilon.$$

Proof. For x and y, these identities follow immediately from

$$\kappa(x) \cdot x = x \cdot \kappa(x) = 1 \text{ and } \kappa(y) \cdot y = y \cdot \kappa(y) = 1.$$

For z,

$$m(\kappa \otimes id)\Delta(z) = \kappa(x) \cdot z + \kappa(z) \cdot y = 0 = \varepsilon(z)1,$$

and

$$m(id \otimes \kappa)\Delta(z) = x \cdot \kappa(z) + z \cdot \kappa(y) = 0 = \varepsilon(z)1.$$

4. THE QUANTUM ALGEBRA $O(M_q(3,\mathbb{C}))$ AND ITS COACTIONS ON $O(\mathbb{C}_q^3)$ AND $\Lambda(\mathbb{C}_q^3)$

Let $M(3,\mathbb{C})$ denote the vector space of 3x3 matrices with complex entries. Each matrix $T \in M(3,\mathbb{C})$ acts naturally as a

linear transformation on the complex vector space $\mathbb{C}^3$. The classical matrix space $M(3,\mathbb{C})$ can therefore be described in terms of its coordinate functions, which generate a commutative coordinate algebra denoted by $O(M_q(3,\mathbb{C}))$.

A natural question is how this classical structure can be extended to the quantum setting. The basic idea of quantum deformation is to replace the commutative coordinate algebra by a noncommutative algebra whose defining relations depend on a deformation parameter q. In this way, the classical coordinate algebra $O(M(3,\mathbb{C}))$ is deformed into the quantum coordinate algebra $O(M_q(3,\mathbb{C}))$. The noncommutativity introduced by the deformation is governed by q, and the classical algebra is recovered in the undeformed limit $q \to 1$.

4.1. The Algebra $O(M_q(3,\mathbb{C}))$

Matrices with complex entries act as linear transformations on complex vector spaces. In this section, we will define the q-deformation of the algebra $O(M(3,\mathbb{C}))$.

Let $T = \begin{pmatrix} a & b & c \\ d & e & f \\ g & u & v \end{pmatrix} = (t_{ij})$ be a matrix whose entries belong to a free associative algebra, while the components of the vectors $X = (x,y,z)^t$ and $\Theta = (\theta,\varphi,\phi)^t$ belong to the algebras $O(\mathbb{C}_q^3)$ and $\Lambda(\mathbb{C}_q^3)$, respectively. Consider the transformations $X' = T.X$ and $\Theta' = T.\Theta$. We also assume that $q^2 + 1 \neq 0$. Under these assumptions, we have the following theorem, whose proof follows directly.

Theorem 4.1. Suppose that the matrix elements of T commute with the generators of $O(\mathbb{C}_q^3)$ and $\Lambda(\mathbb{C}_q^3)$. Then a necessary and sufficient condition for the components of X' and Θ' to satisfy relations (3.1) and (3.2), respectively, is that the following relations hold:

$$
\begin{aligned}
a\cdot b &= q^2 b\cdot a, \qquad a\cdot e = e\cdot a + (q^2-1)b\cdot d,\\
a\cdot c &= c\cdot a, \qquad a\cdot f = f\cdot a + (1-q^{-2})c\cdot d,\\
a\cdot d &= d\cdot a, \qquad a\cdot u = u\cdot a + (q^2-1)b\cdot g,\\
a\cdot g &= q^2 g\cdot a, \qquad a\cdot v = v\cdot a + (1-q^{-2})c\cdot g,\\
b\cdot c &= q^{-2}c\cdot b, \quad b\cdot d = q^{-2}d\cdot b, \quad b\cdot e = e\cdot b,\\
b\cdot f &= q^{-2}f\cdot b, b\cdot g = g\cdot b, b\cdot u = q^2 u\cdot b, b\cdot v = v\cdot b,\\
c\cdot d &= d\cdot c, \qquad c\cdot e = e\cdot c + (q^2-1)b\cdot f,\\
c\cdot f &= f\cdot c, \qquad c\cdot u = q^2 u\cdot c + (q^2-1)b\cdot v,\\
c\cdot g &= q^2 g\cdot c, \qquad c\cdot v = q^2 v\cdot c, \qquad (4.1)\\
d\cdot e &= q^2 e\cdot d, \quad d\cdot f = f\cdot d, \quad d\cdot g = g\cdot d,\\
d\cdot u &= q^2 u\cdot d, \quad d\cdot v = v\cdot d,\\
e\cdot f &= q^{-2}f\cdot e, \quad e\cdot g = g\cdot e + (q^{-2}-1)d\cdot u,\\
e\cdot u &= u\cdot e, \qquad e\cdot v = v\cdot e + (q^{-2}-1)f\cdot u,\\
f\cdot u &= q^2 u\cdot f, \quad f\cdot g = q^2 g\cdot f + (1-q^2)d\cdot v,\\
f\cdot v &= v\cdot f,\\
g\cdot u &= q^2 u\cdot g, \quad g\cdot v = v\cdot g, \quad u\cdot v = q^{-2}v\cdot u.
\end{aligned}
$$

Definition 4.2. The algebra $O(M_q(3,\mathbb{C}))$ is defined as the quotient of the free algebra $\mathbb{C}[a,b,c,d,e,f,g,u,v]$ by the two-sided ideal J_q generated by the relations (4.1) in Theorem 4.1.

4.2. Bialgebra Structure on $O(M_q(3,\mathbb{C}))$

The algebra $O(M_q(3,\mathbb{C}))$ introduced in the previous section admits a natural bialgebra structure. This structure is determined by the coproduct and counit maps, which are defined on the generators of the algebra and extended multiplicatively.

Definition 4.3. The coproduct

$$\Delta: O\left(M_q(3,\mathbb{C})\right) \rightarrow O\left(M_q(3,\mathbb{C})\right) \otimes O\left(M_q(3,\mathbb{C})\right)$$

is defined on the generators $t_{ij}, 1 \leq i,j \leq 3$, by

$$\Delta(t_{ij}) = \sum_{k=1}^{3} t_{ik} \otimes t_{kj},$$

and the counit $\varepsilon: O(M_q(3,\mathbb{C})) \rightarrow \mathbb{C}$ is given by

$$\varepsilon(t_{ij}) = \delta_{ij}.$$

The compatibility of these maps with the defining relations of $O(M_q(3,\mathbb{C}))$ leads to the following result.

Theorem 4.4. The maps Δ and ε are well defined on the algebra $O(M_q(3,\mathbb{C}))$. Moreover, they satisfy
$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \ (\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id.$

Consequently, $(O(M_q(3,\mathbb{C})), \Delta, \varepsilon)$ is a bialgebra.

Proof. It is sufficient to verify the required identities on the generators t_{ij}. For the coproduct,

$$(\Delta \otimes id)\Delta(t_{ij}) = \sum_{k,l=1}^{3} t_{il} \otimes t_{lk} \otimes t_{kj},$$

whereas

$$(id \otimes \Delta)\Delta(t_{ij}) = \sum_{k,l=1}^{3} t_{ik} \otimes t_{kl} \otimes t_{lj}.$$

After relabelling the summation indices, these expressions coincide. Similarly,

$$(\varepsilon \otimes id)\Delta(t_{ij}) = \sum_{k=1}^{3} \varepsilon(t_{ik}) t_{kj} = t_{ij},$$

and

$$(id \otimes \varepsilon)\Delta(t_{ij}) = \sum_{k=1}^{3} t_{ik} \varepsilon(t_{kj}) = t_{ij}.$$

Thus, the coproduct is coassociative and the counit identities hold. Hence $O(M_q(3,\mathbb{C}))$ is a bialgebra.

4.3. Coactions on $O(\mathbb{C}_q^3)$ and $\Lambda(\mathbb{C}_q^3)$

The bialgebra structure established above allows us to introduce a natural left coaction of $O\left(M_q(3,\mathbb{C})\right)$ on the coordinate algebra of the quantum space $O(\mathbb{C}_q^3)$. Let $x = x_1$, $y = x_2$, $z = x_3$ denote the generators of $O(\mathbb{C}_q^3)$.

Definition 4.5. A left $O(M_q(3,\mathbb{C}))$-coaction on $O(\mathbb{C}_q^3)$ is a linear map $\delta_L: O(\mathbb{C}_q^3) \to O(M_q(3,\mathbb{C}))\otimes O(\mathbb{C}_q^3)$ defined on the generators by

$$\delta_L(x_i) = \sum_{j=1}^{3} t_{ij}\otimes x_j, \quad i = 1,2,3.$$

The compatibility of this map with the defining relations of $O(\mathbb{C}_q^3)$ gives the following result.

Theorem 4.6. The map δ_L is a well-defined left coaction of the bialgebra $O(M_q(3,\mathbb{C}))$ on $O(\mathbb{C}_q^3)$. In particular, $(\Delta\otimes id)\circ\delta_L = (id\otimes\delta_L)\circ\delta_L$ and $(\varepsilon\otimes id)\circ\delta_L = id$.

Proof. For each generator x_i,

$$(\Delta\otimes id)\delta_L(x_i) = \sum_{j=1}^{3}\Delta(t_{ij})\otimes x_j = \sum_{j,k=1}^{3} t_{ik}\otimes t_{kj}\otimes x_j.$$

On the other hand,

$$(id\otimes\delta_L)\delta_L(x_i) = \sum_{j=1}^{3} t_{ij}\otimes\delta_L(x_j) = \sum_{j,k=1}^{3} t_{ij}\otimes t_{jk}\otimes x_k\,.$$

After relabelling the indices, the two expressions agree. Furthermore,

$$(\varepsilon \otimes id)\delta_L(x_i) = \sum_{j=1}^{3} \varepsilon(t_{ij})x_j = x_i.$$

Thus, δ_L satisfies the coaction axioms. The relations defining $O(\mathbb{C}_q^3)$ are preserved under δ_L by the defining relations of the algebra $O(M_q(3,\mathbb{C}))$. Therefore, the map δ_L extends uniquely to an algebra homomorphism δ_L. Hence it defines a left coaction of $O\left(M_q(3,\mathbb{C})\right)$ on $O(\mathbb{C}_q^3)$.

Definition 4.7. The coaction δ is extended to the exterior algebra by

$$\delta' : \Lambda(\mathbb{C}_q^3) \rightarrow O(GL_q(3)) \otimes \Lambda(\mathbb{C}_q^3),$$

where $\delta'(\Theta_i) = \Sigma_{j=1}{}^{3}\, t_{ij} \otimes \Theta_j$.

Theorem 4.8. The algebra generated by the entries t_{ij} of T, subject to the relations obtained from the invariance of the defining relations of $O(\mathbb{C}_q^3)$ and $\Lambda(\mathbb{C}_q^3)$ under the coactions δ and δ', defines the coordinate algebra of $M_q(3,\mathbb{C})$.

More precisely, the relations among the generators t_{ij} are uniquely determined by requiring $\delta(R) = 0$ for every defining relation R of $O(\mathbb{C}_q^3)$, and $\delta'(\kappa) = 0$ for every defining relation κ of $\Lambda(\mathbb{C}_q^3)$. Consequently, $O(M_q(3,\mathbb{C}))$ acts covariantly on both the quantum coordinate algebra $O(\mathbb{C}_q^3)$ and its exterior algebra $\Lambda(\mathbb{C}_q^3)$.

5. CONCLUSION

In this paper, we introduced a new three-dimensional quantum space $O(\mathbb{C}_q^3)$ and investigated its fundamental algebraic and Hopf-algebraic structures. A Hopf algebra structure compatible with the defining relations of $O(\mathbb{C}_q^3)$ was constructed.

The coproduct, counit, and antipode were explicitly defined and their compatibility conditions were verified. A distinctive feature of the resulting Hopf algebra is the non-primitive coproduct of the generator z, $\Delta(z) = x \otimes z + z \otimes y$, which differs from the usual primitive coproduct encountered in classical and some quantum algebraic constructions. This structure provides an explicit example of a three-dimensional quantum space equipped with a nontrivial coalgebraic structure.

We also considered a q-deformed exterior algebra associated with $O(\mathbb{C}_q^3)$. The defining relations among the exterior generators reduce to the classical anticommutation relations in the limit. Furthermore, the quantum algebra $O(M_q(3,\mathbb{C}))$ was introduced through its coactions on both the quantum coordinate algebra and the associated exterior algebra.

The results obtained in this paper provide the algebraic foundation for a more detailed investigation of the differential geometry of the newly introduced quantum space. In particular, a natural continuation of the present work is the construction of a differential calculus on $O(\mathbb{C}_q^3)$. Such a study would involve determining the commutation relations between the generators x, y, z and their differentials, constructing the corresponding exterior algebra systematically, and identifying the associated differential operators and their algebraic properties. The compatibility of the differential calculus with the Hopf algebra structure and the $O(M_q(3,\mathbb{C}))$-coaction would also provide an important aspect of this investigation.

A forthcoming study will therefore focus specifically on the differential calculus on $O(\mathbb{C}_q^3)$. Building on the Hopf algebra and exterior structures established here, that work will investigate the differential relations, exterior derivative, higher-degree differential forms, and covariance properties associated with the proposed quantum space. In particular, the aim is to develop a

consistent noncommutative differential geometry for $O(\mathbb{C}_q^3)$ and to examine how its differential structure interacts with the Hopf algebraic and $O(M_q(3,\mathbb{C}))$-covariant structures obtained in the present chapter. This continuation will provide a natural extension of the present algebraic construction and will contribute to a more complete geometric understanding of the new quantum space.

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