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Exact Controllability for Nonlinear Implicit Caputo Fractional Models with Hybrid Structure

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Abstract
Existing research on fractional hybrid differential equations (abbrev. FHDEs) has largely focused on results concerning the existence, uniqueness or stability of solutions, often under restrictive assumptions and relying on contraction mappings with constants less than one or auxiliary conditions. However, the controllability of nonlinear hybrid implicit fractional systems with control inputs has only been partially addressed. In this paper, we advance the theory by analysing a generalised class of Caputo FHDEs that include explicit control functions. Our contributions are threefold. Firstly, we establish an equivalence theorem that clarifies the precise notion of a solution, forming the basis for all subsequent arguments. Secondly, we prove the existence of solutions under fixed control using Schauder's fixed-point theorem, with auxiliary constructions supported by the Banach contraction principle. Third, we demonstrate exact controllability under assumptions that are strictly weaker than those required in earlier works. These results broaden the class of admissible systems and highlight the flexibility of fixed-point techniques beyond conventional approaches. A physical example illustrates the applicability of the theoretical findings.

Keywords: *Controllability, Existence of solution, Fractional hybrid differential equation*

AMS Subject Classification (2020): *34A08; 34A12; 47H10; 34K34*

1. Introduction

The use of fractional derivatives and integrals has profoundly reshaped the way complex systems are modeled in science and engineering [1, 2]. Allowing the differentiation order $\alpha \in (0, 1)$ to vary continuously, fractional calculus creates a unified framework that naturally interpolates between steady and dynamic states. This approach yields compact yet physically interpretable models of intricate processes, often with fewer empirical parameters than those required by conventional integer-order methods [3]. Such advances have had a direct impact on control theory, inspiring the design of fractional-order models that typically achieve greater robustness and superior performance.

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Controllability, the ability to guide a system from any initial state to a chosen target, lies at the foundation of control theory. In the context of fractional-order systems, this question has generated a substantial literature such as semi-group methods, resolvent operator theory, and fixed-point techniques [4–6]. More recent contributions account for practical features such as delays, impulses, nonlocal conditions, and memory effects [7–9]. A detailed exposition of this topic can be found in the aforementioned studies and the references therein.

A central motivation of this work is the analysis of fractional-order hybrid differential equations (FHDEs), which naturally arise in applications ranging from curved and multilayer beam deflection to gravity-driven flows. Early contributions by Dhage and Lakshmikantham [10] addressed the existence of solutions for first-order HDEs via fixed-point techniques in Banach algebras. This line of research was extended to the fractional framework in [11], where existence results were obtained for a FHDEs involving the Riemann–Liouville derivative. Building on this, Benchohra and Lazreg [12] studied implicit FDEs with the Caputo derivative and established a result about Ulam stability. More recently, these directions are unified in [13] and [14], which introduce a broader class of FHDEs and proved results on existence, uniqueness, and Ulam stability.

Primarily motivated by the aforementioned works and related contributions, we conduct a thorough analysis of several key aspects of this type of FHDEs with control input. In particular, our study investigates the existence of solutions under fixed control and exact controllability. To this end, we consider the following nonlinear hybrid implicit Caputo fractional differential equation with the order $0 < \alpha < 1$:

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}\left(\frac{\omega(t)-\varphi(t,\omega(t))}{\phi(t,\omega(t))}\right) = G\left(t,\omega(t),{}^{C}\mathcal{D}^{\alpha}\left(\frac{\omega(t)-\varphi(t,\omega(t))}{\phi(t,\omega(t))}\right)\right) + b(t)u(t), \quad t \in [0,T] \\ \omega(0) = \theta\phi(0,\omega(0)) + \varphi(0,\omega(0)) \end{cases} \tag{1.1}$$

where $\varphi : [0,T] \times \mathbb{R} \to \mathbb{R}, \phi : [0,T] \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$, and $G : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $b : [0,T] \to \mathbb{R}\backslash\{0\}$ are continuous functions, and $\theta \in \mathbb{R}$ with the control input $u(\cdot) \in L^1([0,T],\mathbb{R})$. The hybrid structure in (1.1), characterized by the term $\frac{\omega(\cdot)-\varphi(\cdot,\omega(\cdot))}{\phi(\cdot,\omega(\cdot))}$, is one of the key aspects that allows the model to capture systems with state-dependent normalization and shifting. This structure is instrumental in our analysis, as it facilitates the application of fixed-point theorems for establishing controllability, following the approach pioneered in [11] for integer-order systems and extended to fractional-order in [13].

Building upon the foundational works of [11, 13, 14], this paper establishes a comprehensive theoretical basis for the analysis and control of nonlinear hybrid implicit fractional systems. Our principal contributions are as follows.

- Placing a control function on the right-hand side not only strengthens the structural perspective of the problem but also establishes a more comprehensive theoretical basis that advances and unifies earlier works like the papers [13, 14], i.e., this shift from analyzing inherent system properties to enabling active control synthesis represents a significant structural and conceptual advancement.

- Prior to our main theorems, we prove an equivalence lemma that rigorously establishes the equivalence between the original fractional differential equation and a nonlinear Volterra integral equation. This lemma serves as the foundation for all subsequent analysis, providing a precise and workable definition of a solution.

- In Theorem 2.2, we guarantee the existence of a solution for any fixed control input. Although standard approaches in this domain, such as those in [4, 13, 14], rely exclusively on the Banach contraction principle, our proof of existence for a fixed control in this result employs a Krasnoselskii-type fixed-point theorem proposed by Burton in [15]. This strategic choice allows us to handle more complex system dynamics by decomposing the problem into a contraction map and a compact map. It is important to note that the Banach principle remains instrumental in guaranteeing the well-definedness of auxiliary functions within this basis, representing an advanced hybrid methodology that moves beyond conventional techniques.

- Theorem 2.3 demonstrates that the system is exactly controllable. This result is stronger than mere existence, as we explicitly construct the control law $u(\cdot)$ that steers the system to any desired state. Achieved solely through fixed-point theory without Gramian-based auxiliary hypotheses, and under a further relaxed assumption, this result significantly enlarges the scope of admissible systems compared to prior analyses in [4, 5, 7].

The proofs of our results does not rely on extra conditions such as imposing that the contraction constant be strictly less than one, thereby enlarging the scope of admissible systems. Taken together, these findings provide a comprehensive theoretical analyses for controlled fractional implicit systems, establishing both the existence and uniqueness of solutions under minimal regularity and non-degeneracy assumptions and demonstrating exact controllability via the fixed-point method, supported by a physical example for illustration.

2. Main results

This section focuses on proving the existence of solutions and the exact controllability of system (1.1). To this end, we first specify the required assumptions and state a central lemma that plays a crucial role in the forthcoming arguments. We then present the relevant notations, definitions, and preliminary results from [16], which provide the framework for the subsequent analysis.

Definition 2.1. [16] Any $\alpha > 0$, α-order Riemann–Liouville (R-L) integral for a L^1 function $\upsilon : [0,T] \to \mathbb{R}^n$ is given by

$$I^{\alpha}\upsilon : t \mapsto \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\upsilon(s)ds, \quad \text{for a.e. } t,$$

where $\Gamma(\cdot)$ is the Gamma function.

This integral operator provides a mapping from $C([0,T],\mathbb{R}^n)$ to itself and also possesses the following characteristic known as the semigroup property.

Lemma 2.1. *[16] Let υ be a continuous function on $[0,T]$. Then, for any $\alpha,\beta > 0$, the relation below holds:*

$$I^{\alpha}I^{\beta}\upsilon(t) = I^{\beta}I^{\alpha}\upsilon(t) = I^{\alpha+\beta}\upsilon(t), \quad \textit{for every } t \in [0,T].$$

Definition 2.2. [10] The R-L fractional derivative $D^{\alpha}\upsilon$ of a function $\upsilon \in L^1[0,T]$ with order $\alpha \in (0,1)$ is defined as

$$D^{\alpha}\upsilon(t) := \frac{d}{dt} I^{1-\alpha}\upsilon(t), \quad \text{for a.e. } t \in [0,T],$$

provided that $I^{1-\alpha}\upsilon$ is absolutely continuous (abbrev. AC).

Definition 2.3. [10] The Caputo derivative denoted as ${}^{C}\mathcal{D}^{\alpha}\upsilon$ is defined by ${}^{C}\mathcal{D}^{\alpha}\upsilon := D^{\alpha}\big(\upsilon - \upsilon(0)\big)$. This definition is valid when the R-L derivative exists, which requires $\upsilon(0)$ to be exist and $I^{1-\alpha}\upsilon$ to be AC.

As an essential tool in proving our main existence result, we state Krasnoselskii's theorem in a general variant proposed by Burton.

Theorem 2.1. *[15] Let Λ be a closed, convex and bounded subset of a Banach space $(\Omega, \|\cdot\|)$. Assume that $\Phi_1 : \Omega \to \Omega$ and $\Phi_2 : \Lambda \to \Omega$ be two operators such that*

- (i) *Φ_1 is a contraction with constant $L < 1$,*
- (ii) *Φ_2 is continuous and compact,*
- (iii) *$\upsilon = \Phi_1\upsilon + \Phi_2\vartheta$ for all $\vartheta \in \Lambda \Rightarrow \upsilon \in \Lambda$.*

Then the operator equation $\Phi_1\upsilon + \Phi_2\upsilon = \upsilon$ has a solution in Λ.

We now introduce the hypotheses that will be used throughout the paper and referred to in the subsequent results.

- (H_1) There exist constants $M_{\phi}, M_G, M_b, M_u > 0$ satisfying $|u(t)| \le M_u$ for a.e. $t \in [0,T]$ such that

$$|\phi(t,\omega)| \le M_{\phi}, \quad |G(t,\omega,z)| \le M_G$$

  with $|b(t)| \le M_b$ for all $t \in [0,T], \omega, z \in \mathbb{R}$.

- (H_2) The functions φ, ϕ are Lipschitz w.r.t. the second variable with the constants L_{φ}, L_{ϕ}, respectively. Additionally, there exist constants $L_G, L_{G_z} > 0$ satisfying $L_{G_z} < 1$ and $L_{\varphi} + |\theta| L_{\phi} < 1$ such that

$$|G(t,\omega,z) - G(t,\omega',z')| \le L_G\,|\omega - \omega'| + L_{G_z}\,|z - z'| \text{ for all } t \in [0,T], \omega, \omega' \in \mathbb{R}, z, z' \in \mathbb{R}.$$

Now we will give a lemma that will explain what we mean by the solution and will be quite useful in future proofs. But before that, let us pay attention to the following remark.

Remark 2.1. The proof of following lemma relies on two standard properties of the Caputo fractional derivative and Riemann-Liouville fractional integral for continuous functions, as established in fractional calculus literature (e.g., [16]):

(1) If $h \in C([0,T],\mathbb{R})$, then ${}^{C}\mathcal{D}^{\rho}\left(I^{\rho}h\right)(x) = h(x)$ for all $x \in [0,T]$.

(2) If ${}^{C}\mathcal{D}^{\rho}w \in C([0,T],\mathbb{R})$ and $w(0)$ exists, then $I^{\rho}\left({}^{C}\mathcal{D}^{\rho}w\right)(x) = w(x) - w(0)$ for all $x \in [0,T]$.

Lemma 2.2. *The solution to the system (1.1) is equivalent to find the solution to the following nonlinear Volterra integral equation*

$$\omega(t) = \varphi(t,\omega(t)) + \theta\phi(t,\omega(t)) + \frac{\phi(t,\omega(t))}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[G\left(s,\omega(s),{}^{C}\mathcal{D}^{\alpha}\left(\frac{\omega(s)-\varphi(s,\omega(s))}{\phi(s,\omega(s))}\right)+b(s)u(s)\right]ds\ . \tag{2.1}$$

Proof. The equivalence proof relies crucially on the stated regularity conditions:

- The continuity of φ, ϕ, G, and b ensures all compositions and integrals are well-defined
- The non-vanishing property of ϕ guarantees the term $\frac{\omega(t)-\varphi(t,\omega(t))}{\phi(t,\omega(t))}$ is well-defined and allows division operations
- The L^1 integrability of u ensures the control term in the integral equation is properly defined
- These conditions collectively justify the application of fractional calculus properties (1) and (2) stated in the preceding remark.

In this direction, let us prove the equivalence in two parts. Let us begin with differential equation implies integral equation assuming $w(\cdot)$ satisfies the differential equation and initial condition. Now define $z(t) = \frac{\omega(t)-\varphi(t,\omega(t))}{\phi(t,\omega(t))}$. From the initial condition, we have $z(0) = \theta$. Then, the system (1.1) becomes ${}^{C}\mathcal{D}^{\alpha}z(t) = G\left(t,\omega(t),{}^{C}\mathcal{D}^{\alpha}z(t)\right) + b(t)u(t)$. Apply the fractional integral operator I^{α} to both sides of this equality and use the property $I^{\alpha}{}^{C}\mathcal{D}^{\alpha}z(t) = z(t) - z(0)$ for $0 < \alpha < 1$, we get

$$z(t) - \theta = I^{\alpha}\left[G\left(t,\omega(t),{}^{C}\mathcal{D}^{\alpha}z(t)\right) + b(t)u(t)\right]$$

which implies that

$$z(t) = \theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[G\left(s,\omega(s),{}^{C}\mathcal{D}^{\alpha}z(s)\right) + b(s)u(s)\right]ds. \tag{2.2}$$

Substitute back $z(s) = \frac{\omega(s)-\varphi(s,\omega(s))}{\phi(s,\omega(s))}$ and ${}^{C}\mathcal{D}^{\alpha}z(s) = {}^{C}\mathcal{D}^{\alpha}\left(\frac{\omega(s)-\varphi(s,\omega(s))}{\phi(s,\omega(s))}\right)$ into (2.2)

$$\frac{\omega(t)-\varphi(t,\omega(t))}{\phi(t,\omega(t))} = \theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[G\left(s,\omega(s),{}^{C}\mathcal{D}^{\alpha}\left(\frac{\omega(s)-\varphi(s,\omega(s))}{\phi(s,\omega(s))}\right)\right)+b(s)u(s)ds\ .$$

Finally, multiplying both sides by $\phi(t,\omega(t))$ and rearranging related terms, we obtain the desired integral equation. Conversely, assume that $\omega(\cdot)$ satisfies (2.1). Rearranging this equation and dividing both sides with $\phi(t,\omega(t))$, we have

$$\frac{\omega(t)-\varphi(t,\omega(t))}{\phi(t,\omega(t))} = \theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[G(s,\omega(s),z(s)) + b(s)u(s)]ds \tag{2.3}$$

where $z(s) = {}^{C}\mathcal{D}^{\alpha}\left(\frac{\omega(s)-\varphi(s,\omega(s))}{\phi(s,\omega(s))}\right)$. Let us define $y(t) = \frac{\omega(t)-\varphi(t,\omega(t))}{\phi(t,\omega(t))}$. Then (2.3) becomes

$$y(t) = \theta + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[G(s,\omega(s),z(s)) + b(s)u(s)]ds. \tag{2.4}$$

Note that the integral in (2.4) is $I^{\alpha}[G(t,\omega(t),z(t)) + b(t)u(t)]$, so $y(t) = \theta + I^{\alpha}[G(t,\omega(t),z(t)) + b(t)u(t)]$. Now, apply the Caputo derivative ${}^{C}\mathcal{D}^{\alpha}$ to both sides of this equality. Since ${}^{C}\mathcal{D}^{\alpha}\theta = 0$ and ${}^{C}\mathcal{D}^{\alpha}I^{\alpha}\psi(t) = \psi(t)$ for continuous ψ, we obtain that ${}^{C}\mathcal{D}^{\alpha}y(t) = G(t,\omega(t),z(t)) + b(t)u(t)$. But $z(t) = {}^{C}\mathcal{D}^{\alpha}\left(\frac{\omega(t)-\varphi(t,\omega(t))}{\phi(t,\omega(t))}\right) = {}^{C}\mathcal{D}^{\alpha}y(t)$, so the last equality becomes that ${}^{C}\mathcal{D}^{\alpha}y(t) = G\left(t,\omega(t),{}^{C}\mathcal{D}^{\alpha}y(t)\right) + b(t)u(t)$, which is the first equation of the system (1.1). Additionally, evaluating the integral equation (2.1) at $t = 0$, we determine that $\omega(0) = \varphi(0,\omega(0)) + \theta\phi(0,\omega(0)) + 0$, which is the initial condition. Thus the proof is complete. □

2.1 Existence of solutions for fixed control

We now proceed to demonstrate the existence of a solution to our system under a fixed control.

Theorem 2.2. *Assume that (H_1) and (H_2) are satisfied. Then, for any fixed control $u(\cdot)$ there exists a solution $\omega \in C([0,T],\mathbb{R})$ to satisfy the system (1.1).*

Proof. The proof utilizes Krasnoselskii's fixed point theorem in the form stated by Burton in Theorem 2.1. Let us introduce the Banach space $X = C([0,T],\mathbb{R})$ with the supremum norm $\|\cdot\|_\infty$. From the Lemma 2.2, the system is equivalent to the integral equation:

$$\omega(t) = \varphi(t,\omega(t)) + \theta\phi(t,\omega(t)) + \frac{\phi(t,\omega(t))}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[G(s,\omega(s),z(s)) + b(s)u(s)]ds$$

where $z(s) = G(s,\omega(s),z(s)) + b(s)u(s)$. Under (H_2), for each ω, the map $z \mapsto G(s,\omega(s),z) + b(s)u(s)$ is a contraction since $L_{Gz} < 1$, so z_ω is uniquely defined and bounded. Specifically, $|z_\omega(t)| \le M_z = \frac{M_G + M_b M_u}{1 - L_{Gz}}$ for all $t \in [0,T]$.

Define operators $\Phi_1, \Phi_2 : X \to X$ as follows

$$\Phi_1(\omega)(t) = \varphi(t,\omega(t)) + \theta\phi(t,\omega(t))$$
$$\Phi_2(\omega)(t) = \frac{\phi(t,\omega(t))}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[G\left(s,\omega(s),z_\omega(s)\right) + b(s)u(s)\right]ds.$$

Next, let us divide our problem into one suitable for fixed-point analysis.
Step 1: For any $\omega, \omega_\star \in X$ we have

$$|\Phi_1(\omega)(t) - \Phi_1(\omega_\star)(t)| \le |\varphi(t,\omega(t)) - \varphi\left(t,\omega_\star(t)\right)| + |\theta||\phi(t,\omega(t)) - \phi\left(t,\omega_\star(t)\right)| \le \left(L_\varphi + |\theta|L_\phi\right)\|\omega - \omega_\star\|_\infty.$$

Since $L_\varphi + |\theta|L_\phi < 1$ by (H_2), Φ_1 is a contraction.
Step 2: Let us show that continuity and compactness of the operator Φ_2.

(i) Let $\omega_n \to \omega$ in X. Since G and ϕ are continuous, and $z_{\omega_n} \to z_\omega$ uniformly (as shown below), it follows that $\Phi_2\left(\omega_n\right) \to \Phi_2(\omega)$ uniformly. To see that $z_{\omega_n} \to z_\omega$, let us investigate the following inequalities

$$|z_{\omega_n}(t) - z_\omega(t)| \le |G\left(t,\omega_n(t),z_{\omega_n}(t)\right) - G\left(t,\omega(t),z_\omega(t)\right)| \le L_G|\omega_n(t) - \omega(t)| + L_{G_z}|z_{\omega_n}(t) - z_\omega(t)|$$

which implies that

$$|z_{\omega_n}(t) - z_\omega(t)| \le \frac{L_G}{1 - L_{G_z}}|\omega_n(t) - \omega(t)|$$

which implies uniform convergence. Thus, Φ_2 is continuous.

(ii) In this part, we will show compactness. For this goal, let us define the closed, convex, and bounded subset of the Banach space $C([0,T],\mathbb{R})$ equipped with the supremum norm

$$\Lambda = \left\{\omega \in X : \|\omega\|_\infty \le R\right\},$$

where

$$R = \frac{M_\varphi + |\theta|M_\phi + C}{1 - L_\varphi},$$

with $M_\varphi = \sup_{t\in[0,T]}|\varphi(t,0)|$ (finite since φ is continuous) and $C = \frac{M_\phi(M_G + M_b M_u)T^\alpha}{\Gamma(\alpha+1)}$. Now our aim is to show that Φ_2 maps the bounded set Λ into a relatively compact set using the Arzelà–Ascoli theorem. Let $H_\omega(t) = G(t,\omega(t),z_\omega(t)) + b(t)u(t)$. Then $|H_\omega(t)| \le M_F := M_G + M_b M_u$ for all $\omega \in \Lambda$ and $t \in [0,T]$. Consider the fractional integral operator I^α. For any function $F \in C([0,T])$ with $\|F\|_\infty \le M_F$, we have the following explicit bounds.

- **For uniform boundedness:**

$$|I^\alpha F(t)| = \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)ds\right| \le \frac{M_F}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds = \frac{M_F t^\alpha}{\Gamma(\alpha+1)} \le \frac{M_F T^\alpha}{\Gamma(\alpha+1)}.$$

- **For equicontinuity with explicit constants:** For any $t, s \in [0,T]$ with $t > s$, we have

$$|I^\alpha F(t) - I^\alpha F(s)| \leq \frac{M_F}{\Gamma(\alpha+1)}|t-s|^\alpha.$$

This bound follows from the Hölder continuity of fractional integrals and can be derived as

$$|I^\alpha F(t) - I^\alpha F(s)| \leq \frac{M_F}{\Gamma(\alpha)}\left|\int_0^t (t-\tau)^{\alpha-1}d\tau - \int_0^s (s-\tau)^{\alpha-1}d\tau\right| = \frac{M_F}{\Gamma(\alpha+1)}|t^\alpha - s^\alpha| \leq \frac{M_F}{\Gamma(\alpha+1)}|t-s|^\alpha,$$

where the last inequality uses the fact that $|t^\alpha - s^\alpha| \leq |t-s|^\alpha$ for $0 < \alpha < 1$.

These bounds show that the set $\{I^\alpha F : \|F\|_\infty \leq M_F\}$ is uniformly bounded by $\frac{M_F T^\alpha}{\Gamma(\alpha+1)}$ and equicontinuous with modulus of continuity $\frac{M_F}{\Gamma(\alpha+1)}|t-s|^\alpha$. By the Arzelà–Ascoli theorem, this set is relatively compact in $C([0,T])$. Now, since $|H_\omega(t)| \leq M_F$ for all $\omega \in \Lambda$, the set $\{I^\alpha H_\omega : \omega \in \Lambda\}$ is relatively compact. Finally, since

$$\Phi_2(\omega)(t) = \frac{\phi(t,\omega(t))}{\Gamma(\alpha)} I^\alpha H_\omega(t),$$

and $|\phi(t,\omega(t))| \leq M_\phi$, the set $\Phi_2(\Lambda)$ is also relatively compact. Therefore, Φ_2 is compact.

Step 3: In this step, we verify invariance condition. Suppose $\omega = \Phi_1(\omega) + \Phi_2(\vartheta)$ for some $\vartheta \in \Lambda$. Then,

$$|\omega(t)| \leq |\varphi(t,\omega(t))| + |\theta||\phi(t,\omega(t))| + |\Phi_2(\vartheta)(t)| \leq |\varphi(t,0)| + L_\varphi|\omega(t)| + |\theta|M_\phi + C,$$

then arranging this inequality we have

$$|\omega(t)|\,(1 - L_\varphi) \leq M_\varphi + |\theta|M_\phi + C,$$

which implies that $\|\omega\|_\infty \leq R$. Thus, $\omega \in \Lambda$. Finally, by the Theorem 2.1, there exists $\omega \in \Lambda$ such that $\omega = \Phi_1(\omega) + \Phi_2(\omega)$, which is a solution to the integral equation and hence to the original system (1.1). This proves the existence of a solution for the fixed control $u(\cdot)$. □

Remark 2.2. The constants M_ϕ, M_G, M_b, M_u in (H_1) and $M_\varphi = \sup_{t\in[0,T]}|\varphi(t,0)|$ play crucial roles in ensuring the well-posedness and convergence of our method: M_ϕ bounds the hybrid scaling function ϕ, ensuring the term $\frac{\omega-\varphi}{\phi}$ remains well-defined and bounded. M_G bounds the nonlinear function G, controlling its growth and ensuring the fixed-point operator remains bounded. M_b and M_u bound the control coefficient and input, respectively, ensuring the control term contributes finitely to the system dynamics. M_φ bounds the baseline value of φ, which together with its Lipschitz constant L_φ controls the growth of the solution.

These constants collectively determine the radius R of the invariant set Λ

$$R = \frac{M_\varphi + |\theta|M_\phi + \frac{M_\phi(M_G + M_b M_u)T^\alpha}{\Gamma(\alpha+1)}}{1 - L_\varphi}.$$

This explicit bound demonstrates that all solutions remain within a ball of radius R in $C([0,T],\mathbb{R})$, and the iterative scheme converges within this bounded set. The dependence on the fractional order α appears through the term $T^\alpha/\Gamma(\alpha+1)$, which arises from the fractional integral operator and decreases as α increases, reflecting the regularizing effect of fractional integration.

2.2 Exact controllability

In this section, we present the exact controllability result for system (1.1) by deeply using fixed-point theory, without the need for auxiliary hypotheses such as those typically required like Gramian-based approach. While the preceding theorem established the existence of a solution under a fixed control input, the result here is stronger: we explicitly construct the control function that steers the system to the prescribed target state.

Definition 2.4. A control system is exact controllable on $[0,T]$ if for any desired state $\omega_d \in \mathbb{R}$, there exists a control function $u \in L^1([0,T],\mathbb{R})$ such that the solution $\omega \in C([0,T],\mathbb{R})$ of the system satisfies $\omega(T) = w_d$.

In the following result, let us use the following weakened version of the hypothesis (H_2).

$\overline{(H_2)}$ The functions φ and ϕ are Lipschitz w.r.t. their second argument, with constants L_φ and L_ϕ, respectively, such that $L_\varphi + |\theta| L_\phi < 1$.

Let us state our results related to exact controllability.

Theorem 2.3. *Assume that $\overline{(H_2)}$ is satisfied. Then the system (1.1) is exact controllable on $[0,T]$.*

Proof. Let us begin by clearly stating our goal in this proof. From the equivalence Lemma 2.2, the system is equivalent to the following integral

$$\omega(t) = \varphi(t,\omega(t)) + \theta\phi(t,\omega(t)) + \frac{\phi(t,\omega(t))}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(G(s,\omega(s),z(s)) + b(s)u(s)\right) ds, \tag{2.5}$$

where

$$z(s) = {}^C\mathcal{D}^\alpha \left(\frac{\omega(s) - \varphi(s,\omega(s))}{\phi(s,\omega(s))} \right).$$

Moreover, again by the Lemma 2.2, we know that our original system (1.1) satisfies following equality

$$z(s) = G(s,\omega(s),z(s)) + b(s)u(s). \tag{2.6}$$

Now, our aim is to choose $u(\cdot)$ so that $\omega(T) = \omega_d$. For this, let us begin to evaluate (2.5) at the time $t = T$ and set $\omega(T) = \omega_d$.

$$\omega_d = \varphi(T,\omega_d) + \theta\phi(T,\omega_d) + \frac{\phi(T,\omega_d)}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} [G(s,\omega(s),z(s)) + b(s)u(s)] ds.$$

By using (2.6) above, we have

$$\omega_d = \varphi(T,\omega_d) + \theta\phi(T,\omega_d) + \frac{\phi(T,\omega_d)}{\Gamma(\alpha)} \underbrace{\int_0^T (T-s)^{\alpha-1} z(s) ds}_{:=C}.$$

where C is a constant. This constant comes from the following calculations

$$\int_0^T (T-s)^{\alpha-1} z(s) ds = \frac{\Gamma(\alpha)}{\phi(T,\omega_d)} \left(\omega_d - \varphi(T,\omega_d) - \theta\phi(T,\omega_d)\right) = C. \tag{2.7}$$

Note here that $\phi(T,\omega_d) \neq 0$ by the problem statement, so C is well-defined.
If we choose $z(\cdot)$ to be constant function like $z(s) = \frac{C\alpha}{T^\alpha}$ for all $s \in [0,T]$. Then, obviously the equality (2.7) is satisfied. Now, let us determine the state function $\omega(\cdot)$ by using this z. When we think again equation (2.5) with the idea $\int_0^t (t-s)^{\alpha-1} z(s) ds = \frac{C\alpha}{T^\alpha} \cdot \frac{t^\alpha}{\alpha} = \frac{Ct^\alpha}{T^\alpha}$, we obtain the following estimation

$$\omega(t) = \varphi(t,\omega(t)) + \theta\phi(t,\omega(t)) + \frac{\phi(t,\omega(t))Ct^\alpha}{\Gamma(\alpha)T^\alpha}. \tag{2.8}$$

Now, let us show that there is a solution of (2.8). For this, call $F(t,\omega(t))$ of the right-hand side of it. By the hypothesis $\overline{(H_2)}$, φ and ϕ are Lipschitz in the second argument, the operator

$$(F\omega)(t) = \varphi(t,\omega(t)) + \theta\phi(t,\omega(t)) + \frac{\phi(t,\omega(t))Ct^\alpha}{\Gamma(\alpha)T^\alpha}$$

satisfies

$$\|F\omega_1 - F\omega_2\|_\infty \leq \left(L_\varphi + |\theta| L_\phi + \frac{L_\phi |C|}{\Gamma(\alpha)} \frac{\tau^\alpha}{T^\alpha} \right) \|\omega_1 - \omega_2\|_\infty.$$

By using $L_\varphi + |\theta| L_\phi < 1$ and choosing $\tau > 0$ sufficiently small ensures that the coefficient is less than 1 , so F is a contraction on $C([0,\tau])$ with $\tau \leq T$. By the Banach fixed-point theorem, a unique solution exists on $[0,\tau]$, and by

iterating this argument step by step, we obtain a unique solution $\omega \in C([0,T])$ for equation (2.8). Now let us verify that $\omega_z(T) = \omega_d$. At $t = T$, the concerning equation (2.8) gives us the following

$$\omega_z(T) = \varphi\left(T, \omega_z(T)\right) + \theta\phi\left(T, \omega_z(T)\right) + \frac{\phi(T, \omega_z(T))Ct^\alpha}{\Gamma(\alpha)T^\alpha}. \tag{2.9}$$

Now, define the function as follows

$$\Psi(\cdot) = \varphi(T, \cdot) + \theta\phi(T, \cdot) + \frac{\phi(T, \cdot)Ct^\alpha}{\Gamma(\alpha)T^\alpha}.$$

Then (2.9) says us that $\omega_z(T) = \Psi\left(\omega_z(T)\right)$, so we determine $\omega_z(T)$ is a fixed point of the operator Ψ. Let us check that ω_d is also a fixed point:

$$\Psi\left(\omega_d\right) = \varphi\left(T, \omega_d\right) + \theta\phi\left(T, \omega_d\right) + \frac{\phi\left(T, \omega_d\right)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha)}{\phi\left(T, \omega_d\right)} \left(\omega_d - \varphi\left(T, \omega_d\right) - \theta\phi\left(T, \omega_d\right)\right) = \omega_d.$$

Moreover, by the $\overline{(H_2)}$, our new operator Ψ is also a contraction as the case by previous operator. Hence, Ψ has a unique fixed point, so $\omega_z(T) = \omega_d$. At the final step, with all of these constructions, let us explicitly determine the control function $u(\cdot)$. By the equality (2.6) and determined functions $z(\cdot)$ and $w_z(\cdot)$ above, let us define the following control

$$u(s) := \frac{z(s) - G(s, \omega_z(s), z(s))}{b(s)}.$$

To rigorously verify that $u(\cdot) \in L^1([0,T], \mathbb{R})$, we note the following. Firstly, since $b(t)$ is continuous on the compact interval $[0,T]$ and $b(t) \neq 0$ for all $t \in [0,T]$, by the extreme value theorem, there exists $b_0 > 0$ such that $|b(t)| \geq b_0$ for all $t \in [0,T]$. Consequently, $1/b(t)$ is continuous and bounded on $[0,T]$. Secondly, since $z(s)$ is continuous by the definition, $\omega_z(s)$ is continuous (as the fixed point solution), and G is continuous by assumption (H_1) then the function $z(s) - G(s, \omega_z(s), z(s))$ is continuous on $[0,T]$. Finally, since $u(s) = \frac{z(s) - G(s, \omega_z(s), z(s))}{b(s)}$ is the quotient of two continuous functions with non-vanishing denominator, $u(s)$ is continuous on $[0,T]$. By the boundedness theorem, continuous functions on compact intervals are bounded, so $u(s)$ is bounded. Therefore, $u \in C([0,T], \mathbb{R}) \subset L^1([0,T], \mathbb{R})$. To summarize everything that has been done here: with $u(\cdot)$ defined by above, and ω_z the solution of the equation (2.8), we conclude that (i) ω_z satisfies the integral equation (2.5) with this $u(\cdot)$, and (ii) equality $\omega_z(T) = \omega_d$ hold. Therefore, the system is exact controllable. □

2.3 An example

In robotics, particularly in systems with viscoelastic components such as prosthetic limbs or soft robots, the dynamics often exhibit memory effects that can be modeled using fractional calculus. The following example demonstrates the exact controllability of a robotic arm joint with fractional-order dynamics, where the goal is to achieve a desired angular position at a specific time. In this sense, let us consider the joint angle $\omega(t)$ of the robotic arm is governed by the following nonlinear hybrid implicit Caputo fractional differential equation with order $\alpha = 0.5$ on the time interval $[0,1]$.

$$\begin{cases} {}^C\mathcal{D}^{0.5}\left(\frac{\omega(t) - \varphi(t, \omega(t))}{\phi(t, \omega(t))}\right) = G\left(t, \omega(t), {}^C\mathcal{D}^{0.5}\left(\frac{\omega(t) - \varphi(t, \omega(t))}{\phi(t, \omega(t))}\right)\right) + b(t)u(t), \quad t \in [0,1] \\ \omega(0) = \theta\phi(0, \omega(0)) + \varphi(0, \omega(0)) \end{cases}$$

where $\varphi(t, \omega) = 0$ (no transformation), $\phi(t, \omega) = 1$ (constant scaling), $G(t, \omega, z) = -\omega - 0.5z$ (spring and fractional damping terms), $b(t) = 1$ (constant control gain), $\theta = 0$ (initial condition), which are chosen for simplicity and applicability. Then the system simplifies to the following

$${}^C\mathcal{D}^{0.5}\omega(t) = -\omega(t) - 0.5{}^C\mathcal{D}^{0.5}\omega(t) + u(t).$$

Rearranging terms we get the following system with the initial condition $\omega(0) = 0$

$${}^C\mathcal{D}^{0.5}\omega(t) = -\frac{2}{3}\omega(t) + \frac{2}{3}u(t).$$

As can be seen that all functions φ, ϕ, G, and b are continuous. φ, ϕ are Lipschitz with the constant 0, G is Lipschitz with constants $L_G = 1$ with $L_{G_z} = 0.5 < 1$, and thus $L_\varphi + |\theta| L_\phi = 0 < 1$. Hence we see by the Theorem 2.3, our system is exact controllable. Let us verify it more explicit way by using it. For a desired state $\omega_d \in C([0,1], \mathbb{R})$,

$$C = \frac{\Gamma(0.5)}{\phi(1, \omega_d)} \left(\omega_d - \varphi(1, \omega_d) - \theta\phi(1, \omega_d)\right) = \Gamma(0.5)\omega_d = \sqrt{\pi}\omega_d \approx 1.772\omega_d$$

Then,

$$L_\varphi + |\theta| L_\phi + \frac{|C| L_\phi}{\Gamma(0.5)} = 0 + 0 + 0 = 0 < 1.$$

To achieve exact controllability, we want to find a control $u(t)$ such that $\omega(1) = \omega_d$. From the last theorem, we define the auxiliary variable $z(s) = {}^C\mathcal{D}^{0.5}\omega(s)$ and set it to a constant

$$z(s) = \frac{C\alpha}{T^\alpha} = \frac{\sqrt{\pi}\omega_d \cdot 0.5}{1^{0.5}} = \frac{\sqrt{\pi}}{2}\omega_d \approx 0.886\omega_d.$$

Then the concerning differential equation

$$z(t) = -\frac{2}{3}\omega(t) + \frac{2}{3}u(t)$$

Solving for $u(t)$ we have that

$$u(t) = \frac{3}{2}z(t) + \omega(t) = \frac{3}{2} \cdot 0.886\omega_d + \omega(t) = 1.329\omega_d + \omega(t).$$

The state $\omega(t)$ is found from the integral equation

$$\omega(t) = \frac{1}{\Gamma(0.5)} \int_0^t (t-s)^{-0.5} z(s)\, ds = \frac{1}{\sqrt{\pi}} \int_0^t (t-s)^{-0.5} \cdot 0.886\omega_d\, ds$$

Evaluating the integral $\int_0^t (t-s)^{-0.5}\, ds = 2t^{0.5}$, we obtain

$$\omega(t) = \frac{0.886\omega_d}{\sqrt{\pi}} \cdot 2t^{0.5} = \frac{1.772\omega_d}{\sqrt{\pi}} t^{0.5} = \omega_d t^{0.5}$$

since $\sqrt{\pi} \approx 1.772$. Therefore, $\omega(t) = \omega_d t^{0.5}$, and at $t = 1$, $\omega(1) = \omega_d$. Here our control law is as follows

$$u(t) = 1.329\omega_d + \omega_d t^{0.5} = \omega_d(1.329 + t^{0.5}).$$

It means that this control law ensures that the robotic arm reaches the desired angle ω_d at time $t = 1$, see the following figure.

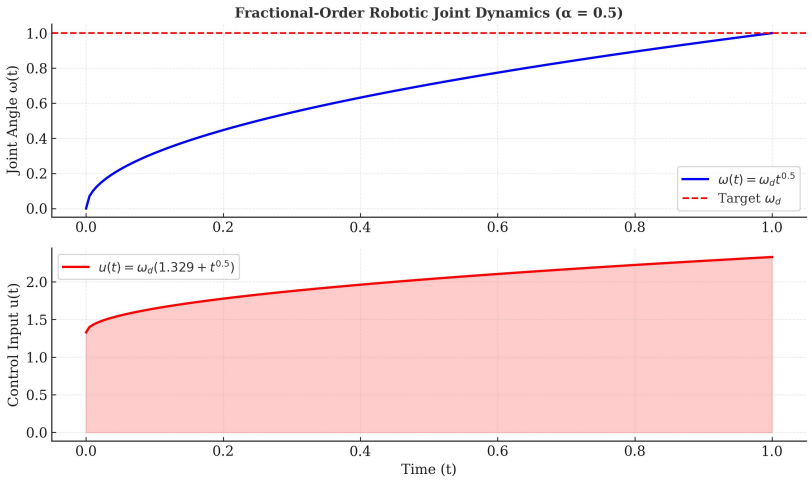


Figure 1. Illustration of the system's behavior in terms of both the state trajectory $\omega(t)$ and the control input $u(t)$.

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